23 July 2001

ELSEVIER

PHYSICS LETTERS A

Physics Letters A 286 (2001) 153–160

www.elsevier.com/locate/pla

New symmetries for a model of fast diffusion

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Received 13 March 2001; received in revised form 31 May 2001; accepted 5 June 2001
Communicated by C.R. Doering

Abstract

In this Letter we prove that, for some partial differential equations that model diffusion, by using the nonclassical method we obtain several new solutions which are not invariant under any Lie group admitted by the equations and consequently which are not obtainable through the classical Lie method. For these partial differential equations that model fast diffusion new classes of symmetries are derived. These *nonclassical potential* symmetries allow us to increase the number of exact explicit solutions of these nonlinear diffusion equations. These solutions are neither nonclassical solutions of the diffusion equation nor solutions arising from classical potential symmetries. Some of these solutions exhibit an interesting behavior as a shrinking pulse formed out of the interaction of two kinks.

1. Introduction

It is known [23] that the diffusion processes appear in many physics processes such as plasma physics, kinetic theory of gases, solid state [3] and transport in porous medium. One of the mathematical model for diffusion processes is

$$u_t = \left(u^{-\alpha} u_x\right)_x. \tag{1}$$

In many metals and ceramic metals, the thermal coefficient of conductivity or diffusion coefficient, if u represents mass concentration, can be approximated as $u^{-\alpha}$. Its divergence for small u causes a much faster spread of heat than in the linear case. In [23] Rosenau presented a number of remarkable features of the fast diffusion processes; for $1 \leqslant \alpha \leqslant 2$, the family of fast diffusion (1) coexists with a subclass of superfast diffusions where the whole process terminates within a finite time. The special case with $\alpha = 1$ emerges in plasma physics and reveals a surprising richness of new physicomathematical phenomena. For $\alpha = 0$ Eq. (1) becomes the heat equation and for $\alpha < 0$ Eq. (1) represents the porous medium equation.

In the past years we can observe a significant progress in application of symmetries to the study of nonlinear partial differential equations of physical importance, as well as in finding exact solutions for such equations. Lie classical symmetries admitted by nonlinear PDEs are useful for finding invariant solutions, as well as to discover whether or not the equation can be linearized by an invertible mapping and construct an explicit linearization when one exists. The first physically meaningful nonlinear evolution equation to be solved exactly was Burger's equation. This equation was mapped to the heat equation [12,19], by a rather simple transformation. It should be remarked that this equation possesses neither solitons nor infinitely many conservations laws. Instead possess infinitely many symmetries [20]. It seems [13] that the

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0375-9601/01/$ – see front matter
PII: S0375-9601(01)00405-4

possession of infinitely many symmetries is the defining feature of an exact solvable equation. In [13] Fokas apply the Lie–Bäcklund symmetry approach to show that the most general equation of the form

$$u_t = g(u)u_{xx} + f(u, u_x)$$

with $g_u \neq 0$ which is exactly solvable of the normal type is equivalent to the equation

$$u_t = \left[(\beta u + \gamma)^{-2} u_x\right]_x + \alpha(\beta u + \gamma)^{-2} u_x,$$

where α, β, γ are arbitrary constants parameters.

A complete group classification for the nonlinear heat equation (1), with $\alpha \neq 1$, was derived by Ovsiannikov [21] by considering the PDE as a system of PDEs, and by Bluman [5]. A classification for Lie–Bäcklund symmetries was obtained by Bluman and Kumei [6]. The importance of the effect of space-dependent parts on the overall dynamics of (2) is well known. In [15] a group classification problem was solved for equation

$$u_t = \left(u^n\right)_{xx} + f(x)u^s u_x + g(x)u^m, \tag{2}$$

with $n \neq 0$, by studying those spatial forms which admit the classical symmetry group. Both the symmetry group and the spatial dependence were found through consistent application of the Lie group formalism.

Motivated by the facts that symmetry reductions for many PDEs are known that are not obtained using the classical Lie group method, or that the Lie classical symmetry groups are rather trivial including at most space and time translations and scale transformations there have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole [5] developed the nonclassical method to study the symmetry reductions of the heat equation. The basic idea of the method is to require that the PDE and the invariance surface condition

$$\xi u_x + \tau u_t - \phi = 0, \tag{3}$$

which is associated with the vector field

$$V = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u, \tag{4}$$

are both invariant under the transformation with infinitesimal generator (4). Since then, a great number of papers have been devoted to the study of nonclassical symmetries of nonlinear PDEs in both one and several dimensions. Classical and nonclassical symmetries of the nonlinear equation (2) with $n = 1$ and $g(x) =$ constant are considered by Clarkson and Mansfield [9], and by Arrigo et al. [1] constructing several new exact solutions. In [2] Arrigo and Hill applied the nonclassical method of Bluman and Cole to study the nonlinear diffusion equation with a nonlinear source

$$u_t = \left[D(u)u_x\right]_x + Q(u). \tag{5}$$

The authors determine those source terms admitting nonclassical symmetry reductions for the case of exponential and power law diffusivity, that is,

$$D(u) = u^m, \qquad D(u) = e^u.$$

They only considered the case for which $\tau \neq 0$. In previous works, we have obtained nonclassical symmetries for

$$u_t = \left(u^n\right)_{xx} + g(x)u^m, \tag{6}$$

a porous medium equation with absorption [17], and for

$$u_t = \left(u^n\right)_{xx} + f(x)u^s u_x, \tag{7}$$

a porous medium with convection [18].

Recently it has been shown [22] that the nonlinear diffusion equation

$$u_t = \left[D(u)u_x\right]_x \tag{8}$$

can be reduced to Fujita's equation (that is, $D(u) = 1/(a_1 + a_2 u + a_3 u^2)$) if it admits a class of generalized conditional symmetries and some new exact solutions of (1) have been obtained for $\alpha = 1$ and $\alpha = 2$ via the method of generalized conditional symmetries developed by Fokas and Liu.

In [24], Zhdanov and Lahno have applied the nonclassical (or conditional) method to the one-dimensional porous medium equation

$$u_t - (uu_x)_x = 0. \tag{9}$$

The nonclassical method in the case for which $\tau \neq 0$ does not lead to any new symmetry but the classical Lie symmetries.

According to these authors [24] the nonclassical method for (9) as well as for the parabolic type PDEs is inefficient. They claim that once obtained new nonclassical symmetries in the case for which $\tau = 0$, performing the symmetry reductions gives rise to invariant solutions corresponding to the Lie symmetries of (9).

One of the aims of this Letter is to prove that the nonclassical method, in the case for which $\tau = 0$, applied to (1) gives rise to new solutions of (1) which are not group-invariant and consequently cannot be obtained by Lie classical symmetries. This result is a counterexample of the statement done in [24].

Nevertheless, an obvious limitation of group-theoretic methods based in local symmetries is that many PDEs do not have local symmetries. It turns out that PDEs can admit nonlocal symmetries whose infinitesimal generators depend on the integrals of the dependent variables in some specific manner. It also happens that if a nonlinear scalar PDE does not admit an infinite-parameter Lie group of contact transformations it is not linearizable by an invertible contact transformation. However, most of the interesting linearizations involve noninvertible transformations, such linearizations can be found by embedding given nonlinear PDEs in auxiliary systems of PDEs [7].

In [6], Bluman introduced a method to find a new class of symmetries for a PDE when it can be written in a conserved form. These symmetries are neither point symmetries nor Lie–Bäcklund symmetries, they are nonlocal symmetries which are called *potential* symmetries. Potential symmetries were obtained in [16] for the porous medium equation (2) when it can be written in a conserved form.

Knowing that an associated system to the Boussinesq equation has the same classical symmetries as the Boussinesq equation, Clarkson [10] proposed as an open problem if an auxiliary system of the Boussinesq equation does possess more or less nonclassical symmetries than the equation itself. Bluman claims [4] that the ansatz to generate nonclassical solutions of the associated system could yield solutions of the original equation which are neither nonclassical solutions nor solutions arising from potential symmetries.

In this Letter we will derive, for (1), a new class of potential symmetries called *nonclassical potential* symmetries, which are realized as nonclassical symmetries of an associated system. The significance of these symmetries will be pointed out by the fact that there are equations of great interest as some diffusion equations (1) that model fast processes for which *no* classical potential symmetries are admitted. Nevertheless, the nonclassical method applied to the corresponding associated potential system lead to new symmetries *nonclassical potential* symmetries as well as to new exact solutions. Some of these solutions exhibit an interesting behaviour as a shrinking pulse formed out of the interaction of two kinks.

2. Nonclassical symmetries

By requiring that both (1) and (3) are invariant under the transformation with infinitesimal generator (4) one obtains an overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The number of determining equations arising in the nonclassical method is smaller than for the classical method. Consequently, the set of solutions is in general, larger than for the classical method as in this method one requires only the subset of solutions of (1) and (3) to be invariant under the infinitesimal generator (4).

To obtain nonclassical symmetries of (1) we apply the algorithm described in [11] for calculating the determining equations. We can distinguish two different cases:

In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,u) = 1$. The nonclassical method applied to (1) give rise to four nonlinear determining equations.

After solving the determining system we can assure that for $\alpha \neq 1/2$ we only recover the classical symmetries, consequently a complete classification of the nonclassical symmetries of the governing equation has been performed for $\tau \neq 0$ and we we can state:

Eq. (1) *admits proper nonclassical symmetries with* $\tau = 1$ *only for* $\alpha = 1/2$.

The corresponding determining equations gives rise to

$$\xi = \xi_1(x,t),$$
$$\phi = \phi_1(x,t)u^{1/2} + \phi_2(x,t)u,$$

where $\xi_1(x,t)$, $\phi_1(x,t)$ and $\phi_2(x,t)$ are related by the following conditions:

$$\frac{\partial \phi_1}{\partial t} + \phi_1\phi_2 + 2\phi_1\frac{\partial \xi_1}{\partial x} + \frac{\partial^2 \phi_2}{\partial x^2} = 0,$$
$$2\frac{\partial^2 \phi_1}{\partial x^2} - \phi_1^2 = 0,$$

$$2\frac{\partial\phi_2}{\partial t}+\phi_2^2+4\frac{\partial\xi_1}{\partial x}\phi_2=0,$$
$$\xi_1\phi_2+4\xi_1\frac{\partial\xi_1}{\partial x}+\frac{\partial\xi_1}{\partial t}=0,$$
$$\xi_1\phi_1+2\frac{\partial\phi_2}{\partial x}+2\frac{\partial^2\xi_1}{\partial x^2}=0.$$

Choosing $\xi_1=\phi_2=0$ and $\phi_1=\phi_1(x)$ we obtain the nonclassical symmetry

$$\xi=0,\qquad \tau=1,\qquad \phi=\frac{12}{x^2}u^{1/2},$$

and we get the nonclassical reduction that yields to the exact solution

$$u=\left(\frac{6t}{x^2}+\mathrm{k}_1x^3+\frac{\mathrm{k}_2}{x^2}\right)^2. \tag{10}$$

This solution (10), which has been already obtained in [1], is not group-invariant, consequently it is unobtainable by Lie classical symmetries.

In the case $\tau=0$, without loss of generality, we may set $\xi=1$ and the determining equation for the infinitesimal ϕ is

$$\phi_t u^{\alpha+2}-\left(\phi_{xx}+\phi^2\phi_{uu}+2\phi\phi_{ux}\right)u^2 + \left(3\alpha\phi\phi_x+2\alpha\phi^2\phi_u\right)u-\alpha(\alpha+1)\phi^3=0. \tag{11}$$

The complexity of this equation is the reason why we cannot solve (11) in general. Thus we will proceed, by making several ansatzs on the form of $\phi(x,t,u)$, to solve (11). Due to the invariance under temporal and spatial translations, the solutions that we obtain will also be solutions by substituting x by $x-x_0$ and t by $t-t_0$. Due to the fact that for $\alpha=2$ (1) can be linearized and it is exactly solvable we are considering $\alpha\neq 2$. As far as we know, this is the first time that nonclassical symmetries with $\tau=0$ have been derived for (1), in previous works [1] only the case with $\tau\neq 0$ has been considered.

Case 1. $\alpha\neq 2$. Choosing $\phi=\eta_1(x,t)u^{\alpha+1}+\eta_2(x,t)u^{\alpha}$ and setting $\alpha\neq 2$ leads to the nonclassical generator

$$\phi=u^{\alpha+1}\left(\frac{x}{(\alpha-2)(t+\mathrm{k}_1)}+\frac{\mathrm{k}_2}{t+\mathrm{k}_1}\right)+\frac{\mathrm{k}_3u^{\alpha}}{t+\mathrm{k}_1}.$$

Solving the surface condition yields to the nonclassical reductions leading to the exact solution

$$u=Kt^{1/\alpha}\left(x^2+\beta t^{2/(2-\alpha)}\right)^{-1/\alpha} \tag{12}$$

with $K=(2(2-\alpha)/\alpha)^{1/\alpha}$ and $\beta=$ constant. In the particular case $\alpha=1$, this is the well known source solution. For some special values of parameter α, besides solution (12) we can obtain the following:

Case 2. $\alpha=1$. Choosing

$$\phi=\eta_1(x,t)u^2+\eta_2(x,t)u$$

we get that $\eta_1(x,t)$ and $\eta_2(x,t)$ must satisfy the following conditions:

$$\eta_2\frac{\partial\eta_2}{\partial x}-\frac{\partial^2\eta_2}{\partial x^2}=0, \tag{13}$$
$$\eta_1\frac{\partial\eta_2}{\partial x}+\frac{\partial\eta_2}{\partial t}-\frac{\partial\eta_1}{\partial x}\eta_2-\frac{\partial^2\eta_1}{\partial x^2}=0, \tag{14}$$
$$\frac{\partial\eta_1}{\partial t}-\eta_1\frac{\partial\eta_1}{\partial x}=0. \tag{15}$$

By solving (13)–(15) several cases appear. In the following we show the infinitesimal generators ϕ corresponding to η_1 constant and the corresponding exact solution u.

Subcase 2.1. If $k_1>0$, setting $\sqrt{k_1}=k$ we get

$$\phi=\sqrt{2}ku\tan\left(\frac{\sqrt{2}k(x+k_2t+k_3)}{2}\right)-k_2u^2,$$
$$u=\frac{\sqrt{2}k\cos(\sqrt{2}kk_2t)}{k_2(\sin(\sqrt{2}kx)-\sin(\sqrt{2}kk_2t))}.$$

Subcase 2.2. If $k_1<0$, setting $\sqrt{-k_1}=k$ we get

$$\phi=-\sqrt{2}ku\tanh\left(\frac{\sqrt{2}k(x+k_2t+k_3)}{2}\right)-k_2u^2,$$
$$u=\frac{\sqrt{2}k\cosh(\sqrt{2}kk_2t)}{k_2(\sinh(\sqrt{2}kx)-\sinh(\sqrt{2}kk_2t))}.$$

Subcase 2.3. If $k_1=0$, we get

$$\phi=-\frac{2}{x+k_2t}u-k_2u^2,$$
$$u=\frac{2(t+k_4)}{(x-k_2t-2k_2k_4)(x+k_2t)}.$$

Subcase 2.4.

$$\phi=\frac{k_1u}{t}-\frac{u^2(x-k_2)}{t},$$
$$u=\frac{k_1^2e^{k_1x/t}}{(k_1x-t)e^{k_1x/t}+k_2t}.$$

These new solutions are unobtainable by Lie classical symmetries and this fact proves that the method of nonclassical symmetries with $\tau = 0$ is *effective* for the parabolic type equations in contradiction with the statement done in [24]. "The conditional symmetries of parabolic equations yield solutions which are nothing else than group-invariant solutions."

3. Proof of inexistence of other solutions, in the case $\alpha = 1$

Let us find the general solution, if it exists, of system (13)–(15). One first replaces (13) by its x-primitive

$$-2G_2^2 + \frac{1}{2}\eta_2^2 - \eta_{2,x} = 0, \tag{16}$$

in which the function G_2 only depends on t. Then one solves system (16), (14), (15) as a linear Cramer system for a choice of three derivatives, e.g., the one which eliminates ∂_t:

$$\eta_{2,x} = -2G_2^2 + \frac{1}{2}\eta_2^2, \tag{17}$$

$$\eta_{2,t} = -\eta_1\left(-2G_2^2 + \frac{1}{2}\eta_2^2\right) + \eta_{1,x}\eta_2 + \eta_{1,xx}, \tag{18}$$

$$\eta_{1,t} = \eta_1\eta_{1,x}. \tag{19}$$

Then one enforces all the Schwarz cross-derivative conditions (there is only one)

$$(\eta_{2,x})_t - (\eta_{2,t})_x = -4G_2G_2' + \eta_2^2\eta_{1,x} - \eta_{1,xxx} = 0. \tag{20}$$

The two ODEs (16), (20), when solved successively, are enough to perform the complete discussion of the original system (13)–(15). The general solution η_2 of (16) is

$$\begin{cases} -2G_2(t)\coth(G_2(t)(x - g_0(t))), \\ \quad \text{if } \eta_{2,x} \neq 0, \\ 2G_2(t), \quad \text{if } \eta_{2,x} = 0. \end{cases} \tag{21}$$

The insertion of this value in (20) defines a second-order linear inhomogeneous ODE for $\eta_{1,x}$. One must distinguish four cases, according as whether the solution of each ODE is characteristic or not, i.e., $(\eta_{1,xxx}, \eta_{2,x}) = (\neq 0, \neq 0)$, $(\neq 0, 0)$, $(0, \neq 0)$, $(0, 0)$. The condition $\eta_{1,xxx} = 0$, together with (15), has two solutions

$$\eta_1 = \begin{cases} k_1, \\ -\frac{x - x_0}{t - t_0}. \end{cases} \tag{22}$$

Hence the four cases:

1. $\eta_{1,xxx} \neq 0$, $\eta_{2,x} \neq 0$ (the generic case). Eq. (20) is an inhomogeneous Lamé equation [14] for the unknown $\eta_{1,x}$, with an irrational Fuchsian index n solution of

   $$n(n+1) - 4 = 0, \quad n = \frac{-1 \pm \sqrt{17}}{2}. \tag{23}$$

   Therefore any particular solution η_1 has movable critical singularities (on the manifold $x - g_0(t) = 0$), and must be discarded.
2. $\eta_{1,xxx} \neq 0$, $\eta_{2,x} = 0$. This implies $G_2(t) \neq 0$, Eq. (20) integrates as

   $$\eta_1 = f_1(t) + f_2(t)e^{2xG_2(t)} + f_3(t)e^{-2xG_2(t)}, \tag{24}$$

   but Eqs. (14) and (15) require $f_2 = f_3 = 0$, impossible.
3. $\eta_{1,xxx} = 0$, $\eta_{2,x} \neq 0$. This implies $G_2' = 0$, $g_0' = 0$, this is the solution denoted Subcase 2.1 (or Subcase 2.2, the same in the complex plane, or its degeneracy Subcase 2.3).
4. $\eta_{1,xxx} = 0$, $\eta_{2,x} = 0$. This provides either $\eta_1 = k_1$, $\eta_2 = k_2$, or the solution denoted Subcase 2.4.

4. Nonclassical potential symmetries

We consider the associated auxiliary system given by

$$\begin{aligned} v_x &= u, \\ v_t &= u^{-\alpha}u_x, \end{aligned} \tag{25}$$

augmented with the invariance surface condition

$$\xi v_x + \tau v_t - \psi = 0, \tag{26}$$

which is associated with the vector field

$$\begin{aligned} V = {} & \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t \\ & + \phi(x,t,u,v)\partial_u + \psi(x,t,u,v)\partial_v. \end{aligned} \tag{27}$$

By requiring both (25) and (26) to be invariant under the transformation with infinitesimal generator (27) one obtains an over determined, nonlinear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\psi(x,t,u,v)$, $\phi(x,t,u,v)$. When at least one of the generators of the group depend explicitly of the potential, that is, if

$$\xi_v^2+\tau_v^2+\phi_v^2\neq 0, \tag{28}$$

then (27) yields a nonlocal symmetry of (1).

A nonclassical potential symmetry of (1) *is a nonclassical symmetry of the associated potential system* (25) *that satisfies* (28).

We can distinguish two different cases: $\tau=0$ and $\tau\neq 0$. For $\tau=0$, although we are able to derive some new generators, we do not get any new solution for (1).

In the second case, we set $\tau=1$ without loss of generality. The nonclassical method, with $\tau\neq 0$, applied to (25) give rise to two nonlinear determining equations for the infinitesimals. If we require that $\xi_u=\psi_u=0$, we obtain

$$\phi=-\xi_v u^2+(\psi_v-\xi_x)u+\psi_x, \tag{29}$$

where $\xi(x,t,v)$ and $\psi(x,t,v)$ are related by the following nonlinear condition:

$$\begin{aligned}&\xi\xi_v(\alpha-2)u^{\alpha+2}+\xi_{vv}u^3+\alpha\psi\psi_x u^{\alpha-1}\\&\quad-(\xi\alpha\psi_v+\xi_v\alpha\psi-2\xi_v\psi-\xi\xi_x\alpha+2\xi\xi_x\\&\qquad+\xi_t)u^{\alpha+1}\\&\quad-(\xi\alpha\psi_x-\alpha\psi\psi_v-\psi_t+\xi_x\alpha\psi-2\xi_x\psi)u^\alpha\\&\quad-(\psi_{vv}-2\xi_{vx})u^2-(2\psi_{vx}-\xi_{xx})u-\psi_{xx}=0.\end{aligned}$$

In this equation the coefficients of the different powers of u must be zero. After solving the determining system a complete classification of the nonclassical symmetries of the governing equation has been performed for $\tau\neq 0$ and we we can state:

Case 1. $\alpha\neq 1,2$. The nonclassical method applied to (25) does not yield any new symmetry different from the ones obtained by Lie classical method.

Case 2. $\alpha=2$. We must remark that in this case equation $v_t=v_{xx}/v_x^2$ may be linearized and transformed into the linear heat equation. Hence a nonclassical point symmetry of the linear diffusion equation is a nonclassical, nonlocal symmetry of (1). For example, applying the nonclassical method applied to (25) give rise to the nonclassical reduction

$$\xi_1=-\frac{3x}{v^2},\qquad \tau=1,\qquad \psi_2=-\frac{3}{v}. \tag{30}$$

These infinitesimals, as ξ depends explicitly on $v=\int u(x)\,dx$, correspond to a nonclassical potential symmetry. Nevertheless, the nonlocal nonclassical reduction (30) corresponds to a nonclassical symmetry obtained for the diffusion equation [1]. Hence any nonclassical (or classical) nonlocal result for $\alpha=2$ is linked to an equivalent nonclassical (or classical) point symmetry of the linear diffusion equation.

Case 3. $\alpha=1$. By applying the nonclassical method we get that ϕ adopts form (29), where ξ and ψ are related by the following conditions:

$$\begin{aligned}&\frac{\partial^2\xi}{\partial v^2}-\xi\frac{\partial\xi}{\partial v}=0,\\&-\frac{\partial^2\psi}{\partial v^2}-\xi\frac{\partial\psi}{\partial v}+\frac{\partial\xi}{\partial v}\psi-\xi\frac{\partial\xi}{\partial x}+2\frac{\partial^2\xi}{\partial v\partial x}-\frac{\partial\xi}{\partial t}=0,\\&-\xi\frac{\partial\psi}{\partial x}-2\frac{\partial^2\psi}{\partial v\partial x}+\psi\frac{\partial\psi}{\partial v}+\frac{\partial\psi}{\partial t}+\frac{\partial\xi}{\partial x}\psi+\frac{\partial^2\xi}{\partial x^2}=0,\\&\psi\frac{\partial\psi}{\partial x}-\frac{\partial^2\psi}{\partial x^2}=0.\end{aligned}$$

Despite the fact that the former equations are too complicated to be solved in general, special solutions can be obtained:

1. For

   $$\xi=k,\qquad \psi=2\tan(v+kt+k_1),$$

   we obtain the similarity variable $z=v-kt$ and the family of invariant solutions is defined, implicitly, by

   $$\frac{\log\sec(z-2v-c)}{k}-x-f(z)=0,$$

   where f satisfies the ODE

   $$kf''-k^2(f')^2-1=0.$$

   Therefore, setting $\zeta=v+kt$, the family of invariant solutions is defined, implicitly, by

   $$-x+\frac{\log\sec(\zeta+k_1)}{k}-\frac{\log\sec(z+k_2)}{k}-k_3=0.$$

Setting $k_1 = k_2 = 0$ and $k = 1$ we get the explicit solution

$$v = 2\arctan\left(\frac{1}{e^x - 1}\Phi(x,t)\right),$$

where $\Phi = \sec(t)\alpha(x,t) - \tan(t)(e^x + 1)$ and $\alpha(x,t) = (e^{2x} - 2\cos(2t)e^x + 1)^{1/2}$.

2. For

$$\xi = k, \qquad \psi = 2\tan(x + v + k_1),$$

we get the similarity variable $z = v - x$ and, setting $\zeta = -x - v$, the family of invariant solutions is defined, implicitly, by

$$\log\sin(\zeta + k_1) - \log\sec(z + k_2) - 4t - 4k_3 = 0.$$

This solution can also be written in explicit form.

We must remark that in these two cases the infinitesimals depend explicitly on $v = \int u(x)\,dx$, consequently they correspond to *nonclassical potential* symmetries. The new exact solutions obtained are unobtainable by using the classical or nonclassical symmetries of (1).

3. For

$$\xi = k, \qquad \psi = -2k_1\tanh\big(k_1(x + kt + k_2)\big),$$

we obtain the similarity variable $z = x - kt$. Setting $\zeta_1 = x + kt + k_2$ and $\zeta_2 = x - kt + k_3$ we get the explicit solution

$$v = \frac{1}{2k}\big(\log 2 - \log\big(\cosh(2k_1\zeta_1)\big) + 1\big) - \frac{1}{k}\big(\log\big(\operatorname{sech}(k_1\zeta_2)\big) + k_3\big).$$

The corresponding exact solution of (1) adopts the form

$$u = \frac{k_1(\tanh(k_1(x - kt)) - \tanh(k_1(x + kt)))}{k}.$$

We point out that although in this case the infinitesimals do not depend on $v = \int u(x)\,dx$, and they do not correspond to a nonclassical potential symmetry, they do no project on to any of the infinitesimals corresponding to the classical or nonclassical generators of (1), consequently u is a *new* exact solution of (1) which *cannot* be obtained by using classical or nonclassical symmetries of (1).

This solution, which describes an unusual diffusion process caused by flux suction at infinity, has been derived by Rosenau in [23] using a different procedure, and looks like the elastic interaction of two kinks giving a shrinking appearance to diffusion. Its derivative looks like the interaction of two solitons solutions. These are of special interest since such solutions are in general associated with integrable equations. Moreover, in general two solitons solutions are associated with Lie–Bäcklund transformations, and although some of these solutions have been obtained [8] by means of nonclassical symmetries, to our knowledge this is the first time that this kind of solution has arisen from a nonclassical reduction of the associated potential system.

5. Concluding remarks

We have proved that for the parabolic type equation (1) the nonclassical method yields to symmetry reductions which are unobtainable by using the Lie classical method and the exact solutions obtained are not group invariant solutions. Consequently, in contradiction with the statement done in [24], we have proved that the nonclassical method is *effective* for PDEs of the parabolic type.

Furthermore, we have introduced new classes of symmetries for some diffusion equations. If these equations are written in a conserved form, then a related system (25) may be obtained. The ansatz to generate nonclassical solutions of (25) yields solutions of (1) which are neither nonclassical solutions of (1) nor solutions arising from potential symmetries. Some of the solutions obtained by reduction from the associated potential system exhibit an interesting behaviour as a "shrinking pulse" formed out of the interaction of two kinks. As far as I know this is the first time that this kind of solution has arisen from a nonclassical reduction of the associated potential system.

Acknowledgements

It is a pleasure to thank P. Rosenau for bringing the fast diffusion equation to my attention, as well as to thank R. Conte for his help and useful suggestions.

References

[1] D.J. Arrigo, J.M. Hill, P. Broadbridge, IMA J. Appl. Math. 52 (1994) 1.

[2] D.J. Arrigo, J.M. Hill, Stud. Appl. Math. 94 (1995) 21.

[3] J.G. Berryman, C.J. Holland, J. Math. Phys. 23 (1982) 983.

[4] G.W. Bluman, in: Proceedings of NATO Advanced Research Workshop, Kluwer, Exeter, 1992.

[5] G.W. Bluman, J. Cole, J. Math. Mech. 18 (1969) 1025.

[6] G.W. Bluman, S. Kumei, J. Math. Phys. 21 (1980) 1019.

[7] G.W. Bluman, S. Kumei, Symmetries and Differential Equations, Springer, Berlin, 1989.

[8] P.A. Clarkson, E.L. Mansfield, Nonlinearity 7 (1993) 975.

[9] P.A. Clarkson, E.L. Mansfield, Physica D 70 (1993) 250.

[10] P.A. Clarkson, Chaos Solitons Fractals (1994).

[11] P.A. Clarkson, E.L. Mansfield, SIAM J. Appl. Math. 55 (1994) 1693.

[12] J.D. Cole, Quart. Appl. Math. 9 (1951) 225.

[13] A.S. Fokas, Y.C. Yortsos, SIAM J. Appl. Math. 42 (1982) 318.

[14] E.L. Ince, Ordinary Differential Equations, Longmans, Green, and Co., London, 1926, Reprinted: Dover, New York, 1956.

[15] M.L. Gandarias, J. Phys. A: Math. Gen. 29 (1996) 607.

[16] M.L. Gandarias, J. Phys. A: Math. Gen. 29 (1996) 5919.

[17] M.L. Gandarias, J. Phys. A: Math. Gen. 30 (1997) 6081.

[18] M.L. Gandarias, J.L. Romero, J.M. Diaz, J. Phys. A: Math. Gen. 32 (1999) 1461.

[19] E. Hopf, Comm. Pure Appl. Math. 3 (1950) 201.

[20] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer, Berlin, 1986.

[21] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.

[22] C.Z. Qu, IMA J. Appl. Math. 62 (1999) 283.

[23] P. Rosenau, Phys. Rev. Lett. 7 (1995) 1056.

[24] R.Z. Zhdanov, V.I. Lahno, Physica D 122 (1998) 178.