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Computational Scheme of a Center Manifold for Neutral Functional Differential Equations

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This work addresses a computational algorithm of terms of a center manifold for neutral functional differential equations. The Bogdanov–Takens and the Hopf singularities are considered. Finally, as an illustration of our scheme, we give an example where the second term of a center manifold is explicitly determined.
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1. INTRODUCTION

Let $r \geq 0$. We denote by $\mathscr{C} = \mathscr{C}([-r, 0], \mathbb{R}^n)$ the Banach space of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ endowed with the supremum norm $\|\phi\| = \sup_{\theta \in [-r,0]} |\phi(\theta)|$, for $\phi \in \mathscr{C}$. If x is a continuous function taking $[\sigma - r, \sigma + a]$, we let $x_t \in \mathscr{C}$ be defined by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$ and $t \in [\sigma, \sigma + a]$. Our main concern throughout this paper is

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with the autonomous neutral functional differential equations of the form

$$\frac{d}{dt}[D(x_t) - G(x_t)] = Lx_t + F(x_t), \tag{1}$$

where L and D are bounded linear operators from $\mathscr{C}$ into $\mathbb{R}^n$; F and G are sufficiently smooth functions mapping $\mathscr{C}$ into $\mathbb{R}^n$ such that $F(0) = G(0) = 0$ and $F'(0) = G'(0) = 0$ (F' and G' denote the Frechet derivatives of F and G, respectively). A solution $x = x(\phi)$ of (1) through a point ϕ in $\mathscr{C}$ is a continuous function taking $[-r, a)$, $a > 0$, into $\mathbb{R}^n$ such that $x_0 = \phi$ and $D(x_t) - G(x_t)$ is continuously differentiable and satisfies (1) for t in $(0, a)$.

In [8], the authors have considered Eq. (1) for which they proved the existence of a local center manifold by using a fixed point theorem. A center manifold is hard to determine. Center manifolds of a certain regularity share at least one common feature: they have the same Taylor expansion up to the order of regularity; a Taylor expansion may give an idea of the local dynamics near the steady state. It is, in most cases, the only information that can be extracted at the expense of nontrivial algorithmic procedures. Examples of such algorithms can be found in [1, 6, 7, 10]. In [2], we dealt with a Hopf singularity, where a center manifold is generated by two conjugate simple eigenvalues; the result in [2] can be extended to functional differential equations of neutral type.

In this work, we are concerned with a Bogdanov–Takens singularity. We assume that the existence of a center manifold associated with (1) is ensured by zero as a double eigenvalue. Then, we derive an algorithmic scheme which allows us, by means of a recursive procedure, to compute at each step the term of order $k \geq 2$ of the Taylor expansion of a center manifold: In fact, we prove that the coefficients of the homogeneous part of degree k of a local center manifold satisfy an initial value problem in finite dimension, whose parameters depend on terms of the same order in Eq. (1) and the terms of a center manifold—of a lower order—already computed.

The paper is organized as follows: The preliminary Section 2 provides, among others, elementary notions and a background about the theory of neutral functional differential equations (for more detail see [8, 9]). In Section 3, we state the theorem which leads us to an analytic characterization of a local center manifold. This characterization is explored in Section 4 for deriving our computational scheme of a center manifold. In Section 5, we consider the delay differential equation considered in [10], for which we compute the second term in the Taylor expansion of a center manifold in order to confirm the result of Hale and Huang in [10]. The paper ends with some concluding remarks regarding other types of singularity.

2. NOTATIONS AND BACKGROUND

Suppose that η and μ are $n \times n$ matrix valued functions of bounded variation in $\theta \in [-r, 0]$ and define

$$L\phi = \int_{-r}^{0} [d\eta(\theta)]\phi(\theta),$$

$$g(\phi) = \int_{-r}^{0} [d\mu(\theta)]\phi(\theta),$$

and

$$D(\phi) = \phi(0) - g(\phi),$$

for all $\phi \in \mathscr{C}$. Also, assume that G depends only upon values of $\phi(\theta)$ for $\theta \in [-r, 0)$ and there exists continuous scalar function $\gamma(s)$, for $s \geq 0$, with $\gamma(0) = 0$ such that

$$\left| \int_{-s}^{0} [d\mu(\theta)]\phi(\theta) \right| \leq \gamma(s)\|\phi\|.$$

Together with (1), we consider the linearized equation

$$\frac{d}{dt} D(x_t) = L(x_t). \tag{2}$$

If we denote by $x(\,.\,, \phi)$ the unique solution of Eq. (2) with initial function ϕ at zero, then Eq. (2) determines a C_0-semigroup of bounded linear operators given by

$$T(t)\phi = x_t(\,.\,, \phi), \qquad \text{for } t \geq 0,$$

where x is the solution of (2) with $x_0 = \phi$. Denote by A the infinitesimal generator of $(T(t))_{t \geq 0}$. The spectrum of A consists of those $\lambda \in \mathbb{C}$ which satisfy the characteristic equation

$$p(\lambda) = \det \Delta(\lambda) = 0, \tag{3}$$

where $\Delta(\lambda) = \lambda[\mathbf{I}_{\mathbb{R}^n} - \int_{-r}^{0} e^{\lambda\theta}\, d\mu(\theta)] - \int_{-r}^{0} e^{\lambda\theta}\, d\eta(\theta)$.

Let $(T_D(t))_{t \geq 0}$ be the strongly continuous semigroup of the linear transformations associated with the solution of

$$\frac{d}{dt} D(x_t) = 0,$$

and define

$$a_D = \inf\{a \in \mathbb{R} : \exists K(a) > 0 \text{ with } \|T_D(t)\varphi\| \leq K(a)e^{at}\|\varphi\|,$$
$$\varphi \in \mathscr{C}, D\varphi = 0\}.$$

Assume that $a_D < 0$ so that there exists no sequence λ_n of roots of (3) with $\mathrm{Re}(\lambda_n) \to 0$ as $n \to \infty$ and all roots of (3) have nonpositive real parts. Then, it follows from [8] that the set

$$\{\lambda \in \mathbb{C} : p(\lambda) = 0 \text{ with } \mathrm{Re}(\lambda) = 0\}$$

is finite. Throughout this paper we assume that the following hypothesis is satisfied.

(H) zero is a double root of (3) and $\{\lambda : p(\lambda) = 0\} \cap \mathbb{R}i = \{0\}$.

The hypothesis (H) yields a decomposition of the state space in the form

$$\mathscr{C} = X_c \oplus X_s,$$

where X_c is the two dimensional generalized eigenspace associated to zero and X_s is its unique $T(t)$ invariant complement subspace in $\mathscr{C}$. Denote by $\Phi = (\phi_1, \phi_2)$ a basis of X_c. Projection operators taking $\mathscr{C}$ on X_c and X_s are determined by means of the adjoint differential equation

$$\frac{d}{ds}\left[y(s) - \int_{-r}^{0} y(s-\theta)\, d\mu(\theta)\right] = -\int_{-r}^{0} y(s-\theta)\, d\eta(\theta) \tag{4}$$

and the bilinear form

$$\langle \psi, \varphi \rangle = \psi(0)D(\varphi) - \int_{-r}^{0}\int_{0}^{\theta} \psi(s-\theta)\, d\eta(\theta)\varphi(s)\, ds$$
$$+ \int_{-r}^{0}\int_{0}^{\theta} \psi'(s-\theta)\, d\mu(\theta)\varphi(s)\, ds$$

for every $\varphi \in \mathscr{C}$ and $\psi \in \mathscr{C}^* = \mathscr{C}([0,r], \mathbb{R}^{n*})$ such that $\psi' \in \mathscr{C}^*$.

In the same way as above, Eq. (4) generates a C_0-semigroup of bounded linear operators in $\mathscr{C}^*$, defined by

$$T^T(t)\psi = y^{-t}(\,.\,, \psi), \qquad \text{for } t \geq 0 \quad \text{and} \quad \psi \in \mathscr{C}^*,$$

where $y^s(\tau) = y(s+\tau)$, for $\tau \in [0,r]$. Let A^T be its infinitesimal generator, let X_c^T be the two dimensional generalized eigenspace associated to zero, and let $\Psi = \mathrm{col}(\psi_1, \psi_2)$ be the basis of X_c^T such that $\langle \Psi, \Phi \rangle = (\langle \psi_i, \varphi_j \rangle)_{i,j=1,2} = \mathbf{I}_{\mathbb{R}^2}$. Then, the space X_s is given by $X_s = \{\phi \in \mathscr{C} : \langle \psi_i, \phi \rangle = 0, \text{ for } i = 1,2\}$. Moreover, for every $\phi \in \mathscr{C}$, we have $\phi = \xi_1\varphi_1 + \xi_2\varphi_2 + \phi_s$, where $\xi_1 = \langle \psi_1, \phi \rangle$ and $\xi_2 = \langle \psi_2, \phi \rangle$.

3. CHARACTERIZATION OF A LOCAL CENTER MANIFOLD

First of all, we recall the definition of a local center manifold associated to Eq. (1).

DEFINITION 1 [8]. Given a C^1 map h from $\mathbb{R}^2$ into X_s, the graph of h is said to be a local center manifold associated to Eq. (1) if and only if $h(0) = 0$, $Dh(0) = 0$, and there exists a neighborhood V of zero in $\mathbb{R}^2$ such that, for each $\xi \in V$, there exists $\delta = \delta(\xi) > 0$ and a function x defined on $]-\delta - r, \delta[$ such that $x_0 = \Phi\xi + h(\xi)$ and x verifies Eq. (1) on $]-\delta, \delta[$ and satisfies the identity

$$x_t = \Phi z(t) + h(z(t)), \qquad \text{for } t \in [0, \delta[, \tag{5}$$

where $z(t)$ is the unique solution of the ordinary differential equation

$$\begin{aligned} \frac{dz(t)}{ft} &= Bz(t) + \Psi(0) f(\Phi z(t) + h(z(t))) \\ &\quad + B\Psi(0) G(\Phi z(t) + h(z(t))) \\ z(0) &= \xi \end{aligned} \tag{6}$$

with

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Geometrically speaking, a center manifold is a graph of a given function mapping a neighborhood of zero in X_c into X_s which is tangent to X_c and locally invariant under the semi-flow generated by Eq. (1). In this section, we derive necessary and sufficient analytic conditions on a given function h under which its graph satisfies the characterization given in the above definition. To this end we state the following result.

THEOREM 2. *Given a C^1 map h from $\mathbb{R}^2$ into X_s with $h(0) = 0$ and $Dh(0) = 0$, a necessary and sufficient condition on the graph of h to be a local center manifold of Eq.* (1) *is that there exists a neighborhood V of zero in $\mathbb{R}^2$ such that, for each $\xi \in V$, there exists $\delta = \delta(\xi) > 0$ satisfying the relations*

$$\Phi(\theta) z(t) + h(z(t))(\theta) = \Phi(t + \theta)\xi + h(\xi)(t + \theta), \quad \textit{for } t + \theta \le 0 \quad \textit{and} \quad 0 \le t < \delta, \tag{7}$$

$$\Phi(\theta) z(t) + h(z(t))(\theta) = \Phi(0) z(t + \theta) + h(z(t + \theta))(0), \quad \textit{for } t + \theta \ge 0 \quad \textit{and} \quad 0 \le t < \delta, \tag{8}$$

and

$$D\left(\frac{\partial h(\xi)}{\partial\theta}\right) = Lh(\xi) + F(\Phi\xi + h(\xi)) + G'(\Phi\xi + h(\xi))\left(\Phi B\xi + \frac{\partial h(\xi)}{\partial\theta}\right), \tag{9}$$

where $z(t)$ is the unique solution of (6).

Proof. Necessity: If h is a local center manifold of Eq. (1), then there exists a neighborhood V of zero in $\mathbb{R}^2$ such that, for each $\xi \in V$, there exists $\delta = \delta(\xi) > 0$ such that the solution of (1) with initial data $\Phi\xi + h(\xi)$ exists on the interval $]-\delta - r, \delta[$ and it is given by

$$x_t = \Phi z(t) + h(z(t)), \qquad \text{for } t \in]-\delta, \delta[.$$

The relations (7) and (8) are immediately deduced from the translation property of the semi-flow $t \mapsto x_t = \Phi z(t) + h(z(t))$ generated by Eq. (1) in a local center manifold. Moreover, this semi-flow has a backward extension to $]-\delta, 0]$. Then, it follows that

$$L(\Phi\xi + h(\xi)) + F(\Phi\xi + h(\xi)) = D\left(\Phi B\xi + \frac{\partial h(\xi)}{\partial\theta}\right) - R(\xi), \tag{10}$$

where

$$R(\xi) = G'(\Phi\xi + h(\xi))\left(\Phi\xi + \frac{\partial h(\xi)}{\partial\theta}\right)(B\xi + \Psi(0)F(\Phi\xi + h(\xi)) + B\Psi(0)G(\Phi\xi + h(\xi))).$$

On the other hand, for $t > 0$, by differentiating relation (7) with respect to t, we get

$$\begin{aligned}\Phi(t+\theta)B\xi + \frac{\partial}{\partial\theta}h(\xi))(t+\theta) &= \left[\Phi(\theta) + \frac{\partial h(z(t))}{\partial\xi}(\theta)\right] \\ &\quad \times [Bz(t) + \Psi(0)F(\Phi z(t) + h(z(t))) \\ &\quad + B\Psi(0)G(\Phi z(t) + h(z(t)))].\end{aligned}$$

Let t go to zero. Then,

$$\frac{\partial}{\partial\theta}(h(\xi))(\theta)$$
$$= \left(\frac{\partial h(\xi)}{\partial\xi}(\theta)\right)\left[B\xi + \Psi(0)F(\Phi\xi + h(\xi)) + B\Psi(0)G(\Phi\xi + h(\xi))\right]$$
$$+ \Phi(\theta)\left[\Psi(0)f(\Phi\xi + h(\xi)) + B\Psi(0)G(\Phi\xi + h(\xi))\right]. \quad (11)$$

So, by substituting the right hand of (11) by $\frac{\partial}{\partial\theta}(h(\xi))(\theta)$ the relation (9) follows immediately.

Sufficiency: From the formulae (7) and (8), one can see that there exists a continuous function y: $]\delta - r, \delta[\to \mathbb{R}^n$ such that

$$y_t = \Phi z(t) = h(z(t)), \qquad \text{for } t \in]-\delta, \delta[.$$

So, it remains to prove that y is a solution of (1). In fact,

$$\frac{d}{dt}(Dy_t - Gy_t) = D\left(\Phi + \frac{\partial h(z(t))}{\partial\xi}\right)\frac{dz(t)}{dt}$$
$$= -G'(\Phi z(t) + h(z(t)))\left(\Phi + \frac{\partial h(z(t))}{\partial\xi}\right)\frac{dz(t)}{dt}. \quad (12)$$

Letting $\xi = z(t)$ and $\theta = 0$ in (11) and substituting $\frac{\partial}{\partial\theta}(h(z(t)))(\theta)$ for the right hand side of (11) into (12), we obtain

$$\frac{d}{dt}(Dy_t - Gy_t) = D(\Phi Bz(t)) + D\left(\frac{\partial hz(t)}{\partial\theta}\right)$$
$$- G'(\Phi\xi + h(\xi))\left(\Phi B\xi + \frac{\partial h(\xi)}{\partial\theta}\right)$$
$$= L(y_t) + F(y_t),$$

which completes the proof of the theorem. ∎

Equation (11) (a direct consequence of (7) and the condition (8) will be of great interest in developing our computational scheme of the coefficients of the Taylor expansion of h.

4. THE COMPUTATIONAL SCHEME

In view of the assumed smoothness on F and G, for every $m \in \mathbb{N}$, we can write

$$h(\xi) = \sum_{k=2}^{m} h_k(\xi) + \chi(\xi), \qquad \text{for } \xi \in V,$$

where h_k is the homogeneous part of degree k of h and $\chi(\xi) = o(|\xi|^m)$.

Let $k \in \mathbb{N}$. The homogeneous part of degree k of Eq. (11) is given by

$$\frac{\partial h_k(\xi)}{\partial \theta} = \frac{\partial h_k(\xi)}{\partial \xi} B\xi + \sum_{i=2}^{k-1} \frac{\partial h_{k-i+1}(\xi)}{\partial \xi} H_i(\xi) + \Phi H_k(\xi), \tag{13}$$

where H_i is the homogeneous part of degree i of

$$H(\xi) = \Psi(0) f(\Phi\xi + h(\xi)) + B\Psi(0) G(\Phi\xi + h(\xi)).$$

Denote by $H_{i-1,i}$ the homogeneous part of degree i of

$$\Psi(0) f\left(\Phi\xi + \sum_{j=2}^{i-1} h_j(\xi)\right) + B\Psi(0) G\left(\Phi\xi + \sum_{j=2}^{i-1} h_j(\xi)\right).$$

We have the following proposition.

PROPOSITION 3. *Under the standing hypothesis* (H), *it follows that*

$$H_i = H_{i-1,i}, \qquad \textit{for } i \geq 2.$$

Proof. The proof of this proposition is similar to the one given in [2]. ∎

As a consequence of the above proposition, Eq. (13) reads

$$\frac{\partial h_k(\xi)}{\partial \theta} = \frac{\partial h_k(\xi)}{\partial \xi} B\xi + \mathscr{F}^{k-1}(\xi), \tag{14}$$

where

$$\mathscr{F}^{k-1}(\xi)(\theta) = \sum_{i=2}^{k} \frac{\partial h_{k-i+1}(\xi)}{\partial \xi}(\theta) H_{i-1,i}(\xi) + \Phi(\theta) H_{k-1,k}(\xi)$$
$$= \sum_{i=0}^{k} F_i^{k-1}(\theta) \xi_1^{k-i} \xi_2^{i}, \qquad \text{for some } F_i^k \in \mathscr{C}. \tag{15}$$

For notational convenience, put

$$h_k(\xi) = \sum_{i=0}^{k} a_i^k \xi_1^{k-i} \xi_2^i \qquad \text{and} \qquad X^k = \operatorname{col}\left(a_0^k, \dots, a_k^k\right),$$

where the a_i^k are elements of X_s. So, in view of Eq. (14), it follows that for $\theta \in [-r, 0]$, we have

$$\frac{dX^k(\theta)}{d\theta} = A_k X^k(\theta) + F^{k-1}(\theta), \tag{16}$$

where

$$A_k = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ k\mathbf{I}_{\mathbb{R}^n} & 0 & 0 & \ddots & & \vdots \\ 0 & (k-1)\mathbf{I}_{\mathbb{R}^n} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mathbf{I}_{\mathbb{R}^n} & 0 \end{bmatrix}$$

and

$$F^{k-1}(\theta) = \operatorname{col}\left(F_0^{k-1}(\theta), \dots, F_k^{k-1}(\theta)\right), \qquad \text{for } \theta \in [-r, 0]. \tag{17}$$

It may be useful to underline the fact that Eq. (16) is obtained by identifying two expressions of a same vector in the space of polynomials in (ξ_1, ξ_2), homogeneous of degree k, with coefficients in $\mathscr{C}$.

LEMMA 4. *Relation* (11) *is equivalent to the ordinary differential equation*

$$\frac{dX^k(\theta)}{d\theta} = A_k X^k(\theta) + F^{k-1}(\theta),$$

$$\textit{for } \theta \in [-r, 0] \quad \textit{and} \quad k \in \mathbb{N}, \quad k \geq 2.$$

where A_k, F^{k-1}, X^k *are as above*.

On the other hand, the homogeneous part of degree k of Eq. (9) leads to

$$D\left(\frac{\partial h_k(\xi)}{\partial \theta}\right) = Lh_k(\xi) + F\left(\Phi\xi + \sum_{j=2}^{i-1} h_j(\xi)\right)$$
$$+ G'\left(\Phi\xi + \sum_{j=2}^{i-1} h(\xi)\right)\left(\Phi B\xi + \sum_{j=2}^{i-1} \frac{\partial h(\xi)}{\partial\theta}\right). \tag{18}$$

For notational convenience, put

$$F\left(\Phi\xi + \sum_{j=2}^{i-1} h_j(\xi)\right) + G'\left(\Phi\xi + \sum_{j=2}^{i-1} h_j(\xi)\right)\left(\Phi B\xi + \sum_{j=2}^{i-1} \frac{\partial h_j(\xi)}{\partial\theta}\right)$$
$$= \sum_{j=0}^{k} M_j^{k-1}\xi_1^{k-j}\xi_2^j,$$

$M_j^k \in \mathbb{R}^n$, and $M^{k-1} = \mathrm{col}(M_0^{k-1}, \ldots, M_k^{k-1})$. Then, Eq. (18) reads

$$D\left(\frac{dX^k}{d\theta}\right) - LX^k = M^{k-1}. \tag{19}$$

A solution of (16) is

$$X^k(\theta) = \exp(\theta A_k)X^k(0) + S^{k-1}(\theta), \qquad \text{for } \theta \in [-r, 0] \tag{20}$$

where $S^{k-1}(\theta) = \int_0^\theta \exp((\theta - s)A_k)F^{k-1}(s)\,ds$. So, by substituting the above expression of X^k in (19), it follows that the vector $X^k(0)$ is a solution of the linear system

$$B_k X^k(0) = E^{k-1}, \tag{21}$$

where

$$B_k = D(A_k \exp(.A_k)) - L(\exp(.A_k)) \tag{22}$$

and

$$E^{k-1} = M^{k-1} - D\left(\frac{dF^{k-1}(\theta)}{d\theta}\right) - LS^{k-1}. \tag{23}$$

Summarizing, we have proved that the coefficients of the homogeneous part of degree k of h satisfy the initial value problem

$$\begin{aligned} \frac{dX^k(\theta)}{d\theta} &= A_k X^k(\theta) + F^{k-1}(\theta), \qquad \text{for } \theta \in [-r, \theta] \\ B_k X^k(0) &= E^{k-1} \qquad \text{with } X^k \in (X_s)^{k+1} \end{aligned} \tag{24}$$

in which E^{k-1} and F^{k-1} are determined in terms of $(h_i)_{i=2}^{k-1}$. We are now in a position to state the fundamental theorem providing us with the computational scheme of a center manifold.

THEOREM 5. *Assume that* (H) *holds. Let* $(a_i^k)_{i=0,\ldots,k} \in (X_s)^{k+1}$ *be the coefficients of the homogeneous part of degree* k *of a local center manifold* *associated with Eq.* (1). *For every* $k \geq 2$, *if we assume that the terms* $(h_j)_{2 \leq j \leq k-1}$ *of a center manifold are known, then the coefficients* $(a_i^k)_{i=0,\dots,k}$ *are uniquely determined by Eq.* (24), *in which* B_k *is given by* (22) *and the vectors* F^k *and* E^{k-1} *respectively* (15) *and* (23) *in terms of* $(h_j)_{2 \leq j \leq k-1}$.

Proof. So far, we know that if $(a_i^k)_{i=0,\dots,k}$ is such a family of coefficients, then it satisfies Eq. (24). What remains to be seen is that Eq. (24) leads to a unique set of coefficients. To this end, it is sufficient to prove that $Y^k \equiv 0$ is the unique solution of the homogeneous problem

$$\begin{aligned} \frac{dY^k(\theta)}{d\theta} &= A_k Y^k(\theta), \qquad \text{for } \theta \in [-r, 0] \\ B_k Y^k(0) &= 0 \qquad \text{with } Y^k \in (X_s)^{k+1}. \end{aligned} \tag{25}$$

So, we have to give more information about the kernel of B_k. We need the following lemma.

LEMMA 6. *The exponential matrix* $\exp(\theta A_k)$ *is given by*

$$\exp(\theta A_k) = \begin{bmatrix} C_0^k \mathbf{I}_{\mathbb{R}^n} & 0 & \cdots & \cdots & 0 \\ \theta C_1^k \mathbf{I}_{\mathbb{R}^n} & C_0^{k-1} \mathbf{I}_{\mathbb{R}^n} & \ddots & & \vdots \\ \vdots & \theta C_1^{k-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \theta^k C_k^k \mathbf{I}_{\mathbb{R}^n} & \theta^{k-1} C_{k-1}^{k-1} \mathbf{I}_{\mathbb{R}^n} & \cdots & \cdots & C_0^0 \mathbf{I}_{\mathbb{R}^n} \end{bmatrix}$$

Proof. The matrix A_k is nilpotent. So, it is easy to verify that

$$\exp(\theta A_k) = \sum_{m=0}^{k} \frac{\theta^m (A_k)^m}{m!}.$$

In a recursive way, one can easily prove that $(A_k)^m = (\alpha_{ij} \mathbf{I}_{\mathbb{R}^n})_{1 \leq ij \leq n(k+1)}$, where

$$\alpha_{ij} = \begin{cases} m! C_m^{k-j+1}, & \text{if } i - j = k \\ 0, & \text{if } i - j \neq k, \end{cases}$$

which completes the proof of the lemma. ∎

By using the above lemma, one can see that

$$B_k = \begin{bmatrix} C_0^k \Delta(0) & 0 & \cdots & \cdots & 0 \\ C_1^k \Delta^{(1)}(0) & C_0^{k-1}\Delta(0) & & & \vdots \\ \vdots & C_1^{k-1}\Delta^{(1)}(0) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ C_k^k \Delta^{(k)}(0) & C_{k-1}^{k-1}\Delta^{(k-1)}(0) & \cdots & \cdots & C_0^0 \Delta(0) \end{bmatrix}.$$

Observe that the above matrix is similar to the matrix characterizing the generalized eigenspace associated with the eigenvalue zero [8]. More explicitly, if we consider the block diagonal matrix

$$D = \operatorname{diag}\big(k!\mathbf{I}_{\mathbb{R}^n}, (k-1)!\mathbf{I}_{\mathbb{R}^n}, \dots, \mathbf{I}_{R^n}\big),$$

then it is easy to verify that

$$\mathscr{B}_k = D_k B_k D_k^{-1}$$

$$= \begin{bmatrix} \Delta(0) & 0 & \cdots & \cdots & 0 \\ \Delta^{(1)}(0) & \Delta(0) & 0 & & \vdots \\ \vdots & \Delta^{(1)}(0) & \ddots & \ddots & \vdots \\ \dfrac{1}{(k-1)!}\Delta^{(k-1)}(0) & \vdots & \ddots & \Delta(0) & 0 \\ \dfrac{1}{k!}\Delta^{(k)}(0) & \dfrac{1}{(k-1)!}\Delta^{(k-1)}(0) & \cdots & \Delta^{(1)}(0) & \Delta(0) \end{bmatrix}.$$

Remark. Note that $\mathscr{B}_k$ is exactly the matrix whose kernel gives us an explicit base of X_c. Moreover, $\operatorname{Ker}(B_K) = \operatorname{Ker}(\mathscr{B}_k)$. So, according to the Folk theorem (see [8, 12]), $\dim \operatorname{Ker}(B_k) = 2$ and is given by

$$\operatorname{Ker}(B_k) = \operatorname{span}\left(V_1^k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ X \end{bmatrix}; V_2^k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ X \\ Y \end{bmatrix} \right),$$

where X and Y are elements of $\mathbb{R}^n$ satisfying

$$\Delta(0)X = 0 \qquad \text{and} \qquad \Delta^{(1)}(0)X + \Delta(0)Y = 0.$$

Moreover, a basis $\Phi = (\varphi_1; \varphi_2)$ of X_c is given by

$$\varphi_1(\theta) = X \qquad \text{and} \qquad \varphi_2(\theta) = \theta X + Y, \qquad \text{for } \theta \in [-r, 0].$$

Now we are in position to prove the well posedness of problem (24). We have the following lemma.

LEMMA 7. *The homogeneous equation* (25) *has only zero as a solution in* $(X_s)^{k+1}$.

Proof. Suppose on the contrary that there exists $Y^k(0) \in \mathrm{Ker}(B_k) \backslash \{0\}$. Then there exists $(\alpha, \beta) \in \mathbb{R} \backslash \{(0,0)\}$ such that $Y^k(0) = \alpha V_1^k + \beta V_2^k$. So,

$$\begin{aligned} Y^k(\theta) &= \alpha \exp(\theta A_k) V_1^k + \beta \exp(\theta A_k) V_2^k \\ &= \mathrm{col}(0, 0, \dots, 0, \beta\varphi_1(\theta), \alpha\varphi_1(\theta) + \beta\varphi_2(\theta)), \end{aligned}$$

where φ_1 and φ_2 are the continuous functions defined in the above remark. This contradicts the fact that $Y^k \in (X_s)^{k+1}$, except if $(\alpha, \beta) = (0, 0)$. ∎

Remark. Note that the matrix B_k is singular, so in order to compute the initial data for the problem (24) we should take into account the abstract condition $X^k \in X_s$. The only way to do that is the formal adjoint product defined in the preliminary section. More explicitly, the solution of (24) must satisfy $\langle \Psi, X^k \rangle = 0$. But

$$\begin{aligned} \langle \Psi, X \rangle &= \left\langle \Psi, \exp(\cdot A_k) X^k(0) + S^{k-1} \right\rangle \\ &= \left\langle \Psi, \exp(\cdot A_k) X^k(0) \right\rangle + \langle \Psi, S^{k-1} \rangle \\ &= \left\langle \Psi, \exp(\cdot A_k) \right\rangle X^k(0) + \langle \Psi, S^{k-1} \rangle. \end{aligned}$$

Hence, $X^k(0)$ should be a solution of a system of two equations

$$G_k X^k(0) = N^{k-1} \qquad \text{and} \qquad B_k X^k(0) = E^{k-1}, \tag{26}$$

where $G_k = \langle \Psi, \exp(\cdot A_k) \rangle$ is an $n(k+1) \times 2(k+1)$ real matrix and $N^{k-1} = -\langle \Psi, S^{k-1} \rangle \in \mathbb{R}^{(2k+2)}$.

According to the above remark, the system (24) reads

$$\begin{aligned} \frac{dX^k(\theta)}{d\theta} &= A_k X^k(\theta) + F^{k-1}(\theta), \qquad \text{for } \theta \in [-r, 0] \\ B_k X^k(0) &= E^{k-1} \qquad \text{and} \qquad G_k X^k(0) = N^{k-1}. \end{aligned} \tag{27}$$

As it is in the above formula, the system does not lend itself to numerical computation. The next lemma provides us with a more suitable form.

LEMMA 8. *There exists an invertible $n(k+1) \times n(k+1)$ matrix M_k such that the boundary conditions* (26) *are equivalent to*

$$M_k X^k(0) = R^{k-1}, \tag{28}$$

where M_k (resp. R^{k-1}) is computed in terms of B_k and G_k (resp. E_{k-1} and N_{k-1}).

Proof. From the above computations, there exists an invertible $n(k+1) \times n(k+1)$ matrix $P_k = (V_1^k, V_2^k, V_3^k, \dots, V_{k+1}^k)$ where V_1^k and V_2^k are the vectors defined above, such that

$$P_k B_k P_k^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{B}_k \end{bmatrix},$$

where $\widetilde{B_k}$ is an $(n(k+1)-2) \times (n(k+1)-2)$ invertible matrix.

Let $(\tilde{a}_0^k(0), \tilde{a}_1^k(0), \tilde{a}_2^k(0), \dots, \tilde{a}_k^k)$ and $(\tilde{e}_0^{k-1}, \tilde{e}_1^{k-1}, \tilde{e}_2^{k-1}, \dots, \tilde{e}_k^{k-1})$ be respectively the components of $X^k(0)$ and E^{k-1} in the basis $(V_1^k, V_2^k, V_3^k, \dots, V_{k+1}^k)$ of $\mathbb{R}^{n(k+1)}$. The condition $B_k X^k(0) = E^{k-1}$ reads

$$\tilde{B}_k \tilde{X}_0^k(0) = \tilde{E}_{k-1}, \tag{29}$$

where $\tilde{X}_2^k(0) = \mathrm{col}(\tilde{a}_2^k(0), \dots, \tilde{a}_k^k(0))$ and $\tilde{E}_{k-1} = \mathrm{col}(\tilde{e}_2^{k-1}, \dots, \tilde{e}_k^{k-1})$. Now, it remains to compute the components $\tilde{a}_0^k(0)$ and $\tilde{a}_1^k(0)$ of $X^k(0)$. It is easy to see that the second condition in (26) is equivalent to

$$\mathscr{G}_k \tilde{X}^k(0) = N^{k-1}, \tag{30}$$

where $\mathscr{G}_k$ is the $2(k+1) \times n(k+1)$ matrix given by

$$\mathscr{G}_k = G_k P_k = \left[G_k V_1^k, G_k V_2^k, G_k V_3^k, \dots, G_k V_{k+1}^k \right].$$

Moreover,

$$G_k V_1^k = \mathrm{col}(0, \dots, 0, 0, 0, 1, 0)$$

and

$$G_k V_2^k = \mathrm{col}(0, \dots, 0, 1, 0, 0, 1).$$

So, it follows that the components $\tilde{a}_0^k(0)$ and $\tilde{a}_1^k(0)$ can be computed in terms of $(\tilde{a}_i^k(0))_{i=2}^k$ respectively by the kth and the $(k-2)$th rows of the linear system (30). Put $\mathscr{G}_k = \{g_{ij}^k : 1 \le i \le 2(k+1) \text{ and } 1 \le j \le n(k+1)\}$, $N^{k-1} = \mathrm{col}(n_0^{k-1}, \dots, n_k^{k-1})$, and $N_1^{k-1} = \mathrm{col}(n_0^{k-1}, n_1^{k-1})$. From (29) it follows that the system (26) is equivalent to

$$\begin{bmatrix} \mathbf{I}_{\mathbb{R}^2} & \tilde{G}_k \\ 0 & \tilde{B}_k \end{bmatrix} \begin{bmatrix} \tilde{X}_1^k(0) \\ \tilde{X}_2^k(0) \end{bmatrix} = \begin{bmatrix} N_1^{k-1} \\ \tilde{E}_{k-1} \end{bmatrix}, \tag{31}$$

where $\tilde{X}_1^k(0) = \mathrm{col}(\tilde{a}_0^k(0), \tilde{a}_1^k(0))$ and $\tilde{G}_k$ is the $2 \times n(k+1)$ given by

$$\tilde{G}_k = \begin{bmatrix} g_{k,3}^k & \cdots & g_{k,k+1}^k \\ g_{k-2,3}^k & \cdots & g_{k-2,k+1}^k \end{bmatrix},$$

which completes the proof of the lemma. ∎

Finally, we proved that the coefficients of a center manifold associated to Eq. (1) are computed by a sequence of ordinary differential equations of the form

$$\frac{dX^k(\theta)}{d\theta} = A_k X^k(\theta) + F^{k-1}(\theta), \qquad \text{for } \theta \in [-r, 0]$$
$$X^k(0) = Q_k^{-1} E^{k-1},$$

where

$$Q_k^{-1} = P_k^{-1} \begin{bmatrix} \mathbf{I}_{\mathbb{R}^2} & \tilde{G}_k \\ 0 & \tilde{B}_k \end{bmatrix}$$

and

$$\overline{E}^{k-1} = \begin{bmatrix} N_1^{k-1} \\ \tilde{E}_{k-1} \end{bmatrix}.$$

This achieves the proof of the theorem.

5. APPLICATION

In this section we consider as an application of our computational scheme an example treated by Hale and Huang [10], where, in view of the study of the qualitative structure of the flow on the center manifold, the authors need more precise information about a center manifold. In particular, they need to know its Taylor expansion up to the second order terms.

It is a singularly perturbed delay equation given by

$$\varepsilon x'(t) = -x(t) + f(x(t-1), \lambda), \tag{32}$$

where ε is a real positive number assumed to be small, λ is a real parameter, and f is an element of $C^k(\mathbb{R}, \mathbb{R})$, $k \geq 3$; more specifically

$$f(x, \lambda) = -(1+\lambda)x + ax^2 + bx^3 + o(x^3), \qquad \text{for some } a, b \in \mathbb{R}.$$

With the change of variables

$$w_1(t) = x(-\varepsilon rt) \qquad \text{and} \qquad w_2(t) = x(-\varepsilon rt + 1 + \varepsilon r),$$

Eq. (32) reads

$$\begin{aligned} \frac{dw_1}{dt} &= rw_1(t) - rf(w_2(t-1), \lambda) \\ \frac{dw_2}{dt} &= rw_2(t) - rf(w_1(t-1), \lambda). \end{aligned} \tag{33}$$

If $r = 1$ and $\lambda = 0$, then the characteristic equation associated to (33) is

$$(\mu - 1)^2 - e^{-2\mu} = 0, \qquad \text{for } \mu \in \mathbb{C}.$$

The above equation has zero as a double root and the remaining roots have a negative real part. This ensures the existence of a two dimensional center manifold associated to Eq. (33) with $(r, \lambda) = (1, 0)$.

Throughout this section we use the notation presented in Section 2 for Eq. (33) with $r = 1$ and $\lambda = 0$. The bases of X_c and X_c^T are respectively

$$\Phi = \begin{bmatrix} -1 & -\frac{1}{3} - \theta \\ 1 & \frac{1}{3} + \theta \end{bmatrix} \qquad \text{and} \qquad \Psi = \begin{bmatrix} s & -s \\ -1 & 1 \end{bmatrix},$$

$$\text{for } (\theta, s) \in [-1, 0] \times [0, 1].$$

It is easy to verify that $\langle \Psi, \Phi \rangle = \mathbf{I}_{\mathbb{R}^2}$, and

$$A\Phi = \Phi B, \qquad \text{with } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In view of the assumed smoothness on the function f, Eq. (33) has a local center manifold, at least of class C^3, and

$$h(\xi) = a_0^2 \xi_1^2 + a_1^2 \xi_1 \xi_2 + a_2^2 \xi_2^2 + \chi(\xi),$$

$$\text{for some } a_i^2 \in X_s \quad \text{and} \quad \chi(\xi) = o(|\xi|^2).$$

Put $X^2 = \mathrm{col}(a_0^2, a_1^2, a_2^2)$. We showed in Section 4 that the computation of X^2 can be obtained by solving the differential equation

$$\begin{aligned} \frac{dX^2(\theta)}{d\theta} &= A_2 X^2(\theta) + F^1(\theta), \qquad \text{for } \theta \in [-r, 0] \\ B_2 X^2(0) &= E^1 \qquad \text{and} \qquad G_2 X^2(0) = N^1, \end{aligned} \tag{34}$$

where B_2 is the 6×6 real matrix given by

$$B_2 = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 2 & 2 & -1 & -1 & 0 & 0 \\ 2 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

Now, we will consider the computation of the parameters of the above system. According to relation (15) we have $\mathscr{F}^1(\xi)(\theta) = 0$, for $\theta \in [-1, 0]$. So, $F^1(\theta) = 0$, for $\theta \in [-1, 0]$ and as a consequence, we have $N^1 = -\langle \Psi, S^1 \rangle = 0$. Moreover, relation (22) leads to $E^1 = \mathrm{col}(-a, -a, \frac{4a}{3}, \frac{4a}{3}, \frac{-4a}{9}, \frac{-4a}{9})$. On the other hand, by Lemma 6, it is easy to see that the 6×6 real matrix G_2 is given by

$$G_2 = \langle \Psi, \exp(\cdot A_2) \rangle = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{-1}{3} & \frac{-1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ \frac{-1}{12} & \frac{1}{12} & \frac{1}{6} & \frac{-1}{6} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{-1}{3} & \frac{-1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Consequently, the two boundary conditions $G_2 X^2(0) = N^1$ and $B_2 X^2(0) = E^1$ lead to

$$X^2(0) = \mathrm{col}\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{3}, \frac{a}{3}, \frac{11a}{36}, \frac{11a}{36}\right)$$

and for $\theta \in [-1, 0]$, we have

$$a_0^2(\theta) = \frac{a}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad a_1^2(\theta) = a\left(\theta + \frac{1}{3}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$a_2^2(\theta) = a\left(\frac{\theta^2}{2} + \frac{\theta}{3} + \frac{11}{36}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This confirms the result established by Hale and Huang in [10].

6. SUPPLEMENTARY RESULTS

If we assume that $\pm i$ are the only imaginary roots of the characteristic equation (3) associated to Eq. (1), one can prove, in the spirit of this work, that the coefficients of a center manifold in the Taylor expansion satisfy the boundary value problem

$$\begin{aligned} &\frac{dX^k(\theta)}{d\theta} = A_k X^k(\theta) + F^{k-1}(\theta), \qquad \text{for } \theta \in [-r, 0] \\ &B_k X^k(0) = E^{k-1} \qquad \text{and} \qquad G_k X^k(0) = N^{k-1}, \end{aligned} \tag{35}$$

where

$$A_k = \begin{bmatrix} 0 & \mathbf{I}_{\mathbb{R}^n} & 0 & \cdots & \cdots & 0 \\ -k\mathbf{I}_{\mathbb{R}^n} & 0 & 2\mathbf{I}_{\mathbb{R}^n} & \ddots & & \vdots \\ 0 & -(k-1)\mathbf{I}_{\mathbb{R}^n} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & k\mathbf{I}_{\mathbb{R}^n} \\ 0 & \cdots & \cdots & 0 & -\mathbf{I}_{\mathbb{R}^n} & 0 \end{bmatrix}$$

and $B_k = D(A_k \exp(.A_k)) - L(\exp(.A_k))$. As in [2], we distinguish two cases: when k is even, the matrix B_K is invertible and the initial data of (35) can be computed without using the second condition $G_k X^k(0) = N^{k-1}$. Hence, $X^k(0) = B_k^{-1} E^{k-1}$, but, when k is odd, B_k is a noninvertible matrix. More precisely, we have $\dim \operatorname{Ker}(B_k) = 2$ and there exists an $n(k+1) \times n(k+1)$ invertible matrix P_k such that

$$\begin{aligned} \operatorname{Ker}(B_k) = \operatorname{span}(V_0 = P_k \operatorname{col}(u, v, 0, \ldots, 0); \\ V_1 = P_k \operatorname{col}(v, -u, 0, \ldots, 0)), \end{aligned}$$

where u and v are elements of $\mathbb{R}^N$ different from zero satisfying $\Delta(i)(u + iv) = 0$. So, we are obliged to use the second boundary condition of (35) to compute $X^k(0)$. In the same way as the above lemma, one can obtain an analogous result in this case (i.e., reducing the resolution of the two linear systems into one whose matrix is invertible).

In [2], the scheme is stated with the abstract condition $X^k \in (X_s)^{k+1}$. The above idea allows us avoid to the condition and to present the scheme in a suitable form for numerical computations of a center manifold.

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