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The symmetry reductions of a turbulence model

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Abstract

In this paper we obtain symmetry reductions of the system of two coupled parabolic partial differential equations which model the evolution of turbulent bursts using the classical Lie method of infinitesimals. The reduction to systems of ordinary differential equations (ODEs) are obtained from the optimal system of subalgebras. These systems admit symmetries which lead to further reductions. An algorithm presented by Bluman for reducing the order of ODEs allows us to reduce some of these systems, invariant under a two-parameter group, directly to first-order ODEs systems. The hidden symmetries of some of these systems are obtained and some new exact solutions have been derived.

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1. Introduction

The propagation of turbulent bursts is a problem of great interest, Barenblatt [3] proposed an equation as a mathematical model for the propagation of turbulent bursts from a plane source, Kamin and Vazquez [19] and Hastings and Peletier [16] have studied this one-equation turbulence model.

The aim of this paper is to study the following system of partial differential equations (PDEs) which models the evolution of turbulent bursts:

$$u_t = a\left(\frac{u^2 u_x}{v}\right)_x - v \qquad v_t = b\left(\frac{u^2 v_x}{v}\right)_x - c\frac{v^2}{u}. \tag{1.1}$$

Here x is the spatial coordinate, t the temporal coordinate, $u(x, t)$ the turbulent energy density, $v(x, t)$ the dissipation rate of turbulent energy, and a, b, c are positive dimensionless constants. In the literature this model is called the b–ε model which is the original notation introduced by Kolmogorv [20] and is also referred to as the k–ε model.

The system (1.1) is derived (cf [5, 21]) under the assumptions that:

(i) the average velocity of the flow is identically zero; and

(ii) there exists some kind of statistical space homogeneity which reduces the model to an one-dimensional system.

The system consists of two coupled parabolic PDEs which contain the singular functions u^2/v and v^2/u. As far as we know, hardly any mathematical results are available for the system (1.1). Barenblatt *et al* [4] found that in the case when $a = b = 1$ and $c > \frac{3}{2}$ the system (1.1) possesses a three-parameter family of similarity solutions. Recently Bertsch *et al* [5] presented a mathematical analysis of this system. They proved that for $a = b$ and $c > 1$, the system (1.1) has a solution if initially u and v has the same support. If $c > 0$ the system has a family of self-similar solutions and the solutions of the system (1.1) converges to one of these self-similar solutions. If $a \neq b$, to our knowledge, there are only a few results for (1.1) available in the literature; some numerical results were obtained by Barenblatt *et al* [4].

Symmetries of nonlinear PDEs systems may be used to reduce the number of independent variables of the PDEs; in particular, we might reduce the PDEs to ordinary differential equations (ODEs). The ODE systems may also have symmetries that allow us to reduce the order of one or of both ODEs, and we can integrate to find exact solutions. Often solutions of the nonlinear PDE system will be asymptotic to the symmetry solutions. Explicit solutions (such as those found by symmetry methods) can also play an important role in the design and testing of numerical integrators [22].

In this paper we discuss symmetry reductions of the system (1.1), using the classical Lie method of infinitesimals. The fundamental basis of this technique is that, when a system of differential equations is invariant under a Lie group of transformations, a reduction transformation exists. The machinery of Lie group theory provides the systematic method to search for these special group-invariant solutions. For systems of PDEs with two independent variables, as it is the system (1.1), a single group reduction transforms the system of PDEs into a system of ODEs, which are generally easier to solve than the original system. Most of the required theory and a description of the method can be found in [8, 9, 17, 18, 22–24].

To apply the classical method to the system (1.1) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u, v) given by

$$\begin{aligned} x^* &= x + \varepsilon X(x,t,u,v) + \mathcal{O}(\varepsilon^2) \\ t^* &= t + \varepsilon T(x,t,u,v) + \mathcal{O}(\varepsilon^2) \\ u^* &= u + \varepsilon U(x,t,u,v) + \mathcal{O}(\varepsilon^2) \\ v^* &= v + \varepsilon V(x,t,u,v) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{1.2}$$

where ε is the group parameter. One then requires that this transformation leaves invariant the set of solutions of the system (1.1). This yields to an overdetermined, linear system of equations for the infinitesimals $X(x, t, u, v)$, $T(x, t, u, v)$, $U(x, t, u, v)$ and $V(x, t, u, v)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\boldsymbol{v} = X(x,t,u,v)\frac{\partial}{\partial x} + T(x,t,u,v)\frac{\partial}{\partial t} + U(x,t,u,v)\frac{\partial}{\partial u} + V(x,t,u,v)\frac{\partial}{\partial v}. \tag{1.3}$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface conditions

$$\Phi_1 \equiv X\frac{\partial u}{\partial x} + T\frac{\partial u}{\partial t} - U = 0 \qquad \Phi_2 \equiv X\frac{\partial v}{\partial x} + T\frac{\partial v}{\partial t} - V = 0. \tag{1.4}$$

In general, if a system of differential equations admits a Lie group $\mathcal{G}_r$ and its Lie algebra $\mathcal{L}_r$ is of dimension $r > 1$, one could, in principle, consider invariant solutions based on one, two, etc,-dimensional subalgebras of $\mathcal{L}_r$. However, there is an infinite number of subalgebras, for example, one-dimensional subalgebras. This problem becomes manageable by recognizing that if two subalgebras are similar, i.e., they are connected with each other by a transformation from the symmetry group (with Lie algebra $\mathcal{L}_r$), then their corresponding invariant solutions are connected with each other by the same transformation. Therefore, it is sufficient to put into one class all similar subalgebras of a given dimension, say s, and select a representative from each class. The set of all these representatives of all these classes is called an *optimal system of orders* [23]. In order to find all invariant solutions with respect to s-dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order s. The set of invariant solutions obtained in this way is called an *optimal system of invariant solutions*. Of course the form of these invariant solutions depends on the choice of the representatives.

Since the system (1.1) has two independent variables, we only consider one-parameter subgroups. We have already seen that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. Although in general this latter problem can still be quite complicated, for one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. The construction of the one-dimensional optimal system appears in [23] using a global matrix for the adjoint transformation. Olver [22] uses a slightly different technique which we will follow. Using this we construct a table showing the separate adjoint actions of each element in $\mathcal{L}_r$ as it acts on all other elements, this construction is easily done by summing the corresponding Lie series. We then consider a general element in a basis of $\mathcal{L}_r$ and ask whether it can be transformed into a new element of a simpler form by subjecting it, iteratively, to various adjoint transformations. For further details and proofs see [13, 22].

The second-order ODE systems obtained from the optimal system of subalgebras admit symmetries that lead to further reductions. The invariance of a system of two second-order ODEs under a one-parameter group allow us to reduce the order of *one* of the equations by 1 [22]. However, if the system is invariant under a two-parameter Lie group and we reduce the order of one of the equations we may lose or gain Lie group symmetries which are called hidden symmetries. In case the system of ODEs is invariant under a two-parameter Lie group we apply an algorithm proposed by Bluman [6] that allow us to reduce directly the system of second-order ODEs to a system of two first-order ODEs. It also happens that the invariance under a normal subgroup allow us to reduce the second-order ODE system to a first-order PDE system, which inherits some additional symmetry allowing us to reduce this system to a first-order ODE system.

The structure of this paper is as follows. In section 2 we study the Lie symmetries of the system (1.1), its Lie algebra as well as the corresponding optimal system. We also report the reduction obtained from the optimal system of subalgebras. These systems admit symmetries which lead to further reductions, we find some hidden symmetries of these systems and obtain some exact solutions. In section 3 we study the Lie symmetries for the special case $a = b$, by reducing the system to an unique PDE. Finally in section 4 we draw some conclusions.

2. Lie symmetries for the system (1.1)

Applying the classical method to system (1.1) yields a system of 16 equations which lead to a four-parameter Lie group. Associated with this Lie group we have a Lie algebra which can be represented by the generators, these generators are

$$\boldsymbol{v}_1 = \frac{\partial}{\partial x} \qquad \boldsymbol{v}_2 = \frac{\partial}{\partial t}$$

$$\boldsymbol{v}_3 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} \qquad \boldsymbol{v}_4 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 3v\frac{\partial}{\partial v}.$$

In order to construct the one-dimensional optimal system $\{\boldsymbol{u}_i\}$, following Olver, we construct a table showing the separate adjoint actions of each element in $\boldsymbol{v}_i$, $i = 1 \dots 4$, as it acts on all other elements. This construction is done easily by summing the Lie series.

Table 1. Commutator table for the Lie algebra $\{\boldsymbol{v}_i\}$.

	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$
$\boldsymbol{v}_1$	0	0	$\boldsymbol{v}_1$	0
$\boldsymbol{v}_2$	0	0	0	$\boldsymbol{v}_2$
$\boldsymbol{v}_3$	$-\boldsymbol{v}_1$	0	0	0
$\boldsymbol{v}_4$	0	$-\boldsymbol{v}_2$	0	0

Table 2. Adjoint table for the Lie algebra $\{\boldsymbol{v}_i\}$.

Ad	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$
$\boldsymbol{v}_1$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3 - \varepsilon\boldsymbol{v}_1$	$\boldsymbol{v}_4$
$\boldsymbol{v}_2$	$\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4 - \varepsilon\boldsymbol{v}_2$
$\boldsymbol{v}_3$	$e^{\varepsilon}\boldsymbol{v}_1$	$\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$
$\boldsymbol{v}_4$	$\boldsymbol{v}_1$	$e^{\varepsilon}\boldsymbol{v}_2$	$\boldsymbol{v}_3$	$\boldsymbol{v}_4$

Table 3. Infinitesimal generators $\{\boldsymbol{u}_i\}$ of the optimal system and similarity variable and similarity solutions.

	$\boldsymbol{u}_i$	z	u	v
1	$\boldsymbol{v}_3 + \mu\boldsymbol{v}_4$	$xt^{-1/\mu}$	$t^{-2-2/\mu}f(z)$	$t^{-3-2/\mu}g(z)$
2	$\mu\boldsymbol{v}_2 + \boldsymbol{v}_3$	$xe^{-t/\mu}$	$e^{2t/\mu}f(z)$	$e^{2t/\mu}g(z)$
3	$\mu\boldsymbol{v}_1 + \boldsymbol{v}_4$	$x - \mu\ln t$	$t^{-2}f(z)$	$t^{-3}g(z)$
4	$\mu\boldsymbol{v}_1 + \boldsymbol{v}_2$	$x - \mu t$	$f(z)$	$g(z)$
5	$\boldsymbol{v}_3$	t	$x^2 f(t)$	$x^2 g(t)$
6	$\boldsymbol{v}_4$	x	$t^{-2}f(x)$	$t^{-3}g(x)$

Table 4. ODE systems to which the PDE systems are reduced to by $\boldsymbol{u}_i$.

	$E_i^1(f, g, f', g', f'') = 0$	$E_i^2(f, g, f', g', g'') = 0$
S_1	$\left(\frac{zf}{\mu} + \frac{af^2f'}{g}\right)' + 2f - \frac{3f}{\mu} - g = 0$	$\left(\frac{zg}{\mu} + \frac{bf^2g'}{g}\right)' + 3g - \frac{3g}{\mu} - \frac{cg^2}{f} = 0$
S_2	$\left(\frac{1}{\mu}zf + a\frac{f^2f'}{g}\right)' - \frac{3}{\mu}f - g = 0$	$\left(\frac{1}{\mu}zg + b\frac{f^2g'}{g}\right)' - \frac{3}{\mu}g - c\frac{g^2}{f} = 0$
S_3	$\left(\mu f + a\frac{f^2f'}{g}\right)' + 2f - g = 0$	$\left(\mu g + b\frac{f^2g'}{g}\right)' + 3g - c\frac{g^2}{f} = 0$
S_4	$\left(\frac{af^2f'}{g} + \mu f\right)' - g = 0$	$\left(\frac{bf^2g'}{g} + \mu g\right)' - \frac{cg^2}{f} = 0$
S_5	$f' + g - \frac{6af^3}{g} = 0$	$g' - 6bf^2 + \frac{cg^2}{f} = 0$
S_6	$\left(\frac{af^2f'}{g}\right)' + 2f - g = 0$	$\left(\frac{bf^2g'}{g}\right)' + 3g - \frac{cg^2}{f} = 0$

In table 3, we list the nontrivial optimal system $\{\boldsymbol{u}_i\}$ with $i = 1, \ldots, 6$, where $\mu \in \mathbb{R}^*$ is arbitrary. We also list the corresponding similarity variables and similarity solutions.

In table 4 we list the system of ODEs obtained when the system (1.1) is reduced by means of $\{\boldsymbol{u}_i\}$, $i = 1, \ldots, 6$; note: $' \equiv d/dz$.

We observe that the similarity variable and similarity solutions are the same as those obtained for the porous medium equation with n arbitrary [14]. In several cases, the reduced systems of ODEs admit symmetries which lead to further reductions and we shall again use the

Table 5. Special values of a, b and c and exact solutions.

a	b	c	u	v
Arbitrary	Arbitrary	$\neq 1$	$x^{1/2}t^{1/(1-c)}$	$\dfrac{x^{1/2}t^{c/(1-c)}}{c-1}$
Arbitrary	Arbitrary	$3-\dfrac{b}{a}$	$-\dfrac{x^2}{6at^2}$	$-\dfrac{x^2}{6at^3}$
Arbitrary	Arbitrary	1	$\mathrm{e}^{3t/(2\mu)}x^{1/2}$	$-\dfrac{3}{2\mu}\mathrm{e}^{3t/(2\mu)}x^{1/2}$

techniques of Lie group theory. The system S_i, $i = 1, \ldots, 6$ gives the following symmetries:

$$\mathrm{S}_1:\ \boldsymbol{v}_{11} = z\frac{\partial}{\partial z} + 2f\frac{\partial}{\partial f} + 2g\frac{\partial}{\partial g}$$

$$\mathrm{S}_2:\ \boldsymbol{v}_{21} = z\frac{\partial}{\partial z}$$

$$\mathrm{S}_3:\ \boldsymbol{v}_{31} = \frac{\partial}{\partial z}$$

$$\mathrm{S}_4:\ \boldsymbol{v}_{41} = \frac{\partial}{\partial z} \qquad \boldsymbol{v}_{42} = -z\frac{\partial}{\partial z} + g\frac{\partial}{\partial g} \qquad [\boldsymbol{v}_{41}, \boldsymbol{v}_{42}] = -\boldsymbol{v}_{41}$$

$$\mathrm{S}_5:\ \boldsymbol{v}_{51} = \frac{\partial}{\partial z} \qquad \boldsymbol{v}_{52} = z\frac{\partial}{\partial z} - 2f\frac{\partial}{\partial f} - 3g\frac{\partial}{\partial g} \qquad [\boldsymbol{v}_{51}, \boldsymbol{v}_{52}] = \boldsymbol{v}_{51}$$

$$\mathrm{S}_6:\ \boldsymbol{v}_{61} = \frac{\partial}{\partial z} \qquad \boldsymbol{v}_{62} = z\frac{\partial}{\partial z} + 2f\frac{\partial}{\partial f} + 2g\frac{\partial}{\partial g} \qquad [\boldsymbol{v}_{61}, \boldsymbol{v}_{62}] = \boldsymbol{v}_{61}.$$

The invariance of an second-order system under a one-parameter group allows us to reduce the order of *one* of the equations in the system by 1. Although we have not been able to integrate the systems S_i, $i = 1, \ldots, 3$, in full generality, in table 5 we list some exact solutions.

Olver [22] gave an existence theorem which shows that if an nth order ODE admits an r-parameter solvable Lie group of transformations, then its general solution can be found by quadratures from the general solution of an $(n - r)$th order ODE. However, this existence theorem does not yield an iterative reduction algorithm. An iterative algorithm was presented by Bluman [6] for reducing an nth order ODE to an $(n - r)$th order ODE plus r quadratures when it admits an r-parameter solvable Lie group of transformations. As far as we know there are not explicit results for systems of second order ODEs. We applied an algorithm proposed by Bluman for reducing the order of ODEs for this system of ODEs.

In accordance with this method, we write the first extension of $\boldsymbol{v}_{42}$ in terms of the invariants of $\boldsymbol{v}_{41}^{(1)}$: $f = g_1$, $g = g_2$, $f' = h_1$, $g' = h_2$. Then, we have

$$\boldsymbol{v}_{42}^{(1)} = g_2\frac{\partial}{\partial g_2} + h_1\frac{\partial}{\partial h_1} + 2h_2\frac{\partial}{\partial h_2}.$$

Now, S_4 can be reduced to a first-order system of ODEs in terms of the invariants of $\boldsymbol{v}_{42}^{(1)}$. This means using the new variables: $w = g_1$, $h_1 = g_2 M(g_1)$, $h_2 = g_2^2 N(g_1)$. In terms of these new variables S_4 becomes

$$M\left(2ag_1M + ag_1^2\frac{\mathrm{d}M}{\mathrm{d}g_1} + \mu\right) - 1 = 0 \tag{2.1}$$

$$g_1\left(bg_1^2N^2 + \mu N + 2bg_1MN + bg_1^2M\frac{\mathrm{d}N}{\mathrm{d}g_1}\right) - c = 0. \tag{2.2}$$

The invariance of S_4 under the one-parameter group with the infinitesimal generator $\boldsymbol{v}_{41}$ allows us to reduce the order of *one* of the equations by 1, the invariance of S_4 under $\boldsymbol{v}_{41}$ also leads to the system of first-order PDEs. Both systems under v_{42} lead to (2.1). However, the invariance of S_4 under v_{42} lead to a different system of first-order PDEs which does not inherit the symmetry v_{41}.

This is illustrates an important point. If we reduce the order of one of the equations of a system of second-order ODEs, or if we reduce the second-order ODE system to a system of first-order PDEs we may lose any additional symmetry properties present in the full group. The existence of these kind of symmetries was pointed out by Olver in the reduction by the non-normal subgroup variables of a second-order ODE, and was denoted by Abraham-Shrauner and Guo [1] as type I hidden symmetry. Only special types of subgroups, namely the normal subgroups, which in this case is v_{41}, will enable us to retain the full symmetry properties under reduction. It should also be noted that although the system obtained by reducing S_4 under v_{42} has no symmetry properties, which reflect the symmetry of S_4 under the group generated by v_{41}, we are able to reduce it to (2.1) by first changing it into the second-order ODE system, and then reducing this system.

As system S_5 is an autonomous first-order system, we can reduce it to a single first-order equation plus a quadrature. We have the equivalent system

$$\frac{\mathrm{d}g}{\mathrm{d}f} = \frac{g(cg^2 - 6bf^3)}{f(g^2 - 6af^3)} \qquad \frac{\mathrm{d}z}{\mathrm{d}f} = \frac{g}{6af^3 - g^2}. \tag{2.3}$$

For $b \neq \frac{3}{2}a$ and $c \neq \frac{3}{2}$ the solution of (2.3) is in implicit form

$$f^{2c/(2c-3)}[(2c-3)g^2 + (18a - 12b)f^3]^{2ac-2b/[(4b-6a)c-6b+9a]} - k\, g^{2a/(2b-3a)} = 0$$

where k is an arbitrary constant. Some particular cases for which we can write explicit solutions for the system (1.1) are as follows.

If $k = 0$ and $c \neq b/a$ we have

$$u(x,t) = \frac{(2\,b - 3\,a)\ (2\,c - 3)\ x^2}{6\ (a\,c - b)^2\ t^2} \qquad v(x,t) = -\frac{(2\,b - 3\,a)^2\ (2\,c - 3)\ x^2}{6\ (a\,c - b)^3\ t^3}$$

whilst if $b = 2a$, and $c = 2$ we obtain

$$u(x,t) = -\frac{k_1^2 x^2}{\exp(k_1 t + k_2) - 6a} \qquad v(x,t) = -\frac{6ak_1^3 x^2}{\left[\exp(k_1 t + k_2) - 6a\right]^2}.$$

For $b = \frac{3}{2}a$ and $c = \frac{3}{2}$ the corresponding solution for (1.1) is given by

$$u(x,t) = \frac{4kx^2}{(k - 6a)^2 t^2} \qquad v(x,t) = \frac{8k^2 x^2}{(k - 6a)^3 t^3}.$$

Using the same algorithm in S_4, means that $v_{62}^{(1)}$ in terms of the invariants of v_{61}, can be written as

$$v_{62}^{(1)} = 2g_1\frac{\partial}{\partial g_1} + 2g_2\frac{\partial}{\partial g_2} + h_1\frac{\partial}{\partial h_1} + h_2\frac{\partial}{\partial h_2}$$

and S_6 can be reduced to a first-order system of ODEs by using the invariants of $v_{62}^{(1)}$ namely $w = g_1/g_2$, $h_1 = g_1^{1/2}M(w)$, $h_2 = g_2^{1/2}N(w)$. Then we have

$$\begin{aligned} &5aw^2M^2 + 2aw^3M\frac{\mathrm{d}M}{\mathrm{d}w} - 2aw^{7/2}N\frac{\mathrm{d}M}{\mathrm{d}w} - 2aw^{5/2}MN + 4w - 2 = 0 \\ &4bw^{5/2}MN - bw^3N^2 + 2bw^{5/2}M\frac{\mathrm{d}N}{\mathrm{d}w} - 2bw^4N\frac{\mathrm{d}N}{\mathrm{d}w} + 6w - 2c = 0. \end{aligned} \tag{2.4}$$

The invariants of v_{61} also lead to a system of first-order PDEs and the invariance of this PDE system under v_{62} leads to (2.4). Nevertheless the invariance of S_6 under v_{62} leads to a different

Table 6. Optimal system, similarity variables and similarity solutions.

	u_i	z	v
1	v_1	t	$h(z)$
2	v_2	t	$x^2 h(z)$
3	$\mu v_2 + v_3$	$x\,t^{-\mu}$	$t^{2\mu-3}\,h(z)$
4	$\mu v_1 + v_3$	$x - \mu \ln t$	$t^{-3}\,h(z)$
5	$\mu v_2 + v_4$	$x \exp\left\{-\mu \frac{c-1}{(2c-3)} t^{(2c-3)/(c-1)}\right\}$	$t^{-c/(c-1)} \exp\left\{2\mu \frac{c-1}{2c-3} t^{(2c-3)/(c-1)}\right\} h(z)$
6	$\lambda v_1 + v_4$	$x - \lambda \frac{c-1}{2c-3} t^{(2c-3)/(c-1)}$	$t^{-c/(c-1)} h(z)$
7	$\mu v_2 + \tilde{v}_4$	$x(\ln t)^{-\mu}$	$t^{-3}(\ln t)^{2\mu-1} h(z)$
8	$\lambda v_1 + \tilde{v}_4$	$x - \lambda \ln \lvert \ln t \rvert$	$t^{-3}(\ln t)^{-1} h(z)$

system of first-order PDEs which is not invariant under v_{61} and thus this symmetry is a hidden symmetry.

We must remark that many engineering and science problems may be reduced to nonlinear systems of ODEs, which are invariant only under a one-parameter Lie group. However, the reduced system may be invariant under an additional one-parameter group which is not a symmetry group of the original ODE system. This group invariance will not be found by the usual Lie method applied to the original system, and is consequently a hidden symmetry of the second-order ODE system. This kind of hidden symmetries were obtained for ODEs by Abraham-Shrauner and Guo [2], and are called hidden symmetries of type II. We have not find any of these symmetries in the S_i systems, $i = 1, \ldots, 6$.

3. New Lie symmetries and reductions for the special case $a = b$

In the special case when $a = b$, it is easy to see that by imposing $u = (c-1)tv$ the system (1.1) is reduced to the following equation:

$$(c-1)tv_t - a(c-1)^3 t^3 (vv_{xx} + v_x^2) + cv = 0. \tag{3.1}$$

This equation is invariant under a four-parameter Lie group whose infinitesimals are as follows.

If $c \neq \frac{3}{2}$:

$$v_1 = \frac{\partial}{\partial x} \qquad v_2 = x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}$$
$$v_3 = t\frac{\partial}{\partial t} - 3v\frac{\partial}{\partial v} \qquad v_4 = t^{c/(1-c)}\frac{\partial}{\partial t} - t^{(2c-3)/(1-c)} v\frac{\partial}{\partial v}$$

whilst if $c = \frac{3}{2}$:

$$v_1 = \frac{\partial}{\partial x} \qquad v_2 = x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}$$
$$v_3 = t\frac{\partial}{\partial t} - 3v\frac{\partial}{\partial v} \qquad \tilde{v}_4 = t\,\ln t\frac{\partial}{\partial t} - (3\ln t + 1)\,v\frac{\partial}{\partial v}.$$

By proceeding in the same way as for system (1.1), we can now construct the one-dimensional optimal system. We list this in table 6, for $c \neq \frac{3}{2}$ it is given by $\{u_i\}$ with $i = 1, \ldots, 6$, whilst for $c = \frac{3}{2}$ the optimal system is given by $\{u_i\}$ with $i = 1, \ldots, 4, 7$ and 8; where $\mu \in \mathbb{R}^*$ and $\lambda \in \mathbb{R}$ are arbitrary constants. We also list the corresponding similarity variables and similarity solutions.

In table 7, we list the ODEs to which (3.1) is reduced by $\{u_i\}$, $i = 1, \ldots, 8$.

Some exact solutions have been obtained for these equations as follows.

Table 7. ODE_i.

	ODE_i
1	$(c-1)zh' + ch = 0$
2	$(c-1)zh' - 6a(c-1)^3z^3h^2 + ch = 0$
3	$a(c-1)^3(hh')' + \mu(c-1)zh' - [(c-1)(2\mu-3) + c]h = 0$
4	$a(c-1)^3(hh')' + (c-1)\mu h' + (2c-3)h = 0$
5	$a(c-1)^2(hh')' + \mu zh' - 2h = 0$
6	$a(c-1)^2(hh')' + \lambda h' = 0$
7	$a(hh')' + 4\mu zh' + 4(1-2\mu)h = 0$
8	$a(hh')' + 4\lambda h' + 4h = 0$

- ODE_1 is a linear first-order equation, then it can be trivially solved, and the corresponding solution for equation (3.1) is given by

$$v(x,t) = k\, t^{c/(1-c)}$$

where k is an arbitrary constant.
- ODE_2 is a Bernouilli equation, the corresponding solution for (3.1) is given by

$$v(x,t) = \frac{(2c-3)x^2}{kt^{c/(1-c)} - 6a(c-1)^3t^3} \qquad \text{if} \quad c \neq \frac{3}{2}$$
$$v(x,t) = \frac{2x^2}{(2k - 3a\ln t)t^3} \qquad \text{if} \quad c = \frac{3}{2}.$$

- If $c \neq \frac{3}{2}$ we can choose $\mu = \frac{2c-3}{3(c-1)}$ in ODE_3, then it can be easily integrated once, and we find for (3.1) the one-parameter family of solutions

$$v(x,t) = t^{-3}\left[kt^{2\mu} - \frac{\mu x^2}{2a(c-1)^2}\right]$$

with k being an arbitrary constant.
- In the case of $c = \frac{3}{2}$ one can get solutions for ODE_4, in particular a solution of (3.1) obtained in this way is given by

$$v(x,t) = -\frac{4\mu}{at^3}\,(x - \mu\ln t)\,.$$

- By choosing $\mu = -2$ in ODE_5 we get for (3.1)

$$v(x,t) = t^{-c/(c-1)}\frac{x^2}{a(c-1)^2}.$$

- The general solution for ODE_6 is given in implicit form by

$$a(c-1)^2\lambda h + a(c-1)^2k_1\ln|\lambda h - k_1| = -\lambda^2(z + k_2) \qquad (3.2)$$

with k_1 and k_2 being arbitrary constants; (3.2) leads to a three-parameter family of solutions of (3.1).
- By choosing $\mu = \frac{1}{3}$ in ODE_7 we get the one-parameter family of solutions of (3.1)

$$v(x,t) = t^{-3}\,(\ln t)^{-1/3}\left[k - \frac{2x^2}{3a(\ln t)^{1/3}}\right].$$

- If we take $\lambda = 0$ in ODE_8, it can be found that

$$\int h(k_1 - h^3)^{-1/2}\,\mathrm{d}h = \pm 2\sqrt{\frac{2}{3a}}(z + k_2).$$

4. Concluding remarks

As far as we know, this is the first symmetry analysis of the system (1.1). In order to understand properly the importance of this symmetry analysis, we point out that this model is derived under the similarity hypothesis of Kolmogorov, which reads: *In processes consisting in exchange and dissipation of turbulent energy, the structure of the field of turbulent bursts is statistically identical in the neighbourhood of each point of the flow.*

According to this, the structure of the field of turbulent burst does not depend on each point. This fact strongly suggests that solutions with physical insight must be asymptotically self-similar solutions. This means that similarity solutions are limiting solutions as $t \to \infty$ Explicit solutions (such as those found by symmetry methods) can also play an important role in the design and testing of numerical integrators [22].

In this paper we have obtained the Lie classical symmetries of (1.1). In general the groups that leave the system (1.1) invariant depend on several parameters, to each one-parameter subgroup there will correspond a family of group-invariant solutions. We desired to minimize the search for group-invariant solutions to that of finding nonequivalent branches of solutions, which leads to the concept of an optimal system of group-invariant solutions from which many other solutions can be derived. To obtain the one-dimensional optimal systems of solutions, following Olver [22], we have looked for the one-dimensional optimal systems of subalgebras. We then constructed all the invariant solutions with respect to the one-dimensional optimal system of subalgebras, as well as all the ODEs to which the system (1.1) is reduced. We list the different similarity variables similarity solutions, as well as the second-order ODE system to which the system (1.1) is reduced. These systems obtained from the optimal system of subalgebras admit symmetries that lead to further reductions. S_1, S_2 and S_3 admit one symmetry, while S_4, S_5 and S_6 admit two symmetries. The invariance of a system of two second-order ODEs under a one-parameter group allows us to reduce the order of *one* of the equations of S_1, S_2, and S_3 by one. The invariance of S_4 and S_6 allows us to reduce them directly to systems of first-order ODEs. This can also be done in two steps, first reducing the order of one of the equations or reducing to a first-order PDE system, and then reducing these systems to first-order ODE system. Using these reduced equations some exact solutions have been derived. For the special case when $a = b$ the system (1.1) may be reduced to a PDE and additional symmetries has been obtained that yield solutions that cannot be derived from classical symmetries of (1.1). A comparison of the numerical and experimental results seems to conclude that the undetermined but fixed parameters satisfy $a = b$, but in [15] we studied equation (3.1) determining some solutions with a more applied character.

As the direct method due to Clarkson and Kruskal [11] has been successfully applied to obtain many new symmetry reductions for several physically significant PDEs (cf [10, 12] and the references therein), we have applied this method to the system (1.1). However, we have found that the symmetry reductions arising from the classical and direct method coincide. Another, more general method is the nonclassical method originally proposed by Bluman and Cole [7]. Nevertheless, for the system (1.1) we find again that it yields the same reductions as the classical method.

We must point out that, although the similarity-type behaviour has similar results to that of the porous medium equation [14], the porous medium equation has different reductions obtained from the nonclassical method. The question proposed by Clarkson [10] is to determine *a priori* which PDEs possess symmetry reductions which are unobtainable using the classical Lie group approach.

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