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Auto-Hodograph Transformations for a Hierarchy of Nonlinear Evolution Equations

N. Euler

Department of Mathematics, Luleå University of Technology, S-971 87 Luleå, Sweden
E-mail: Norbert@sm.luth.se

M. L. Gandarias

Department of Mathematics, Cadiz University, P.O. Box 40, Puerto Real, Cadiz, Spain
E-mail: Marialuz.Gandarias@uca.es

and

M. Euler and O. Lindblom

Department of Mathematics, Luleå University of Technology, S-971 87 Luleå, Sweden
E-mail: Marianna@sm.luth.se, Ove@sm.luth.se

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We introduce nonlocal auto-hodograph transformations for a hierarchy of nonlinear evolution equations. This is accomplished by composing nonlocal transformations (one of which is a hodograph transformation) which linearize the given equations. This enables one to construct sequences of exact solutions for any equation belonging to the hierarchy. © 2001 Academic Press

1. INTRODUCTION

In this paper we describe nonlocal transformations for a hierarchy of evolution equations resulting from the well-known nonlinear diffusion equation

$$u_t = \left(u^{-2}u_x\right)_x. \tag{1.1}$$

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As usual, the subscripts denote partial derivatives. It is well known [1, 2] that (1) admits Lie–Bäcklund symmetries and recursion operators. The simplest nontrivial Lie–Bäcklund symmetry of (1.1) is

$$Z_{\mathrm{LB}} = \left(u^{-3}u_{xxx} - 9u^{-4}u_x u_{xx} + 12u^{-5}u_x^3\right)\frac{\partial}{\partial u}, \tag{1.2}$$

and a recursion operator takes the form

$$R[u] = D_x^2 u^{-1} D_x^{-1}. \tag{1.3}$$

Here D_x is the total derivative operator in x, and

$$D_x^{-1} f(x) = \int^x f(x')\,dx'.$$

It is noteworthy that (1.1) is the only equation of the form

$$u_t = (u^n u_x)_x \tag{1.4}$$

which admits Lie–Bäcklund symmetries and recursion operators [1]. Using its recursion operator, we write down a hierarchy of evolution equations for (1.1). The novelty of our approach is to construct nonlocal auto-transformations, i.e., nonlocal transformations of contact-type, which keep the equations form-invariant. This is possible due to the fact that the constructed hierarchy is linearizable by two corresponding hierarchies of nonlocal transformations, one of which is a hierarchy of nonlocal hodograph transformations.

2. DEFINITION AND EXAMPLE

We give the following

Definition. We call a nonlocal transformation of hodograph type an auto-hodograph transformation if it transforms a given differential equation into itself.

Before discussing the hierarchy, we consider Eq. (1.1) and show how the auto-hodograph transformation is obtained and used in the construction of exact solutions of this equation.

The linear equation

$$U_T = U_{XX} \tag{2.1}$$

may be transformed into (1) by the invertible contact transformation $\mathcal{T}_1$ [1], given by

$$\mathcal{T}_1 : \begin{cases} dx(X,T) = U\,dX + U_X\,dT \\ dt(X,T) = dT \\ u(x,t) = U^{-1}. \end{cases} \tag{2.2}$$

Moreover, the linearization (2.1) of (1.1) (now written in terms of $\tilde{u}(\tilde{x},\tilde{t})$) can also be achieved by the invertible nonlocal hodograph transformation $\mathcal{T}_2$ [3], given by

$$\mathcal{T}_2 : \begin{cases} dX(\tilde{x},\tilde{t}) = \tilde{u}\,d\tilde{x} + \tilde{u}^{-2}\tilde{u}_{\tilde{x}}\,d\tilde{t} \\ dT(\tilde{x},\tilde{t}) = d\tilde{t} \\ U(X,T) = \tilde{x}. \end{cases} \tag{2.3}$$

An auto-hodograph transformation $\mathcal{A}$, mapping (1.1) with variable $(\tilde{x},\tilde{t},\tilde{u})$, into (1.1) with variable (x,t,u), is obtained by the composition

$$\mathcal{A} = \mathcal{T}_1 \circ \mathcal{T}_2 .$$

This gives

$$\mathcal{A} : \begin{cases} dx(\tilde{x},\tilde{t}) = \tilde{x}\tilde{u}\,d\tilde{x} + \left[\tilde{x}\tilde{u}^{-2}\tilde{u}_{\tilde{x}} + \tilde{u}^{-1}\right] d\tilde{t} \\ dt(\tilde{x},\tilde{t}) = d\tilde{t} \\ u(x,t) = \tilde{x}^{-1}. \end{cases} \tag{2.4}$$

The following example shows how the auto-hodograph transformation (2.4) can be used to construct exact solutions of (1.1).

Example. Using $\tilde{u}(\tilde{x},\tilde{t}) = 1$ as the seed-solution of (1.1), and applying $\mathcal{A}$, we obtain the solution

$$u(x,t) = \frac{1}{\sqrt{2}}(x-t)^{-1/2}.$$

Applying $\mathcal{A}$ again we get the solution

$$x = -\frac{1}{6}\left(\frac{2}{u(x,t)} - 2t\right)^{3/2} - t\left(\frac{2}{u(x,t)} - 2t\right)^{1/2},$$

which can explicitly be written as

$$u(x,t) = (8t^3 + 9x^2)^{-1/2}\left[(8t^3 + 9x^2)^{1/2} + 3x\right]^{-1/3}$$
$$\times\left\{\left[(8t^3 + 9x^2)^{1/2} + 3x\right]^{2/3} + 2t\right\}.$$

Applying $\mathscr{A}$ on the above, we obtain

$$x = \frac{2}{3}\left((8t^3 + 9u^{-2})^{1/2} + 3u^{-1}\right)^{-2/3} + \frac{1}{24}\left((8t^3 + 9u^{-2})^{1/2} + 3u^{-1}\right)^{4/3}$$
$$+ \frac{2}{3}t^4\left((8t^3 + 9u^{-2})^{1/2} + 3u^{-1}\right)^{-4/3}$$
$$+ \frac{t}{6}\left((8t^3 + 9u^{-2})^{1/2} + 3u^{-1}\right)^{2/3} - \frac{1}{2}t^2.$$

Here u may explicitly be written as

$$u(x,t) = \frac{6q^{3/2}}{q^3 - 8t^3},$$

where q is a solution of the quartic equation

$$q^4 + 4tq^3 - 12(2x + t^2)q^2 + 16t^3q + 16t^4 = 0.$$

3. A HIERARCHY OF AUTO-HODOGRAPH TRANSFORMATIONS

We now make use of the recursion operator (1.3) and construct a hierarchy of auto-hodograph transformations for the resulting hierarchy of evolution equations. We recall that an important property of recursion operators $R[u]$ of evolution equations is that one can generate symmetries

$$\eta(x,t,u,u_x,u_{xx},\dots)\frac{\partial}{\partial u}$$

for the evolution equation by acting η on compositions of $R[u]$, i.e.,

$$R^m[u]\eta, \qquad m = 0,1,2,\dots,$$

where $R^m[u]$ denotes m compositions of the given recursion operator and $R^0[u] := 1$. Moreover, a hierarchy of evolution equations can be obtained from the given evolution equation

$$u_t = F(x, t, u, u_x, u_{xx}, \dots),$$

by

$$u_t = R^m[u]F(x, t, u, u_x, u_{xx}, \dots).$$

For (1.1) the first three equations in the hierarchy $u_t = R^m[u](u^{-2}u_x)_x$ take the form

$$\begin{aligned} u_t &= \left(u^{-2}u_x\right)_x \\ u_t &= \left(u^{-3}u_x\right)_{xx} \\ u_t &= \left(\frac{1}{u}\left(u^{-3}u_x\right)_x\right)_x . \end{aligned}$$

We now prove the following

PROPOSITION. *For the hierarchy of evolution equations*

$$u_t = R^m[u]\left(u^{-2}u_x\right)_x, \tag{3.1}$$

a hierarchy of auto-hodograph transformations $\mathscr{A}_m$ is given by

$$\mathscr{A}_m : \begin{cases} dx(\tilde{x}, \tilde{t}) = \tilde{x}\tilde{u}\, d\tilde{x} + \left\{\tilde{x}D_{\tilde{x}}^{-1}R^m[\tilde{u}]\left(\tilde{u}^{-2}\tilde{u}_{\tilde{x}}\right)_{\tilde{x}} + \tilde{u}^{-1}\right\} d\tilde{t} \\ dt(\tilde{x}, \tilde{t}) = d\tilde{t} \\ u(x, t) = \tilde{x}^{-1}, \end{cases} \tag{3.2}$$

for corresponding m, *where* $m = 0, 1, 2, \dots$. *Here*

$$R[\tilde{u}] = D_{\tilde{x}}^2 \tilde{u}^{-1} D_{\tilde{x}}^{-1}, \qquad D_{\tilde{x}}^0 := 1, \quad R^0[\tilde{u}] := 1.$$

Proof. The linear hierarchy

$$U_T = D_X^m U_{XX} \tag{3.3}$$

is transformed to the nonlinear hierarchy

$$u_t = R^m[u]\left(u^{-2}u_x\right)_x \tag{3.4}$$

by the hierarchy of transformations $\mathcal{T}_{1,m}$. This hierarchy of transformations has the following form:

$$\mathcal{T}_{1,m}: \begin{cases} dx(X,T) = U\,dX + D_X^m U_X\,dT \\ dt(X,T) = dT \\ u(x,t) = U^{-1}(X,T). \end{cases} \tag{3.5}$$

Moreover, the hierarchy

$$\tilde{u}_{\tilde{t}} = R^m[\tilde{u}]\left(\tilde{u}^{-2}\tilde{u}_{\tilde{x}}\right)_{\tilde{x}} \tag{3.6}$$

can be linearized into the hierarchy (3.3) by the hierarchy of nonlocal hodograph transformations $\mathcal{T}_{2,m}$. This hierarchy of transformations has the following form:

$$\mathcal{T}_{2,m}: \begin{cases} dX(\tilde{x},\tilde{t}) = \tilde{u}\,d\tilde{x} + D_{\tilde{x}}^{-1}R^m[\tilde{u}]\left(\tilde{u}^{-2}\tilde{u}_{\tilde{x}}\right)_{\tilde{x}}\,d\tilde{t} \\ dT(\tilde{x},\tilde{t}) = d\tilde{t} \\ U(X,T) = \tilde{x}. \end{cases} \tag{3.7}$$

The composition $\mathcal{T}_{1,m} \circ \mathcal{T}_{2,m}$ gives the auto-hodograph transformation $\mathcal{A}_m$ which transforms (3.6) into (3.4). ∎

4. INVERTING THE TRANSFORMATIONS

We now construct the inverses of $\mathcal{T}_{1,m}$ and $\mathcal{T}_{2,m}$ in the most convenient form, i.e., in terms of the recursion, differential, and integral operators.

First consider the hierarchy of transformations $\mathcal{T}_{2,m}$. The transformation states that

$$\frac{\partial X}{\partial \tilde{x}} = \tilde{u}, \tag{4.1.1}$$

$$\frac{\partial X}{\partial \tilde{t}} = D_{\tilde{x}}^{-1}R^m[\tilde{u}]\left(\tilde{u}^{-2}\tilde{u}_{\tilde{x}}\right)_{\tilde{x}}. \tag{4.1.2}$$

Differentiating $\tilde{x}(X,T) = U$ with respect to $\tilde{x}$ leads to

$$\frac{\partial X}{\partial \tilde{x}} = \left(\frac{\partial U}{\partial X}\right)^{-1}$$

with $U \neq$ constant. The inverse of the hierarchy $\mathscr{T}_{2,m}$ is therefore

$$\mathscr{T}_{2,m}^{-1}: \begin{cases} \tilde{x}(X,T) = U \\ d\tilde{t}(X,T) = dT \\ \tilde{u}(\tilde{x},\tilde{t}) = \left(\dfrac{\partial U}{\partial X}\right)^{-1}, \end{cases} \tag{4.2}$$

where U is not constant. In addition to that, this transformation leads to the identity

$$\left(u^{-1}D_x\right)^{m+1} = u^{-1}D_x^{-1}R^m[u]D_x^2, \qquad m = 0,1,2,\ldots, \tag{4.3}$$

which is of importance for the inverse of the hierarchy $\mathscr{T}_{1,m}$. We therefore first prove this identity: Differentiating $x(X,T) = U$ with respect to t leads to

$$D_X^m U_{XX} = -u^{-1}D_x^{-1}R^m[u]\left(u^{-2}u_x\right)_x, \tag{4.4}$$

where we made use of (3.3) and (4.1.2). It is easy to show that

$$D_X^m U_X = \left(u^{-1}D_x\right)^m u^{-1},$$

so that (4.4) becomes

$$\left(u^{-1}D_x\right)^m \left(u^{-3}u_x\right) = u^{-1}D_x^{-1}R^m[u]\left(u^{-2}u_x\right)_x,$$

i.e., the identity is proved.

Consider now the hierarchy of the transformations $\mathscr{T}_{1,m}$: From the transformation it follows that

$$dX = U^{-1}\,dx - U^{-1}D_x^m U_x\,dT = u\,dx - uD_X^m U_X\,dt.$$

It is easy to show that, for $\mathscr{T}_{1,m}$,

$$D_X^m U_X = -\left(u^{-1}D_x\right)^m \left(u^{-3}u_x\right),$$

so that, by the identity (4.3), the inverse of $\mathscr{T}_{1,m}$ takes the form

$$\mathscr{T}_{1,m}^{-1}: \begin{cases} dX(x,t) = u\,dx + D_x^{-1}R^m[u]\left(u^{-2}u_x\right)_x\,dt \\ dT(x,t) = dt \\ U(X,T) = u^{-1}(x,t). \end{cases} \tag{4.5}$$

5. FINAL REMARKS

Finally, we remark the following: By substituting $u(x,t) = 1/v(x,t)$ in the hierarchy (3.1) one can obtain the hierarchy of equations

$$v_t = R^m[v](v^2 v_{xx}), \tag{5.1}$$

which admits the recursion operator

$$R[v] = v^2 D^2 v D_x^{-1} v^{-2}. \tag{5.2}$$

Writing down the first three equations in this hierarchy, we get

$$\begin{aligned} v_t &= v^2 v_{xx} \\ v_t &= (v^3 v_{xx})_x \\ v_t &= v^2 \left[v v_x^2 + v^2 v_{xx} \right]_{xx}. \end{aligned}$$

One should note that the hierarchy (5.1), just as the hierarchy (3.1), admits an auto-hodograph transformation and an infinite set of Lie–Bäcklund symmetries and potential symmetries. (We refer to the book of Bluman and Kumei [2] for details on potential symmetries.)

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REFERENCES

1. G. Bluman and S. Kumei, On the remarkable nonlinear diffusion equation $\partial/\partial x[a(u+b)^{-2}\partial u/\partial x] - (\partial u/\partial t) = 0$, *J. Math. Phys.* **21** (1980), 1019–1023.
2. G. Bluman and S. Kumei, "Symmetries and Differential Equations," Springer-Verlag, New York, 1989.
3. P. A. Clarkson, A. S. Fokas, and M. J. Ablowitz, Hodograph transformations of linearizable partial differential equations, *SIAM J. Appl. Math.* **49** (1989), 1188–1209.