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New methods of reduction for ordinary differential equations

C. MURIEL[†] AND J. L. ROMERO
Departamento de Matemáticas, Universidad de Cádiz, PO Box 40,
11510 *Puerto Real, Cádiz, Spain*

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We introduce a new class of symmetries, that strictly includes Lie symmetries, for which there exists an algorithm that lets us reduce the order of an ordinary differential equation. Many of the known order-reduction processes, that are not consequence of the existence of Lie symmetries, are a consequence of the invariance of the equation under vector fields of the new class. These vector fields must satisfy a new prolongation formula and there must exist a procedure for determining the vector fields of this class that lead to an equation invariant. We have also found some whose Lie symmetries are trivial, have no obvious order reductions, but can be completely integrated by using the new class of symmetries.

1. Introduction

Classical symmetry groups have been widely used to reduce the order of an ordinary differential equation (ODE) and to reduce the number of independent variables in a partial differential equation (PDE).

Nevertheless, there are many examples of ODEs with trivial Lie symmetries whose order can be reduced, or that can be completely integrated. It is not obvious (Clarkson, 1995) how a theory of order reductions could be achieved from a group-theoretic standpoint. In a recent paper (Hood, 1997) presented an ansatz-based method for the reduction of ODEs, which is somewhat similar to the direct method used for PDEs (Clarkson & Kruskal, 1989). It has also been suggested (González-López, 1988) that dynamical symmetries or Lie–Bäcklund symmetries could explain the reduction and integrability of a family of ODEs which has no Lie symmetries but, however, can be integrated. Other examples related to this situation appear in Olver (1995, p. 182).

The aim of this paper is to show that many of the known order-reduction processes can be explained by the invariance of the equation under some special vector fields that are neither Lie symmetries nor Lie–Bäcklund symmetries, but satisfy a new prolongation formula. The components of these vector fields must satisfy a system of determining equations that depends on an arbitrary function, which can be chosen to solve the system easily. When this arbitrary function is chosen to be null we obtain the classical Lie symmetries.

For this class of vector fields it is still possible, as for Lie symmetries (Olver, 1986, p. 144), to obtain a complete set of functionally independent invariants, by derivation of invariants of lower order. This lets us reduce the order of the equation and explain the reduction process of many ODEs that lack Lie symmetries, but whose order reductions are

[†]E-mail: concepcion.muriel@uca.es.

almost obvious. We have also found some ODEs whose Lie symmetries are trivial, have no obvious order reductions, but can be completely integrated by using the new class of vector fields.

Finally, the new theory allows us to explain several aspects of the loss and the gain of symmetries by order reductions. This has been studied by Abraham-Shrauner & Guo (1993), Abraham-Shrauner *et al.* (1995), Abraham-Shrauner (1996), who mention the difficulty in evaluating hidden symmetries: there is no direct method for determining this kind of symmetry. The vector fields we consider in this paper are essential in this theory and, for instance, exponential vector fields (Olver, 1986, p. 185) are a specific example of our class of vector fields.

2. The concept of $\mathcal{C}^\infty(M^{(1)})$-symmetry

Let v be a vector field defined on an open subset $M \subset X \times U$. We denote by $M^{(k)}$ the corresponding jet space $M^{(k)} \subset X \times U^{(k)}$, for $k \in \mathbb{N}$. Their elements are $(x, u^{(k)}) = (x, u, u_1, \ldots, u_k)$, where, for $i = 1, \ldots, k$, u_i denotes the derivative of order i of u with respect to x.

The two basic tools to obtain Lie symmetries of an ODE

$$\Delta(x, u^{(n)}) = 0 \tag{2.1}$$

are the general prolongation formula (Olver, 1986, p. 113) and the infinitesimal invariance criterion (Olver, 1986, p. 106). The latter characterizes a Lie symmetry of an ODE as a vector field $v = \xi(x,u)\partial/\partial x + \eta(x,u)\partial/\partial u$ that satisfies

$$v^{(n)}(\Delta(x, u^{(n)})) = 0 \qquad \text{if} \quad \Delta(x, u^{(n)}) = 0, \tag{2.2}$$

where $v^{(n)}$ denotes the nth prolongation of v.

For every function $\lambda \in C^\infty(M^{(1)})$, we will define a new prolongation of v in the following way.

DEFINITION 2.1 (New prolongation formula) Let $v = \xi(x,u)\partial/\partial x + \eta(x,u)\frac{\partial}{\partial u}$ be a vector field defined on M, and let $\lambda \in C^\infty(M^{(1)})$ be an arbitrary function. The λ-*prolongation of order n of* v, denoted by $v^{[\lambda,(n)]}$, is the vector field defined on $M^{(n)}$ by

$$v^{[\lambda,(n)]} = \xi(x,u)\frac{\partial}{\partial x} + \sum_{i=0}^{n} \eta^{[\lambda,(i)]}(x, u^{(i)})\frac{\partial}{\partial u_i},$$

where $\eta^{[\lambda,(0)]}(x,u) = \eta(x,u)$ and

$$\begin{aligned}\eta^{[\lambda,(i)]}(x, u^{(i)}) = {} & D_x(\eta^{[\lambda,(i-1)]}(x, u^{(i-1)})) - D_x(\xi(x,u))u_i \\ & + \lambda(\eta^{[\lambda,(i-1)]}(x, u^{(i-1)}) - \xi(x,u)u_i)\end{aligned}$$

for $1 \leqslant i \leqslant n$, where D_x denotes the total derivative operator with respect to x.

Let us observe that, if $\lambda \in C^\infty(M^{(1)})$ and $\lambda = 0$, the λ-prolongation of order n of v is the usual nth prolongation of v.

DEFINITION 2.2 Let $\Delta(x, u^{(n)}) = 0$ be an nth-order ordinary differential equation. We will say that a vector field v, defined on M, is a $C^\infty(M^{(1)})$-*symmetry* of the equation if there exists a function $\lambda \in C^\infty(M^{(1)})$ such that

$$v^{[\lambda,(n)]}(\Delta(x, u^{(n)})) = 0 \qquad \text{when} \quad \Delta(x, u^{(n)}) = 0.$$

In this case we will also say that v is a λ-symmetry.

Let us observe that if v is a 0-symmetry then v is a classical Lie symmetry.

An elegant characterization of Lie symmetries of the equation

$$u_n = F(x, u^{(n-1)}) \tag{2.3}$$

appears in Stephani (1989, p. 22): we associate to this equation the vector field

$$A = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + F(x, u^{(n-1)}) \frac{\partial}{\partial u_{n-1}}.$$

Then, a vector field v is an infinitesimal symmetry of (2.3) when $[v^{(n-1)}, A] = \mu A$ for some $\mu \in \mathcal{C}^\infty(M^{(1)})$.

Our next result presents a characterization of $C^\infty(M^{(1)})$-symmetries which is similar to this one.

THEOREM 2.1

1. Let us suppose that, for some $\lambda \in C^\infty(M^{(1)})$, the vector field v is a λ-symmetry of equation (2.3). Then

$$[v^{[\lambda,(n-1)]}, A] = \lambda \cdot v^{[\lambda,(n-1)]} + \mu \cdot A$$

for some $\mu \in C^\infty(M^{(1)})$.

2. Conversely, if

$$X = \xi(x, u) \frac{\partial}{\partial x} + \eta_0(x, u) \frac{\partial}{\partial u} + \sum_{i=1}^{n-1} \eta_i(x, u^{(i)}) \frac{\partial}{\partial u_i}$$

is a vector field defined on $M^{(n-1)}$ such that

$$[X, A] = \lambda \cdot X + \mu \cdot A$$

for some $\lambda, \mu \in C^\infty(M^{(1)})$, then the vector field

$$v = \xi(x, u) \frac{\partial}{\partial x} + \eta_0(x, u) \frac{\partial}{\partial u},$$

defined on M, is a λ-symmetry of the equation (2.3) and $X = v^{[\lambda,(n-1)]}$.

Proof. 1. Let us determine the expression of the vector field $[v^{[\lambda,(n-1)]}, A]$ in the coordinates $\{x, u, u_1, \ldots, u_{n-1}\}$. By using the prolongation formula that appears in

Definition 2.2 we have

$$\begin{aligned}
[v^{[\lambda,(n-1)]}, A](x) &= -A(\xi(x,u)),\\
[v^{[\lambda,(n-1)]}, A](u) &= \eta^{[\lambda,(1)]}(x,u^{(1)}) - A(\eta(x,u))\\
&= -A(\xi(x,u))\cdot u_1 + \lambda(\eta(x,u) - \xi(x,u)u_1),\\
[v^{[\lambda,(n-1)]}, A](u_1) &= \eta^{[\lambda,(2)]}(x,u^{(2)}) - A(\eta^{[\lambda,(1)]}(x,u^{(1)}))\\
&= -A(\xi(x,u))\cdot u_2 + \lambda(\eta^{[\lambda,(1)]}(x,u^{(1)}) - \xi(x,u)u_2),\\
&\;\;\vdots\\
[v^{[\lambda,(n-1)]}, A](u_i) &= \eta^{[\lambda,(i+1)]}(x,u^{(i+1)}) - A(\eta^{[\lambda,(i)]}(x,u^{(i)}))\\
&= -A(\xi(x,u))\cdot u_{i+1} + \lambda(\eta^{[\lambda,(i)]}(x,u^{(i)}) - \xi(x,u)u_i),\\
&\;\;\vdots\\
[v^{[\lambda,(n-1)]}, A](u_{n-1}) &= v^{[\lambda,(n-1)]}(F(x,u^{(n-1)})) - A(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}))
\end{aligned}$$

and

$$\begin{aligned}
v^{[\lambda,(n)]}(u_n) = {}& D_x(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)})) - D_x(\xi(x,u))u_n\\
&+\lambda(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}) - \xi(x,u)u_n).
\end{aligned} \tag{2.4}$$

Since v is a λ-symmetry,

$$v^{[\lambda,(n)]}(u_n) = v^{[\lambda,(n-1)]}(F(x,u^{(n-1)})) \quad \text{when} \quad u_n = F(x,u^{(n-1)}). \tag{2.5}$$

Hence, if $u_n = F(x,u^{(n-1)})$, equation (2.4) says that

$$\begin{aligned}
v^{[\lambda,(n-1)]}(F(x,u^{(n-1)})) = {}& A(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)})) - A(\xi(x,u))F(x,u^{(n-1)})\\
&+ \lambda(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}) - \xi(x,u)F(x,u^{n-1})).
\end{aligned}$$

If we set $\mu = -A(\xi(x,u)) - \lambda\xi(x,u)$ then we can write

$$\begin{aligned}
[v^{[\lambda,(n-1)]}, A](x) &= \lambda\xi(x,u) + \mu,\\
[v^{[\lambda,(n-1)]}, A](u) &= \lambda\eta(x,u) + \mu\cdot u_1,\\
[v^{[\lambda,(n-1)]}, A](u_1) &= \lambda\eta^{[\lambda,(1)]}(x,u^{(1)}) + \mu\cdot u_2,\\
&\;\;\vdots\\
[v^{[\lambda,(n-1)]}, A](u_i) &= \lambda\eta^{[\lambda,(i)]}(x,u^{(i)}) + \mu\cdot u_{i+1},\\
&\;\;\vdots\\
[v^{[\lambda,(n-1)]}, A](u_{n-1}) &= \lambda\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}) + \mu F(x,u^{(n-1)}).
\end{aligned}$$

Therefore, we conclude that

$$[v^{[\lambda,(n-1)]}, A] = \lambda v^{[\lambda,(n-1)]} + \mu \cdot A.$$

2. Let us suppose that $[X, A] = \lambda X + \mu A$. If we apply both elements of this equation to each coordinate function, we obtain

$$\mu = -A(\xi(x, u)) - \lambda\xi(x, u)$$

and, for $0 \leqslant i \leqslant n-2$, the coordinates $\eta_i(x, u^{(i)})$ of X must satisfy

$$\eta_{i+1}(x, u^{(i+1)}) = D_x(\eta_i(x, u^{(i)})) - u_{i+1}D_x(\xi(x, u)) + \lambda(\eta_i(x, u^{(i)}) - \xi(x, u)u_{i+1}).$$

Hence

$$X = v^{[\lambda,(n-1)]}.$$

If we apply both elements of $[X, A] = \lambda X + \mu A$ to the coordinate function u_{n-1}, we obtain

$$\begin{aligned} &X(F(x, u^{(n-1)})) - A(\eta^{[\lambda,(n-1)]}(x, u^{(n-1)})) \\ &\qquad = \lambda\eta^{[\lambda,(n-1)]}(x, u^{(n-1)}) - (A(\xi(x, u)) + \lambda\xi(x, u)) \cdot F(x, u^{(n-1)}). \end{aligned}$$

This proves that

$$\begin{aligned} X(F(x, u^{(n-1)})) = {}& A(\eta^{[\lambda,(n-1)]}(x, u^{(n-1)})) + \lambda\eta^{[\lambda,(n-1)]}(x, u^{(n-1)}) \\ & -(A(\xi(x, u)) + \lambda\xi(x, u)) \cdot F(x, u^{(n-1)}). \end{aligned} \tag{2.6}$$

Let us check that v satisfies

$$v^{[\lambda,(n)]}(u_n - F(x, u^{(n-1)})) = 0, \quad \text{if} \quad u_n = F(x, u^{(n-1)}).$$

If we evaluate

$$\begin{aligned} v^{[\lambda,(n)]}(u_n - F(x, u^{(n-1)})) = {}& D_x(\eta^{[\lambda,(n-1)]}(x, u^{(n-1)})) - u_n D_x(\xi(x, u)) \\ & + \lambda(\eta^{[\lambda,(n-1)]}(x, u^{(n-1)}) \\ & - \xi(x, u)u_n) - X(F(x, u^{(n-1)})) \end{aligned}$$

when $u_n = F(x, u^{(n-1)})$, we obtain, by (2.6), that

$$v^{[\lambda,(n)]}(u_n - F(x, u^{(n-1)})) = 0, \quad \text{when} \quad u_n = F(x, u^{(n-1)}).$$

Therefore v satisfies

$$0 = v^{[\lambda,(n)]}(u_n - F(x, u^{(n-1)})) = v^{[\lambda,(n)]}(\Delta(x, u^{(n)})), \quad \text{if} \quad \Delta(x, u^{(n)}) = 0,$$

and v is a λ-symmetry of the equation. □

3. Order reductions and $C^\infty(M^{(1)})$-symmetries

In this section we prove that if v is a $C^\infty(M^{(1)})$-symmetry of equation (2.3) then there exists a procedure to reduce the equation to an $(n-1)$th-order equation and a first-order equation. For this goal, the main tools will be Theorem 2.1 and our next result, which explains that if v is a λ-symmetry, with $\lambda \in C^\infty(M^{(1)})$, then we can determine invariants for the λ-prolongation of v by derivation of invariants of lower order.

THEOREM 3.1 Let v be a vector field defined on $M \subset X \times U$ and let $\lambda \in C^\infty(M^{(1)})$. If

$$\alpha = \alpha(x, u^{(k)}), \quad \beta = \beta(x, u^{(k)}) \in C^\infty(M^{(k)})$$

are such that

$$v^{[\lambda,(k)]}(\alpha(x, u^{(k)})) = v^{[\lambda,(k)]}(\beta(x, u^{(k)})) = 0,$$

then

$$v^{[\lambda,(k+1)]}\left(\frac{D_x\alpha(x, u^{(k)})}{D_x\beta(x, u^{(k)})}\right) = 0.$$

Proof. It is clear that

$$[v^{[\lambda,(k+1)]}, D_x] = \lambda v^{[\lambda,(k+1)]} + \mu D_x,$$

where $\mu = -D_x(v(x)) - \lambda v(x) \in C^\infty(M^{(1)})$. Therefore,

$$\begin{aligned} v^{[\lambda,(k+1)]}\left(\frac{D_x\alpha}{D_x\beta}\right) &= \frac{1}{(D_x\beta)^2}(D_x\beta \cdot v^{[\lambda,(k+1)]}(D_x\alpha) - D_x\alpha \cdot v^{[\lambda,(k+1)]}(D_x\beta)) \\ &= \frac{1}{(D_x\beta)^2}(D_x\beta \cdot [v^{[\lambda,(k+1)]}, D_x](\alpha) - D_x\alpha \cdot [v^{[\lambda,(k+1)]}, D_x](\beta)) \\ &= \frac{1}{(D_x\beta)^2}(D_x\beta \cdot (\mu \cdot D_x\alpha) - D_x\alpha \cdot (\mu \cdot D_x\beta)) \\ &= 0. \end{aligned}$$

□

Our next objective is to show how a $C^\infty(M^{(1)})$-symmetry lets us reduce the order of an ODE.

THEOREM 3.2 Let v be a λ-symmetry, with $\lambda \in C^\infty(M^{(1)})$, of the equation $\Delta(x, u^{(n)}) = 0$. Let $y = y(x, u)$ and $w = w(x, u, u_1)$ be two functionally independent first-order invariants of $v^{[\lambda,(n)]}$. The general solution of the equation can be obtained by solving an equation of the form $\Delta_r(y, w^{(n-1)}) = 0$ and an auxiliary equation $w = w(x, u, u_1)$.

Proof. Let $y = y(x, u)$ and $w = w(x, u, u_1)$ be two functionally independent first-order invariants of $v^{[\lambda,(n)]}$ such that w depends on u_1. By Theorem 3.1,

$$w_1 = \frac{D_x w(x, u, u_1)}{D_x y(x, u)}$$

is an invariant for $v^{[\lambda,(n)]}$ which is obviously functionally independent of y and w, because w_1 depends on u_2. From w_1 and y we construct, by derivation, a third-order invariant for $v^{[\lambda,(n)]}$, and so on. Therefore, the set

$$\{y, w, w_1, \ldots, w_{n-1}\}$$

is a complete set of functionally independent invariants of $v^{[\lambda,(n)]}$. Since v is, by hypothesis, a $C^\infty(M^{(1)})$-symmetry of the equation, this can be written in terms of $\{y, w, w_1, \dots, w_{n-1}\}$. The resulting equation is a $(n-1)$th-order equation of the form

$$\Delta_r(y, w^{(n-1)}) = 0. \tag{3.7}$$

We can recover the general solution of the original equation from the general solution of (3.7) and the corresponding first-order auxiliary equation:

$$w = w(x, u, u_1). \qquad \square$$

4. Some applications

In this section we consider some equations whose order can be reduced by using some specific *ansatz*, but lack Lie or Lie–Bäcklund symmetries, and that have appeared in the literature. Some of the corresponding reduction processes can be explained by the existence of $C^\infty(M^{(1)})$-symmetries.

A. Olver (1995, p. 182) considered the equation

$$u_{xx} = [(x + x^2)e^u]_x \tag{4.8}$$

as an example of an equation that can be integrated by quadratures, but that lacks non-trivial Lie symmetries. Equation (4.8) has the form

$$u_{xx} = D_x F(x, u), \tag{4.9}$$

which admits the obvious order reduction

$$u_x = F(x, u) + C, \quad C \in \mathbb{R}, \tag{4.10}$$

but could have no Lie symmetries. For this class of equations we have the following result.

THEOREM 4.1 A second-order differential equation of the form (4.9) admits the λ-symmetry $v = \partial/\partial u$, with $\lambda = F_u(x, u)$. The reduction process of (4.8)–(4.10) is the process described in Theorem 3.2.

Proof. 1. By using the prolongation formula that appears in Definition 2.2, we obtain

$$V = \frac{\partial}{\partial u} + F_u \frac{\partial}{\partial u_x} + (F_u^2 + u_x F_{uu} + F_{xu}) \frac{\partial}{\partial u_{xx}}. \tag{4.11}$$

It is straightforward to check that the vector field $v^{[\lambda,(2)]}$ satisfies $v^{[\lambda,(2)]}(u_{xx} - D_x F(x, u)) = 0$. Hence $v^{[\lambda,(2)]}(u_{xx} - D_x F(x, u)) = 0$ when $u_{xx} - D_x F(x, u) = 0$.

Therefore v is a λ-symmetry of (4.9), with $\lambda = F_u(x, u)$.

2. Two functionally independent invariants for $v^{[\lambda,(2)]}$ are $z = x$ and $w = u_x - F(x, u)$.

3. Another invariant of $v^{[\lambda,(2)]}$ can be obtained by derivation:

$$w_z = \frac{D_x w}{D_x z} = u_{xx} - D_x F(x, u).$$

4. In terms of $\{z, w, w_z\}$ the expression of (4.9) is the trivial first-order differential equation:

$$w_z = 0.$$

Its general solution, $w = C, C \in \mathbb{R}$, allows us to obtain the general solution of (4.9) by solving:

$$u_x = F(x, u) + C. \qquad \square$$

B. González-López (1988) proved that an ODE of the form

$$u_{xx} - (u^{-1}u_x^2 + pg(x)u^p u_x + g'(x)u^{p+1}) = 0, \quad p \neq 0, \tag{4.12}$$

is integrable by quadratures but does not have non-trivial Lie symmetries, unless the function g is in the form:

$$g(x) = k_1 e^{k_2 x}(k_3 + k_4 x)^{k_5}, \quad \text{or} \quad g(x) = k_6 e^{k_7 x^2},$$

where $k_i \in \mathbb{R}$, for $1 \leqslant i \leqslant 7$. For this class of equations we have the result that follows.

THEOREM 4.2 Equation (4.12) admits $v = \frac{\partial}{\partial u}$ as a λ-symmetry, with $\lambda = \frac{pg(x)u^{p+1}+u_x}{u}$. This equation can be reduced by the process described in Theorem 3.2.

Proof. It can be checked, by using the prolongation formula given in Definition 2.2, that

$$\begin{aligned} v^{[\lambda,(2)]} = {} & \frac{\partial}{\partial u} + \left(\frac{u_x}{u} + p\,u^p\,g(x)\right)\frac{\partial}{\partial u_x} \\ & + \Bigg(\frac{u_{xx}}{u} + 2\,p\,u^{-1+p}\,u_x\,g(x) \\ & + p^2\,u^{-1+p}\,u_x\,g(x) + p^2\,u^{2\,p}\,g(x)^2 + p\,u^p\,g'(x)\Bigg)\frac{\partial}{\partial u_{xx}}. \end{aligned} \tag{4.13}$$

Let us denote by $\Delta(x, u, u_x, u_{xx})$ the first member of (4.12).

1. It can be checked that

$$v^{[\lambda,(2)]}(\Delta(x, u, u_x, u_{xx})) = -\frac{{u_x}^2}{u^2} + \frac{u_{xx}}{u} - p\,u^{-1+p}\,u_x\,g(x) - u^p\,g'(x).$$

To evaluate the second member when $\Delta(x, u, u_x, u_{xx}) = 0$, we substitute u_{xx} by $u^{-1}u_x^2 + pg(x)u^p u_x + g'(x)u^{p+1}$, and then we simplify. Accordingly:

$$\begin{aligned} v^{[\lambda,(2)]}(\Delta(x, u, u_x, u_{xx})) &= -\frac{{u_x}^2}{u^2} - p\,u^{-1+p}\,u_x\,g(x) - u^p\,g'(x) \\ &\quad + \frac{{u_x}^2}{u} + p\,u^p\,u_x\,g(x) + \frac{u^{1+p}\,g'(x)}{u} \\ &= 0. \end{aligned}$$

Therefore, $v = \partial/\partial u$ is a λ-symmetry of (4.12), with $\lambda = (pg(x)u^{p+1} + u_x)/u$.

2. Two functionally independent invariants for $v^{[\lambda,(2)]}$ are

$$z = x, \quad w = \frac{u_x}{u} - u^p\, g(x).$$

3. The key feature that allows us to use $v^{[\lambda,(2)]}$ to reduce the order of (4.12) is that we can obtain another invariant of $v^{[\lambda,(2)]}$ from z and w by derivation:

$$w_z = \frac{D_x w}{D_x z} = -\frac{{u_x}^2}{u^2} + \frac{u_{xx}}{u} - p\, u^{-1+p}\, u_x\, g(x) - u^p\, g'(x).$$

The set $\{z, z, w_z\}$ is a complete set of functionally independent invariants for $v^{[\lambda,(2)]}$.

4. The first member of equation (4.12) can be written, in terms of z, w and w_z, as

$$w_z = 0. \tag{4.14}$$

The general solution of (4.14) is $w = C,\ C \in \mathbb{R}$. We recover the general solution of our original equation (4.12) by solving the first-order differential equation:

$$\frac{u_x}{u} - u^p\, g(x) = C,$$

which is a Bernoulli equation and is, therefore, integrable by quadratures. □

C. Since the specific equations that have been considered above can be easily reduced, in order to show the utility of our new symmetries, we must find ODEs which cannot be trivially reduced or integrated and that have no Lie symmetries.

The existence of a $C^\infty(M^{(1)})$-symmetry for a second-order ODE can be used to integrate the equation, by solving two first-order differential equations: one of them is the reduced equation and the other one is the auxiliary equation which appears in Theorem 3.2. The relation between the original equation and the two first-order equations is not clear unless one knows how to calculate $C^\infty(M^{(1)})$-symmetries.

The algorithm to obtain $C^\infty(M^{(1)})$-symmetries will be described using an example and will exhibit another of the advantages of $C^\infty(M^{(1)})$-symmetries: the determining equations to obtain $C^\infty(M^{(1)})$-symmetries are easier to solve than the usual determining equations in the classical Lie method. The reason is clear, the function λ can be chosen in such a way that the corresponding infinitesimals can be calculated. Nevertheless, if you want to find Lie symmetries, you must solve a fixed system.

To show this advantage, the determining equations in the classical method (4.16) should be compared with the new determining equations (4.23) which appear below. The example that follows goes even further: the determining equations in Lie's method has only the trivial solution (the equation has no Lie symmetries) but, with an appropriate selection of λ (not equal to zero, of course), we have found non-trivial solutions for the infinitesimals of a $C^\infty(M^{(1)})$-symmetry.

THEOREM 4.3 The second-order differential equation

$$u_{xx} = -\left(\frac{x^2}{4u^3} + u + \frac{1}{2u}\right) \tag{4.15}$$

has no Lie symmetries.

Proof. If a vector field $v = p(x,u)\partial/\partial x + r(x,u)\partial/\partial u$ is a Lie symmetry of the equation (4.15), its infinitesimals p, r must satisfy the following determining system:

$$\begin{aligned}
&p_{uu} = 0,\\
&r_{uu} - 2p_{xu} = 0,\\
&\frac{3p_u x^2}{4u^3} + 3p_u u + \frac{3p_u}{2u} + 2r_{xu} - p_{xx} = 0,\\
&\frac{-r_u x^2}{4u^3} + \frac{p_x x^2}{2u^3} - \frac{3rx^2}{4u^4} + \frac{px}{2u^3} - r_u u + 2p_x u - \frac{r_u}{2u} + \frac{p_x}{u} - \frac{r}{2u^2} + r_{xx} + r = 0.
\end{aligned} \tag{4.16}$$

The first equation shows that p must be linear in u:

$$p(x,u) = a(x)u + b(x). \tag{4.17}$$

After substituting this value in (4.16) we have:

$$\begin{aligned}
&r_{uu} - 2a' = 0\\
&\frac{3ax^2}{4u^3} + 3au + \frac{3a}{2u} + 2r_{xu} - (a''u + b'') = 0\\
&\frac{-r_u x^2}{4u^3} + \frac{(a'u+b')x^2}{2u^3} - \frac{3rx^2}{4u^4} + \frac{(au+b)x}{2u^3} - r_u u + 2(a'u+b')u - \frac{r_u}{2u} + \frac{a'u+b'}{u}\\
&\qquad - \frac{r}{2u^2} + r_{xx} + r = 0.
\end{aligned}$$

By integrating the first of these equations with respect to u, we obtain that r must have the form

$$r(x,u) = a'(x)u^2 + c(x)u + d(x), \tag{4.18}$$

where a, c and d are functions of x. Then, the remaining equations are now

$$\begin{aligned}
&\frac{3ax^2}{4u^3} + 3au + \frac{3a}{2u} + 2(2a''u + c') - (a''u + b'') = 0,\\
&\frac{-(2a'u+c)x^2}{4u^3} + \frac{(a'u+b')x^2}{2u^3} - \frac{3(a'u^2+cu+d)x^2}{4u^4} + \frac{(au+b)x}{2u^3} - (2a'u+c)u\\
&\quad + 2(a'u+b')u - \frac{2a'u+c}{2u} + \frac{a'u+b'}{u} - \frac{a'u^2+cu+d}{2u^2}\\
&\qquad + a'''u^2 + c''u + d'' + a'u^2 + cu + d = 0.
\end{aligned}$$

We simplify the first equation and then cancel out every coefficient of powers of u; in particular, since the coefficient of u^2 is $a(x)$, it follows that

$$a(x) = 0. \tag{4.19}$$

Therefore,

$$\begin{aligned}
&2c' - b'' = 0,\\
&\frac{-cx^2}{4u^3} + \frac{b'x^2}{2u^3} - \frac{3(cu+d)x^2}{4u^4} + \frac{bx}{2u^3} - cu + 2b'u - \frac{c}{2u} + \frac{b'}{u} - \frac{cu+d}{2u^2}\\
&\qquad + c''u + d'' + cu + d = 0.
\end{aligned}$$

If we multiply the last equation by u^4 then the coefficients of u^2 and u^5 must vanish. Hence $d(x) = 0$ and

$$c(x) = b'(x). \tag{4.20}$$

With these values, the determining equations reduce to

$$b'' = 0, \qquad \frac{-b'x^2}{2u^3} + \frac{bx}{2u^3} + b'''u + 2b'u = 0. \tag{4.21}$$

Equation $(4.21)_1$ gives

$$b(x) = c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R}^2, \tag{4.22}$$

and $(4.21)_2$, gives

$$c_2 x + 4c_1 u^4 = 0.$$

Consequently $c_1 = c_2 = 0$, and $b(x) = 0$. By (4.20), we have $c(x) = 0$. By (4.17), $p(x,u) = 0$. Finally, by (4.18), $r(x,u) = 0$. Therefore, the equation has no Lie symmetries. □

Our next theorem proves that (4.15) admits non-trivial $C^\infty(M^{(1)})$-symmetries that allow us to obtain the general solution of the equation.

THEOREM 4.4 Equation (4.15) admits non-rivial $C^\infty(M^{(1)})$-symmetries and can be completely integrated by the process described in Theorem 3.2.

Proof. If a vector field $v = p(x,u)\partial/\partial x + r(x,u)\partial/\partial u$ is a λ-symmetry of equation (4.15), for some $\lambda \in C^\infty(M^{(1)})$, then their infinitesimals p and r must satisfy the following determining equations:

$$\begin{aligned}
&p_{uu} = 0,\\
&r_{uu} - 2p_{xu} - 2\lambda p_u - \lambda_u p = 0,\\
&\frac{3p_u x^2}{4u^3} + 3p_u u + \frac{3p_u}{2u} + 2r_{xu} - p_{xx} + 2\lambda r_u + \lambda_u r - 2\lambda p_x - \lambda_x p - \lambda^2 p = 0,\\
&\frac{-r_u x^2}{4u^3} + \frac{p_x x^2}{2u^3} - \frac{3rx^2}{4u^4} + \frac{px}{2u^3} - r_u u + 2p_x u - \frac{r_u}{2u} + \frac{p_x}{u} - \frac{r}{2u^2} + r_{xx} + r\\
&\qquad + \frac{\lambda p x^2}{2u^3} + 2\lambda p u + \frac{\lambda p}{u} + 2\lambda r_x + \lambda_x r + \lambda^2 r = 0.
\end{aligned} \tag{4.23}$$

It can be checked that these equations, whose unknowns are p, r and λ, admit the solution $p = 0, r = u, \lambda = x/u^2$. Hence, if $\lambda = x/u^2$, the vector field $v = u\frac{\partial}{\partial u}$ is a λ-symmetry of (4.15).

By using the prolongation formula (2.1), we can determine $v^{[\lambda,(2)]}$ with $\lambda = x/u^2$, and we obtain

$$v^{[\lambda,(2)]} = u\frac{\partial}{\partial u} + \left(u_x + \frac{x}{u}\right)\frac{\partial}{\partial u_x} + \left(\frac{1}{u} + u_{xx} + \frac{x^2}{3}\right)\frac{\partial}{\partial u_{xx}}.$$

It can be checked that

$$y = x, \quad w = -\left(\frac{u_x}{u} + \frac{x}{2u^2}\right)$$

are two functionally independent invariants for $v^{[\lambda,(1)]}$.

By Theorem 3.1, we can calculate an additional invariant by derivation:

$$w_y = \frac{-u + 2uu_x + 2u_x x - 2u^2 u_{xx}}{2u^3}.$$

Equation (4.15) can be written in terms of $\{y, w, w_y\}$ and we obtain the reduced equation

$$1 + w^2 - w_y = 0.$$

This first-order differential equation can be integrated easily: its general solution is:

$$w = \tan(y + c_1), \qquad c_1 \in \mathbb{R}.$$

We recover the general solution of equation (4.15) by solving the auxiliary first-order differential equation

$$2u^2 \tan(x + c_1) + 2uu_x + x = 0.$$

We set $\tilde{u} = u^2$ and then the equation is transformed into the linear differential equation

$$2\tilde{u}\tan(x + c_1) + \tilde{u}_x + x = 0.$$

If we integrate this equation, we obtain the general solution of (4.15):

$$u = \pm\cos(x+c_1)\sqrt{-\log(\cos(x + c_1)) - x\tan(x + c_1) + c_2}, \quad c_1, c_2 \in \mathbb{R}. \qquad \square$$

5. Exponential vector fields and $C^\infty(M^{(1)})$-symmetries

Olver (1986, p. 185) introduced the concept of 'exponential vector field' in order to show that not every method of integrating an ODE comes from the classical Lie method. He defined an *exponential vector field* as a formal expression of the form

$$v^* = \mathrm{e}^{\int P(x,u)\,\mathrm{d}x}\left(\xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}\right),$$

where $\int P(x, u)\,\mathrm{d}x$ is, formally, the integral of the function $P(x, u)$, once one has chosen a function $u = f(x)$. He notes that $v^{*(n)} = \mathrm{e}^{\int P(x,u)\,\mathrm{d}x} V$, where V is an ordinary vector field on $M^{(n)}$, and that an exponential vector field can be used to reduce the order of ODEs. We show here that exponential vector fields are specific $C^\infty(M^{(1)})$-symmetries and that study of λ-symmetries of an equation lets us determine the exponential vector fields admitted by the equation.

LEMMA 5.1 Let us consider the ODE (2.3) and let A be its corresponding vector field. If a vector field v is a λ-symmetry of (2.3), then for every $f \in C^\infty(M)$ the vector field $f \cdot v$ is a $\tilde{\lambda}$-symmetry, with $\tilde{\lambda} = -(A(f)/f) + \lambda$, of (2.3).

Proof. If v is a λ-symmetry of the equation, then:

$$[v^{[\lambda,(n-1)]}, A] = \lambda \cdot v^{[\lambda,(n-1)]} + \mu \cdot A.$$

By the properties of the Lie bracket,

$$\begin{aligned}[f \cdot v^{[\lambda,(n-1)]}, A] &= f \cdot \lambda v^{[\lambda,(n-1)]} + f \cdot \mu A - A(f) \cdot v^{[\lambda,(n-1)]} \\ &= \left(\lambda - \frac{A(f)}{f}\right) \cdot f \cdot v^{[\lambda,(n-1)]} + (f\mu) \cdot A.\end{aligned}$$

Let us denote $\tilde{\lambda} = \lambda - A(f)/f$. Then, the vector field $(f \cdot v)^{[\tilde{\lambda},(n-1)]}$ satisfies:

$$[(f \cdot v)^{[\tilde{\lambda},(n-1)]}, A] = \tilde{\lambda} \cdot (f \cdot v)^{[\tilde{\lambda},(n-1)]} + (f\mu) \cdot A.$$

Therefore

$$(f \cdot v)^{[\tilde{\lambda},(n-1)]} = f \cdot v^{[\lambda,(n-1)]},$$

and, by Theorem 2.1, the vector field $f \cdot v$ is a $\mathcal{C}^\infty(M^{(1)})$-symmetry of the equation. □

Let us observe that if $\lambda \equiv 0$ the vector field v is a Lie symmetry of the equation. In this case, by Lemma 5.1, the vector field $f \cdot v$ is a $\tilde{\lambda}$-symmetry with $\tilde{\lambda} = -A(f)/f$, for every $f \in \mathcal{C}^\infty(M)$.

THEOREM 5.1 Let v^* be the exponential vector field, in Olver's sense,

$$v^* = \mathrm{e}^{\int P(x,u)\,\mathrm{d}x}\left(\xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}\right).$$

Let us suppose that

$$v^{*(n)}(\Delta(x, u^{(n)})) = 0 \quad \text{if} \quad \Delta(x, u^{(n)}) = 0.$$

Then the vector field $v = \xi(x,u)\partial/\partial x + \eta(x,u)\partial/\partial u$ is a λ-symmetry with $\lambda = P(x,u)$.

Proof. It is sufficient to apply Lemma 5.1 to v^* and $f = \mathrm{e}^{-\int P(x,u)\,\mathrm{d}x}$.

Exponential vector fields usually arise, in practice, in reduction processes. When an ODE admits a Lie algebra with two infinitesimal generators v_1 and v_2 such that $[v_1, v_2] = cv_1$, $c \in \mathbb{R}$, one *must* reduce the order by using v_1 and the reduced equation inherits a Lie symmetry from v_2. Nevertheless, if $c \neq 0$, and we use v_2 instead of v_1 to reduce the equation then, in the coordinates of the reduced equation, the vector field $v_1^{(1)}$ is not well defined: it is an exponential vector field. In our language, we say that v_1 can be 'modified' in order to be conserved as a $C^\infty(M^{(1)})$-symmetry of the reduced equation. This is the first step on the theory of the loss and gain of symmetries by successive reductions of order. Additional work dealing with this theory is currently in progress.

Let us consider, again, equation (4.12), which admits $v = \partial/\partial u$ as a λ-symmetry, with $\lambda = pgu^p + u_x/u$. This equation has been considered in Abraham-Shrauner *et al.* (1995), to explain its integrability via hidden symmetries. By means of the transformation

$$u^p = -\frac{w'}{pgw}, \tag{5.24}$$

equation (4.12) can be transformed into the third-order equation

$$w'w'' - (w'')^2 - \left(\frac{g'}{g}\right)'(w')^2 = 0. \tag{5.25}$$

This equation admits $G_1 = \partial/\partial w$ and $G_2 = w\partial/\partial w$ as Lie symmetries. Since $[G_1, G_2] = G_1$, the reduction of (5.25) to (4.12) via (5.24) has been carried by a non-normal subgroup and so G_1 is lost as a point symmetry of (4.12). In fact G_1 is the exponential vector field

$$G_1 = -\frac{1}{p}u \cdot \exp\left(\int pgu^p \,\mathrm{d}x\right)\frac{\partial}{\partial u}.$$

By Lemma 5.1, the vector field $v_1 = -(1/p)u\partial/\partial u$ is a $\tilde{\lambda}$-symmetry of (4.12) with $\tilde{\lambda} = pgu^p$. Essentially, the vector field v_1 is the same λ-symmetry v that we have found in Theorem 4.2. By Lemma 5.1, the vector field $v = \partial/\partial u$ is a λ-symmetry with

$$\lambda = -\left(\frac{-1}{p}u\right)^{-1} A\left(\frac{-1}{p}u\right) + \tilde{\lambda}.$$

It must also be observed that vector fields v or v_1 can be obtained through a well-defined algorithm and there is no need to guess the transformation (5.24).

6. Concluding remarks

There are many known examples of ODEs, without Lie symmetries, that can be completely integrated. Many of the corresponding reduction processes are based on some specific *ansätze* and, as far as we know, they are not based on group theoretic considerations.

We have introduced a new class of vector fields, the $C^\infty(M^{(1)})$-symmetries, that are neither Lie symmetries nor Lie–Bäcklund symmetries but that let us give a reduction process in these equations, using the invariance of the equation under these vector fields. These using can be obtained by solving a set of determining equations that are easier to solve than the determining equations for the classical Lie method.

We illustrate our new theory with some examples of second-order differential equations which have no Lie symmetries (therefore the classical Lie method cannot be used to integrate them) but admit non-trivial $C^\infty(M^{(1)})$-symmetries. The new method of reduction is applied to integrate these equations by solving, in each case, two first-order differential equations.

There are many recent papers dealing with properties of hidden symmetries (Abraham-Shrauner, 1996; Abraham-Shrauner & Guo, 1993; Abraham-Shrauner *et al.*, 1995). In general, we have found that any vector field that is lost, as a point symmetry, after an order reduction, can be modified appropriately (depending on the structure of the Lie algebra) in a such a way that it can be recovered as a $C^\infty(M^{(1)})$-symmetry of the reduced equation.

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