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CHARACTERIZATIONS OF COMPLETENESS OF NORMED SPACES THROUGH WEAKLY UNCONDITIONALLY CAUCHY SERIES

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Abstract. In this paper we obtain two new characterizations of completeness of a normed space through the behaviour of its weakly unconditionally Cauchy series. We also prove that barrelledness of a normed space X can be characterized through the behaviour of its weakly-$*$ unconditionally Cauchy series in X^*.

Keywords: completeness, barrelledness, weakly unconditionally Cauchy series

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1. Introduction

Let X be a real normed space and let $\sigma = \sum\limits_{i=1}^{\infty} x_i$ be a series in X. Let us recall that σ is called unconditionally convergent (uc) (resp. weakly unconditionally Cauchy (wuC)) if $\sum\limits_{i=1}^{\infty} x_{\pi(i)}$ converges (resp. $\Big(\sum\limits_{k=1}^{i} x_{\pi(k)}\Big)_i$ is a weakly Cauchy sequence) for every permutation π of $\mathbb{N}$. It is well known that $\sum\limits_{i=1}^{\infty} x_i$ is wuC if and only if for each $x^* \in X^*$ $\sum\limits_{i=1}^{\infty} |x^*(x_i)| < \infty$, where X^* is the dual space of X.

Many studies have been made on the behaviour of a series of the form $\sum\limits_{i=1}^{\infty} a_i x_i$, where $(a_i)_i$ is a bounded sequence of real numbers. For instance, unconditionally convergent (resp. weakly unconditionally Cauchy) series can be characterized as the series $\sum\limits_{i=1}^{\infty} x_i$ such that $\sum\limits_{i=1}^{\infty} a_i x_i$ is convergent for every bounded sequence (resp. for every null sequence) $(a_i)_i$ (Cf. [2], [3] and [4]). The Banach space of bounded sequences (resp. null sequences) of real numbers, endowed with the sup norm, will be denoted, as usual, by ℓ_∞ (resp. c_0).

For any given series $\sigma = \sum_{i=1}^{\infty} x_i$ in X, let us consider the set $\mathscr{S} = \mathscr{S}(\sigma)$ (resp. $\mathscr{S}_w = \mathscr{S}_w(\sigma)$) of sequences $(a_i)_i \in l_\infty$ such that $\sum_{i=1}^{\infty} a_i x_i$ converges (resp. converges for the weak topology). The set $\mathscr{S}$ (resp. $\mathscr{S}_w$), endowed with the sup norm, will be called the *space of convergence* (resp. *weak convergence*) of the series σ. Clearly $\mathscr{S}$ and $\mathscr{S}_w$ are subspaces of ℓ_∞.

If X is a normed space and $\mathscr{S}$ is a subspace of ℓ_∞ such that $c_0 \subseteq \mathscr{S}$, we will denote

$$X(\mathscr{S}) = \left\{ \overline{x} = (x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} a_i x_i \text{ is convergent for every } (a_i)_{i \in \mathbb{N}} \in \mathscr{S} \right\}.$$

In [1] it is proved that $X(\mathscr{S})$ is a normed space with the norm

$$\|\overline{x}\|_{\mathscr{S}} = \sup \left\{ \left\| \sum_{i=1}^{\infty} a_i x_i \right\| : (a_i)_{i \in \mathbb{N}} \in B_{\mathscr{S}} \right\},$$

where $B_{\mathscr{S}}$ denotes the unit ball in $\mathscr{S}$, and that if X is complete then $X(\mathscr{S})$ is also complete. Some others properties of spaces $X(\mathscr{S})$ have been studied in [1], [6] and [7].

For a given series $\sigma = \sum_{i=1}^{\infty} x_i^*$ in X^*, the set of bounded sequences $(a_i)_i$ of real numbers such that $\sum_{i=1}^{\infty} a_i x_i^*$ is $*$-weakly convergent will be denoted by $\mathscr{S}_{*w}(\sigma)$.

It is well known (see [2], [3] and [5]) that if X is a Banach space then:

1. There exists a wuC series in X which is convergent but which is not unconditionally convergent if and only if X has a copy of c_0.
2. There exists in X a wuC and weakly convergent series which does not converge if and only if X has a copy of c_0.
3. There exists in X^* a $*$-weakly unconditionally Cauchy ($*$-wuC) series which is not unconditionally convergent if and only if X^* has a copy of ℓ_∞.

It is obvious that if X does not have a copy of c_0 then the following conditions are equivalent: 1) The series $\sigma = \sum_{i=1}^{\infty} x_i$ is wuC. 2) The series σ is uc. 3) $\mathscr{S}(\sigma) = \mathscr{S}_w(\sigma) = \ell_\infty$.

In this paper we characterize the completeness of X through the spaces $\mathscr{S}(\sigma)$ and $\mathscr{S}_w(\sigma)$, where σ is a wuC series in X. We also characterize the barrelledness of a normed space X through the spaces $\mathscr{S}_{*w}(\sigma)$, where σ is a $*$-wuC series in X^*.

Let us recall that if X is a normed space then $\sigma = \sum_{i=1}^{\infty} x_i$ is wuC if and only if

$$(2.1) \qquad E = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \ n \in \mathbb{N}, \ |\alpha_i| \leqslant 1, \ i \in \{1, \dots, n\} \right\}$$

is bounded.

Theorem 2.1. *Let X be a Banach space and let $\sigma = \sum_{i=1}^{\infty} x_i$ be a series in X. The space $\mathscr{S}(\sigma)$ is complete if and only if σ is* wuC.

Proof. Let us suppose that σ is wuC. Let E be the set defined by (2.1). Let us suppose that $\|x\| < M$ for every $x \in E$. Let $\{(a_i^{(k)})_i\}_k$ be a sequence in $\mathscr{S}(\sigma)$ that converges to $(a_i^{(0)})_i \in l_\infty$. For any given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $|a_i^{(k)} - a_i^{(0)}| < \frac{\varepsilon}{2M}$, for every $k \geqslant k_0$ and $i \in \mathbb{N}$. If $k > k_0$ then there exists $i_k = i_k(k, \varepsilon) \in \mathbb{N}$ such that $\left\| \sum_{i=q}^{p} a_i^{(k)} x_i \right\| < \frac{\varepsilon}{2}$, for $p > q \geqslant i_k$. Since $\frac{2M}{\varepsilon} \sum_{i=q}^{p} (a_i^{(k)} - a_i^{(0)}) x_i \in E$, we obtain that $\left\| \sum_{i=q}^{p} (a_i^{(k)} - a_i^{(0)}) x_i \right\| \leqslant \frac{\varepsilon}{2}$, for $k > k_0$, and that $\left\| \sum_{i=q}^{p} a_i^{(0)} x_i \right\| < \varepsilon$. This proves that $\mathscr{S}(\sigma)$ is complete.

It is obvious that if $\mathscr{S}(\sigma)$ is complete then σ is wuC. □

Theorem 2.2. *Let X be a normed space. The space X is complete if and only if for every weakly unconditionally Cauchy series $\sigma = \sum_{i=1}^{\infty} x_i$ in X the space $\mathscr{S}(\sigma)$ is complete.*

Proof. If X is not complete then there exists an absolutely convergent series $\sigma = \sum_{i=1}^{\infty} x_i$ in X which is not convergent and is such that $\|x_i\| < \frac{1}{i2^i}$ for every $i \in \mathbb{N}$. Let $\sigma' = \sum_{i=1}^{\infty} z_i$ be the series defined by $z_{2i-1} = i x_i$, $z_{2i} = -i x_i$, for $i \in \mathbb{N}$. It is clear that σ' is wuC.

Let $(a_i)_i \in c_0$ be the sequence defined by $a_{2i-1} = \frac{1}{2i}$, $a_{2i} = -\frac{1}{2i}$, for $i \in \mathbb{N}$. Since the series $\sum_{i=1}^{\infty} a_i z_i$ does not converge we have that $\mathscr{S}(\sigma')$ is not complete, although σ' is wuC. □

Our next result give us some information on the relationship between the spaces $\mathscr{S}(\sigma)$ and $\mathscr{S}(\sigma')$, when σ and σ' are two different series in X. The natural frameworks for this study are the spaces $X(\mathscr{S})$.

Theorem 2.3. *Let X be a Banach space. Let $\sigma = \sum_{i=1}^{\infty} x_i$ be a* wuC *series in X and let $\mathscr{S} = \mathscr{S}(\sigma)$.*

1. *If $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence in $X(\mathscr{S})$ that converges to σ then $\bigcap_{n\in\mathbb{N}} \mathscr{S}(\sigma_n) = \mathscr{S}(\sigma)$.*
2. *If $\sigma_0 \in X(\mathscr{S})$ then the set $\{\sigma' \in X(\mathscr{S}) : \mathscr{S}(\sigma') \neq \mathscr{S}(\sigma_0)\}$ is open in $X(\mathscr{S})$.*

Proof. 1. For $n \in \mathbb{N}$, we denote $\sigma_n = \sum_{i=1}^{\infty} x_i^n$. It is clear that $\mathscr{S}(\sigma_n) \supseteq \mathscr{S}(\sigma)$. Let us suppose that $(a_i)_{i\in\mathbb{N}} \in \bigcap_{n\in\mathbb{N}} \mathscr{S}(\sigma_n)$ and let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $\|\sigma - \sigma_n\|_{\mathscr{S}} < \frac{\varepsilon}{2}$. There exists $i_0 \in \mathbb{N}$ such that $\left|\sum_{i=q+1}^{p} \frac{a_i}{\|(a_i)_{i\in\mathbb{N}}\|} x_i^n\right| < \frac{\varepsilon}{2}$, for $p > q \geqslant i_0$. Then

$$\left|\sum_{i=q+1}^{p} \frac{a_i}{\|(a_i)_{i\in\mathbb{N}}\|} x_i\right| \leqslant \left|\sum_{i=q+1}^{p} \frac{a_i}{\|(a_i)_{i\in\mathbb{N}}\|}(x_i - x_i^n)\right| + \left|\sum_{i=q+1}^{p} \frac{a_i}{\|(a_i)_{i\in\mathbb{N}}\|} x_i^n\right|$$
$$\leqslant \|\sigma - \sigma_n\|_{\mathscr{S}} + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

2. It is clear that $X(\mathscr{S}(\sigma_0))$ is a closed subspace of $X(\mathscr{S})$. The first part of this theorem proves that $\{\sigma' \in X(\mathscr{S}) \colon \mathscr{S}(\sigma') = \mathscr{S}(\sigma_0)\}$ is closed in $X(\mathscr{S}(\sigma_0))$. □

Remark 2.4. Let X be a Banach space and let $\sigma = \sum_{i=1}^{\infty} x_i$ be a wuC series in X. It is clear that $\mathscr{S}(\sigma) = c_0$ if and only if $(x_i)_{i\in\mathbb{N}}$ does not have any null subsequence. In this case $(x_i)_{i\in\mathbb{N}}$ has a basic subsequence that is equivalent to the c_0-base.

Therefore, if σ is wuC then $\mathscr{S}(\sigma) = \mathscr{S}(\sigma')$, for every subseries σ' of σ, if and only if either $\mathscr{S}(\sigma) = \ell_\infty$ or $\mathscr{S}(\sigma) = c_0$.

Nevertheless, if σ_1 and σ_2 are two arbitrary wuC series in X, we do not know any conditions on σ_1 and σ_2 that let us affirm that $\mathscr{S}(\sigma_1) = \mathscr{S}(\sigma_2)$.

If X has a copy of c_0 and $\mathscr{F}$ is a closed subspace of ℓ_∞ such that $c_0 \subseteq \mathscr{F}$, we do not know if there exists a series σ in X such that $\mathscr{S}(\sigma) = \mathscr{F}$ (if X does not have a copy of c_0 and $\mathscr{F} \neq \ell_\infty$ the answer to this question is negative).

3. Completeness through weakly convergent series

It is well known that if a series converges in a Banach space X then this series is weakly convergent. Nevertheless, the converse is, in general, false. A weakly convergent series $\sum_{i=1}^{\infty} x_i$ is not necessarily a weakly unconditionally Cauchy series. We can ask, as in the second section, if the sets $\mathscr{S}_w(\sigma)$ may also be used to characterize the completeness of a normed space X.

Lemma 3.1. *If X is a Banach space and $\sigma = \sum\limits_{i=1}^{\infty} x_i$ is a series in X, then σ is* wuC *if and only if $c_0 \subseteq \mathscr{S}_w(\sigma)$.*

Proof. It is obvious that the condition is necessary. Let us suppose that $c_0 \subseteq \mathscr{S}_w(\sigma)$ and let $(a_i)_i$ be an arbitrary null sequence. The series $\sum\limits_{i=1}^{\infty} a_i x_i$ is weakly convergent. Let $(i_k)_k$ be an increasing sequence of positive integers and let us consider the set $M = \{i_k \colon k \in \mathbb{N}\}$. Let $(b_i)_i$ be the sequence defined by $b_i = a_i$ if $i \in M$, and $b_i = 0$ if $i \notin M$. The series $\sum\limits_{i=1}^{\infty} b_i x_i = \sum\limits_{k=1}^{\infty} a_{i_k} x_{i_k}$ is weakly convergent. Therefore $\sum\limits_{i=1}^{\infty} x_i$ is wuC. □

Theorem 3.2. *Let X be a Banach space and let $\sigma = \sum\limits_{i=1}^{\infty} x_i$ be a series in X. The space $\mathscr{S}_w(\sigma)$ is complete if and only if σ is* wuC.

Proof. Let us suppose that σ is wuC. We will prove that $\mathscr{S}_w(\sigma)$ is complete.

Let $\{(a_i^{(k)})_i\}_k$ be a sequence in $\mathscr{S}_w(\sigma)$ that converges to $(a_i^{(0)})_i \in \ell_\infty$ and let $(z_k)_k$ be a sequence in X such that $\sum\limits_{i=1}^{\infty} a_i^{(k)} x^*(x_i) = x^*(z_k)$, for every $x^* \in X^*$.

Let E be the set defined by (2.1). There exists $M > 0$ such that $\|x\| \leqslant M$, for every $x \in E$. For any given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\|(a_i^{(k)})_i - (a_i^{(0)})_i\| < \frac{\varepsilon}{3M}$, for $k \geqslant k_0$. Hence, $|a_i^{(k)} - a_i^{(0)}| < \frac{\varepsilon}{3M}$, for $i \in \mathbb{N}$ and $k \geqslant k_0$. This proves that

$$\Big\|\sum_{i=1}^{m} (a_i^{(k)} - a_i^{(0)}) x_i\Big\| \leqslant \frac{\varepsilon}{3}, \tag{3.1}$$

for $m \geqslant 1$, and we have $\Big\|\sum\limits_{i=1}^{m} (a_i^{(p)} - a_i^{(q)}) x_i\Big\| \leqslant \frac{2\varepsilon}{3}$, for $p > q \geqslant k_0$ and $m \geqslant 1$.

Therefore $\sum\limits_{i=1}^{m} (a_i^{(p)} - a_i^{(q)}) x^*(x_i) \leqslant \frac{2\varepsilon}{3}$, for every $x^* \in X^*$ such that $\|x^*\| = 1$ and $m \geqslant 1$. There exists $x_0^* \in X^*$ such that $\|x_0^*\| = 1$ and

$$\|z_p - z_q\| = \sum_{i=1}^{\infty} (a_i^{(p)} - a_i^{(q)}) x_0^*(x_i),$$

for every $p > q \geqslant k_0$. Since

$$\sum_{i=1}^{m} (a_i^{(p)} - a_i^{(q)}) x_0^*(x_i) \leqslant \Big\|\sum_{i=1}^{m} (a_i^{(p)} - a_i^{(q)}) x_i\Big\| \leqslant \frac{2\varepsilon}{3},$$

it is clear that $\|z_p - z_q\| < \varepsilon$. Hence there exists $z_0 \in X$ such that $\lim\limits_{k\to\infty} z_k = z_0$.

On the other hand, for any given $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that $\|z_k - z_0\| < \frac{\varepsilon}{3}$, for $k \geqslant k_1$. If $x^* \in X^*$ and $\|x^*\| = 1$ then, by (3.1), $\left|\sum_{i=1}^{m} (a_i^{(0)} - a_i^{(k)}) x^*(x_i)\right| \leqslant \frac{\varepsilon}{3}$, for $m \in \mathbb{N}$ and $k \geqslant k_0$. If $k \geqslant \max\{k_0, k_1\}$ then we have that

$$\left|\sum_{i=1}^{m} a_i^{(0)} x^*(x_i) - x^*(z_0)\right| < \frac{2\varepsilon}{3} + \left|\sum_{i=1}^{m} a_i^{(k)} x^*(x_i) - x^*(z_k)\right|,$$

for $m \in \mathbb{N}$. Since $\sum_{i=1}^{\infty} a_i^{(k)} x^*(x_i) = x^*(z_k)$, there exists $m_0 \in \mathbb{N}$ such that if $m \geqslant m_0$ then $\left|\sum_{i=1}^{m} a_i^{(k)} x^*(x_i) - x^*(z_k)\right| < \frac{\varepsilon}{3}$. Hence $\left|\sum_{i=1}^{m} a_i^{(0)} x^*(x_i) - x^*(z_0)\right| < \varepsilon$. This proves the theorem. □

Lemma 3.3. *Let X be a normed space. If $\sigma = \sum_{i=1}^{\infty} x_i$ is an unconditionally Cauchy series in X then $\mathscr{S}(\sigma) = \mathscr{S}_w(\sigma)$.*

Proof. If $(a_i)_i \in \mathscr{S}_w$ there exists $x \in X$ such that $x^*\left(\sum_{i=1}^{n} a_i x_i\right) \to x^*(x)$, for $x^* \in X^*$. Since σ is an unconditionally Cauchy series, there exists $x^{**} \in X^{**}$ such that $\sum_{i=1}^{\infty} a_i x_i = x^{**}$. Hence $x^*\left(\sum_{i=1}^{n} a_i x_i\right) \to x^{**}(x^*)$, for $x^* \in X^*$. This proves that $x^{**} = x$ and $(a_i)_i \in \mathscr{S}$. □

Theorem 3.4. *A normed space X is complete if and only if for every weakly unconditionally Cauchy series $\sigma = \sum_{i=1}^{\infty} x_i$ in X the space $\mathscr{S}_w(\sigma)$ is complete.*

Proof. Let us suppose that X is not complete. We can find, as in the proof of Theorem 2.2, an absolutely convergent series $\sigma' = \sum_{i=1}^{\infty} z_i$ that is wuC and such that $c_0 \nsubseteq \mathscr{S}(\sigma')$; therefore $\mathscr{S}(\sigma')$ is not complete. Since σ' is an unconditionally Cauchy series, by Lemma 3.3, we have that $\mathscr{S}(\sigma') = \mathscr{S}_w(\sigma')$. □

4. Barrelledness through weak-∗ convergent series in X^*

The study that we have made in sections 2 and 3 can be extended, in a natural way, to series in the dual space X^* of X.

Theorem 4.1. *Let X be a normed space and let $\zeta = \sum_{i=1}^{\infty} x_i^*$ be a series in X^*. Let us consider the following conditions:*

1) *ζ is* wuC.

2) $\mathscr{S}_{*w}(\zeta) = \ell_\infty$.

3) $\sum_{i=1}^{\infty} |x_i^*(x)| < +\infty$ *for every* $x \in X$.

We have that $1 \Rightarrow 2 \Rightarrow 3$.

These three conditions are equivalent for every series $\zeta = \sum_{i=1}^{\infty} x_i^*$ *in* X^* *if and only if* X *is a barrelled normed space.*

Proof. 1) $\Rightarrow$ 2). If $\sum_{i=1}^{\infty} x_i^*$ is wuC and $(a_i)_i \in \ell_\infty$ then $\sum_{i=1}^{\infty} a_i x_i^*$ is also wuC. Hence, $\left(\sum_{i=1}^{n} a_i x_i^*\right)_n$ is a bounded sequence in X^* that is a Cauchy sequence for the weak-$*$ topology on X^*. Hence we have that $\sum_{i=1}^{\infty} a_i x_i^*$ is weak-$*$ convergent.

2) $\Rightarrow$ 3). For every $x \in X$, let us consider the series $\sum_{i=1}^{\infty} x_i^*(x)$. For every $(a_i)_i \in c_0$, the series $\sum_{i=1}^{\infty} a_i x_i^*(x)$ is convergent. Hence $\sum_{i=1}^{\infty} |x_i^*(x)| < +\infty$.

Let us suppose that X is a barrelled normed space and that $\sum_{i=1}^{\infty} x_i^*$ is a series in X^* such that condition 3) is satisfied. Let us consider the set

$$E = \left\{ \sum_{i=1}^{m} \alpha_i x_i^* : m \in \mathbb{N},\ |\alpha_i| \leqslant 1,\ i \in \{1, \dots, m\} \right\}.$$

It is clear that E is pointwise bounded and, therefore, E is bounded for the norm topology of X^*. This proves that $\sum_{i=1}^{\infty} x_i^*$ is wuC.

If X is not barrelled then there exists a weak-$*$ bounded set $\mathscr{F} \subseteq X^*$ which is not bounded. For every $i \in \mathbb{N}$, there exists $y_i^* \in \mathscr{F}$ such that $\|y_i^*\| > 2^{2i}$. Let us write $x_i^* = \frac{1}{2^i} y_i^*$, for $i \in \mathbb{N}$. It is clear that $\sum_{i=1}^{\infty} |x_i^*(x)| < +\infty$ for every $x \in X$. Nevertheless, since $\|x_i^*\| > 2^i$ for every $i \in \mathbb{N}$, the series $\sum_{i=1}^{\infty} \frac{1}{2^i} x_i^*$ does not converge and $\sum_{i=1}^{\infty} x_i^*$ is not a weakly unconditionally Cauchy series. This completes the proof. □

Remark 4.2. If X is a barrelled normed space, $\mathscr{S}_{*w}$ is complete if and only if $\mathscr{S}_{*w} = \ell_\infty$.

Remark 4.3. The proof of Theorem 4.1 shows that X is a barrelled normed space if and only if in X^* the set of weak unconditionally Cauchy series coincides with the set of weak-$*$ unconditionally Cauchy series.

Let us observe that if X is a Banach space, then there exists a weakly unconditionally Cauchy series in X^* which is not unconditionally convergent if and only if X^* has a copy of ℓ_∞.

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