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Symmetries, periodic plane waves and blow-up of λ–ω systems☆

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Abstract

Nonclassical symmetries of one-dimensional reaction diffusion equations, of the λ–ω type, have been studied. The functional forms of λ and ω for which the system admits nonclassical symmetries have been determined and the corresponding reduced systems have been obtained. Some of these reduced systems admit symmetries which lead to further reductions. Among the several classes of exact solutions that have been obtained, asymptotically periodic plane waves appear as similarity solutions of λ–ω systems. We also have obtained a family of solutions that exhibit a blow-up process.

Keywords: λ–ω systems; Nonclassical symmetries; Periodic plane waves; Blow-up solutions

1. Introduction and preliminaries

Reaction diffusion equations whose kinetic ordinary differential equations (ODEs) possess a stable limit cycle have been widely studied in the last two decades. Let us consider a reaction diffusion system of the form

$$u_t = \Delta_x u + f(u, v, \gamma), \qquad v_t = \Delta_x v + g(u, v, \gamma) \tag{1.1}$$

such that the corresponding diffusionless system exhibits a Hopf bifurcation to a limit cycle at the bifurcation value γ_c. Kopell and Howard [11] proved that (1.1) can be transformed into a system of the form

$$u_t = \Delta_x u + \lambda(z)u - \omega(z)v, \qquad v_t = \Delta_x v + \omega(z)u + \lambda(z)v \tag{1.2}$$

near γ_c.

On the other hand, systems of type (1.2) are important, among reaction diffusion systems, by their symmetry properties. In fact, if a system (1.1) is invariant under rotations in (u, v) space, then it has the form (1.2) [1]. The λ–ω systems exhibit many different types of solutions: spiral and scroll waves [8–10], periodic plane waves [11,13], spatiotemporal chaos [12]. Sherratt [18–20] has studied the evolution of many of the above-mentioned types of solutions, from given initial conditions, by considering some specific cases of λ and ω.

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Although group analysis of differential equations has been applied in many fields of mathematical physics, only a few studies have been made for λ–ω systems. Steeb and Strampp [21] studied the particular case $\lambda = c - z^2$, $\omega = 2$. Suhubi and Chowdhury [22] obtained the most general symmetry group admitted by one-dimensional reaction diffusion equations with arbitrary source functions. Archilla et al. [1] obtained classical symmetries of λ–ω systems in two-dimensional media and characterized these systems, among reaction diffusion systems, by their symmetry properties.

In this work, we study the classical and nonclassical symmetries of (1.2) in the one-dimensional case and we reduce (1.2) to systems of ODEs. Besides, by means of additional symmetries of one of these reduced systems, we characterize some families of solutions of (1.2) presenting the following behaviours:

1. Source solutions that lead to asymptotically periodic plane waves propagating with opposed velocities depending on whether $x > x_0$ or $x < x_0$, x_0 being an arbitrary point.
2. Solutions exhibiting a blow-up process at finite time. This blow-up process propagates through a fixed point x_0 with non-constant velocity.
3. Bounded solutions which behave as the sources when t is large enough.

Van Saarloos and Hohenberg [23] have studied the generalized Ginzburg–Landau equation which contains, as a particular case, system (1.2). They are primarily interested in the study of uniformly translating solutions, that have the form $u + \mathrm{i}v = a(\xi)\,\mathrm{e}^{\mathrm{i}(\phi(\xi)-\omega t)}$, and several classes of solutions have been found. Solution (3.60) in [23], that has been obtained by means of a specific ansatz, is physically the same as one of the solutions that appear in our Section 4 and are referred to as source solutions.

The classical method for finding symmetry reductions of partial differential equations (PDEs) is the Lie group method of infinitesimal transformations. The fundamental basis of this technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. The machinery of Lie group theory provides a systematic method to search for these special group invariant solutions. For systems of PDEs with two independent variables, a single group reduction transforms the system of PDEs into a system of ODEs, which, in general, is easier to solve than the original system. Most of the required theory and description of the method can be found in [3,14,17]. To apply the classical method to system (1.2), one looks for an infinitesimal generator, of the form

$$V = \xi(x,t,u,v)\frac{\partial}{\partial x} + \eta(x,t,u,v)\frac{\partial}{\partial t} + \psi_1(x,t,u,v)\frac{\partial}{\partial u} + \psi_2(x,t,u,v)\frac{\partial}{\partial v}, \tag{1.3}$$

that leaves invariant system (1.2). This yields an over-determined linear system for the infinitesimals ξ, η, ψ_1 and ψ_2. Classical symmetry reductions have been derived for the $(2+1)$-dimensional λ–ω system in [1], and for the $(1+1)$-dimensional case in [22], via an isovector approach.

In the last decades several generalizations of the classical Lie group method for symmetry reductions have been formulated. Bluman and Cole [2] developed the nonclassical method to study the symmetry reductions of the heat equation. Clarkson and Mansfield [5] presented an algorithm for calculating the determining equations associated with the nonclassical method. This method has been used, with much success, to generate many new symmetry reductions and exact solutions for several physically significant PDEs. These solutions are not obtainable using the classical Lie method [4,6,7].

In order to apply the nonclassical method to system (1.2), we require only the subset of solutions of system (1.2) and the surface condition

$$\xi u_x + \eta u_t - \psi_1 = 0, \qquad \xi v_x + \eta v_t - \psi_2 = 0 \tag{1.4}$$

to be invariant under the transformation with infinitesimal generator (1.3). These methods were generalized by Olver and Rosenau [15,16] to include "weak symmetries", "side conditions" and "differential constraints", although their methods are too general to be practical.

2. Determination of the λ–ω systems which admit nontrivial symmetries

We impose that the transformation group generated by V, as defined in (1.3), leaves invariant (1.2) when (1.4) is satisfied. We study the problem separately, depending on whether $\eta \neq 0$ or $\eta = 0$.

Case 1. If $\eta \neq 0$, we may set $\eta(x, t, u, v) = 1$ without loss of generality. The nonclassical method applied to (1.2) gives rise to 15 determining equations for the infinitesimals. From these equations, it is easily found that

$$\begin{aligned}
&\xi(x, t, u, v) = \xi_1(x, t),\\
&\psi_1(x, t, u, v) = \psi_{11}(x, t)u + \psi_{12}(t)v + \psi_{13}(x, t),\\
&\psi_2(x, t, u, v) = \psi_{21}(t)u + \psi_{22}(x, t)v + \psi_{23}(x, t).
\end{aligned} \tag{2.1}$$

Besides, if we exclude the case where λ and ω are constant functions, we have that $\psi_{13} = \psi_{23} = 0$, $\psi_{21} = -\psi_{12}$ and $\psi_{22} = \psi_{11}$. By substituting (2.1) into the determining equations, it is found that ξ_1, ψ_{11} and ψ_{12} must satisfy

$$-2\frac{\partial \psi_{11}}{\partial x} + \frac{\partial^2 \xi_1}{\partial x^2} - 2\xi_1 \frac{\partial \xi_1}{\partial x} - \frac{\partial \xi_1}{\partial t} = 0, \tag{2.2}$$

$$-\psi_{11} z \frac{\mathrm{d}\lambda}{\mathrm{d}z} - \frac{\partial^2 \psi_{11}}{\partial x^2} + \frac{\partial \psi_{11}}{\partial t} + 2\psi_{11}\frac{\partial \xi_1}{\partial x} - 2\lambda \frac{\partial \xi_1}{\partial x} = 0, \tag{2.3}$$

$$\psi_{11} z \frac{\mathrm{d}\omega}{\mathrm{d}z} + \frac{\mathrm{d}\psi_{12}}{\mathrm{d}t} + 2\psi_{12}\frac{\partial \xi_1}{\partial x} + 2\omega \frac{\partial \xi_1}{\partial x} = 0. \tag{2.4}$$

From these equations we deduce:

Case 1.1. For λ and ω arbitrary functions, the only symmetries that are admitted by (1.2) are

$$\xi = k_1, \qquad \eta = 1, \qquad \psi_1 = k_2 v, \qquad \psi_2 = -k_2 u, \tag{2.5}$$

where k_1 and k_2 are arbitrary constants. It can be checked that symmetry (2.5) is a classical symmetry. This symmetry just expresses the invariance of (1.2) with respect to translations in the independent variables, and rotations in the phase space (u, v).

Case 1.2. For

$$\lambda(z) = a \log z + b, \qquad \omega(z) = c \log z + d, \tag{2.6}$$

where a, b, c and d are arbitrary constants, two subcases can be considered for which additional symmetries are found:

Case 1.2.1. If $a \neq 0$,

$$\xi = k_1, \qquad \psi_1 = k_2\, \mathrm{e}^{at} u + \left(\frac{ck_2}{a}(1 - \mathrm{e}^{at}) + k_3\right) v, \qquad \psi_2 = -\left(\frac{ck_2}{a}(1 - \mathrm{e}^{at}) + k_3\right) u + k_2\, \mathrm{e}^{at} v. \tag{2.7}$$

Case 1.2.2. If $a = 0$,

$$\xi = k_1, \qquad \psi_1 = k_2 u + (-ck_2 t + k_3)v, \qquad \psi_2 = -(-ck_2 t + k_3)u + k_2 v. \tag{2.8}$$

We remark that (2.7) contains (2.8) (if we take limit as $a \to 0$ in (2.7) we get (2.8)). We also point out that symmetry (2.7) is a classical symmetry.

Case 1.3. For

$$\lambda(z)=az^n+b,\qquad \omega(z)=cz^n+d, \tag{2.9}$$

where a, b, c, d and n are arbitrary constants, we obtain that

$$\psi_{11}=-\frac{2}{n}\xi_{1x},$$

and that ξ_{1x}, ψ_{12} are related by the following conditions:

$$\psi_{12t}+2\xi_{1x}\psi_{12}=0,\quad \xi_{1xxx}-2(\xi_{1x})^2-nb\xi_{1x}-\xi_{1xt}=0,\quad n\xi_{1xx}+4\xi_{1xx}-2n\xi_{1x}\xi_1-n\xi_{1t}=0. \tag{2.10}$$

We can consider two subcases:

Case 1.3.1. If $\psi_{12}\not\equiv 0$ it is easy to see that (2.10) is a compatible system if and only if $b=0$. Then

$$\xi=-\frac{x-x_0}{2(T-t)},\qquad \psi_1=\frac{u}{n(T-t)}-\left(\frac{k_1}{T-t}+d\right)v,\qquad \psi_2=\left(\frac{k_1}{T-t}+d\right)u+\frac{v}{n(T-t)}. \tag{2.11}$$

It can be checked that symmetries (2.11) correspond to classical symmetries.

Case 1.3.2. If $\psi_{12}\equiv 0$, the first equation in (2.10) becomes trivial, and our problem reduces to determining ξ_1. Due to the difficulty to solve this system, we look for solutions of (2.10) in which ξ_1 does not depend on t. Then, $\xi_1(x)$ verifies

$$\xi_1'''-2(\xi_1')^2-bn\xi_1'=0, \tag{2.12}$$

$$(n+4)\xi_1''-2n\xi_1\xi_1'=0. \tag{2.13}$$

This is possible only if $n=2$ and $k_1=-9b$. We can consider two subcases:

Case 1.3.2.1. If $b\neq 0$, i.e. $\lambda(z)=az^2+b$ and $\omega(z)=cz^2+d$, then the infinitesimals for our symmetry group are

$$\xi=-3\sqrt{\frac{b}{2}}\coth\left[\sqrt{\frac{b}{2}}(x+k_2)\right],\qquad \psi_1=-\frac{3}{2}b\,\mathrm{cosech}^2\left[\sqrt{\frac{b}{2}}(x+k_2)\right]u-dv,$$

$$\psi_2=du+\frac{3}{2}b\,\mathrm{cosech}^2\left[\sqrt{\frac{b}{2}}(x+k_2)\right]v. \tag{2.14}$$

This is a nonclassical symmetry which does not correspond to a classical symmetry.

Case 1.3.2.2. If $b=0$, then the infinitesimals are

$$\xi=-\frac{3}{x+k_2},\qquad \psi_1=-\frac{3u}{(x+k_2)^2}-dv,\qquad \psi_2=du-\frac{3v}{(x+k_2)^2}.$$

This result can be obtained from (2.14) by means of taking the limit as $b\to 0$.

Case 2. If $\eta=0$, then we can take, without loss of generality, $\xi=1$. In this case, it is found that:

Case 2.1. For arbitrary λ and ω, the unique admitted group is given by

$$\xi = 1, \qquad \eta = 0, \qquad \psi_1 = k_1 v, \qquad \psi_2 = -k_1 u,$$

which means the invariance of (1.2) with respect to translations in x and rotations in (u, v).

Case 2.2. For $\lambda(z) = a \log z + b,\ \omega(z) = c \log z + d$, we have

$$\psi_1(x, t, u, v) = \psi_{11}(t)u + \psi_{12}(t)v, \qquad \psi_2(x, t, u, v) = -\psi_{12}(t)u + \psi_{11}(t)v, \tag{2.15}$$

where $\psi_{11}(t), \psi_{12}(t)$ must satisfy

$$\dot{\psi}_{11} - a\psi_{11} = 0, \qquad \dot{\psi}_{12} + c\psi_{11} = 0.$$

Therefore,

$$\psi_1 = k_1\,\mathrm{e}^{at}u - \left[\frac{ck_1}{a}(\mathrm{e}^{at} - 1) + k_2\right]v, \qquad \psi_2 = \left[\frac{ck_1}{a}(\mathrm{e}^{at} - 1) + k_2\right]u + k_1\,\mathrm{e}^{at}v. \tag{2.16}$$

When $a = 0$ infinitesimals ψ_1 and ψ_2 can be obtained by taking the limit as $a \to 0$ in (2.16).

3. Symmetry reductions

In this section, we investigate the reduced system of ODEs that corresponds to each one of the symmetries found in Section 2. The reductions that can be derived through the groups of translations and rotations can be obtained as in [1].

The only λ–ω systems which admit nontrivial symmetries are of the form

$$\lambda(z) = a\delta(z) + b, \qquad \omega(z) = c\delta(z) + d,$$

where $\delta(z)$ stands for $\delta(z) = z^n$ or $\delta(z) = \log z$. These two cases will be studied separately.

Before doing this, and for further computations, it is convenient to change from variables (u, v) to polar variables (z, ϕ), where

$$u = z\cos\phi, \qquad v = z\sin\phi.$$

In terms of variables (z, ϕ), (1.2) can be written as

$$z_t = z_{xx} - z\phi_x^2 + z\lambda(z), \qquad \phi_t = \phi_{xx} + \frac{2}{z}z_x\phi_x + \omega(z). \tag{3.1}$$

We may set $d = 0$ without loss of generality. In fact, defining $\varphi = \phi - dt$, then (3.1) is transformed to

$$z_t = z_{xx} - z\varphi_x^2 + az\delta(z) + bz, \qquad \varphi_t = \varphi_{xx} + \frac{2}{z}z_x\varphi_x + c\delta(z). \tag{3.2}$$

The symmetries we have found in Section 2, that are written in terms of variables (u, v), must be then written in polar variables. The similarity variables can be obtained solving the invariant surface condition or, equivalently, solving the characteristic system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \xi, \qquad \frac{\mathrm{d}z^2}{\mathrm{d}t} = 2u\psi_1 + 2v\psi_2, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\frac{v}{u^2 + v^2}\psi_1 + \frac{u}{u^2 + v^2}\psi_2. \tag{3.3}$$

Reduction 1.2. $\eta=1,\ \delta(z)=\log z$. Let us first suppose that $a\neq 0$. From (2.7) and (3.3), we obtain that similarity variables $w,\ \chi$ and k are given by

$$w=x-k_1t,\qquad z=\exp\left[\frac{k_2}{a}(\mathrm{e}^{at}-1)\right]\chi(w),\qquad \varphi=\frac{ck_2}{a^2}(\mathrm{e}^{at}-at-1)-k_3t+k(w),$$

and the ODE reduced system takes the form

$$k_2\chi-k_1\chi'+\chi\zeta^2-\chi''-a\chi\log\chi-b\chi=0,\qquad k_3\chi+k_1\chi\zeta+\chi\zeta'-2\chi'\zeta+c\chi\log\chi=0,$$

where $\zeta=k'$ and $'$ stands for $\mathrm{d}/\mathrm{d}w$. If $a=0$ similarity variables and solutions can be obtained from the above expressions by taking limit as $a\to 0$.

Reduction 1.3.1. $\eta=1,\ \delta(z)=z^n,\ b=0$. From (2.11) and (3.3), we can deduce that the classical symmetry reduction is given by

$$w=\frac{x-x_0}{\sqrt{T-t}},\qquad z=(T-t)^{-1/n}\chi(w),\qquad \varphi=-k_1\log(T-t)+k(w), \tag{3.4}$$

and the reduced system is

$$\chi''-\frac{1}{2}w\chi'-\frac{1}{n}\chi-\chi\zeta^2+a\chi^{n+1}=0,\qquad \zeta'-\frac{1}{2}w\zeta+\frac{2}{\chi}\chi'\zeta+c\chi^n-k_1=0, \tag{3.5}$$

where ζ and $'$ are defined as above.

Reduction 1.3.2. $\eta=1,\ \delta(z)=z^2$. From (2.14) and (3.3), we obtain the nonclassical symmetry reduction

$$w=t+\frac{2}{3b}\log\left[\cosh\left[\sqrt{\frac{b}{2}}(x+k_2)\right]\right],\qquad z=\sqrt{\frac{2}{b}}\tanh\left[\sqrt{\frac{b}{2}}(x+k_2)\right]\chi(w),\qquad \varphi=k(w), \tag{3.6}$$

and the reduced system is

$$9b^2\chi+18a\chi^3-9b\chi'-2\chi\zeta^2+2\chi''=0,\qquad -9b\chi\zeta+18c\chi^3+2\chi\zeta'+4\chi'\zeta=0, \tag{3.7}$$

where ζ and $'$ are defined as above.

We point out that (3.6) and (3.7) stand for $b=0$, by means of taking the limit as $b\to 0$. In fact, when $b=0$, the nonclassical symmetry reduction is given by

$$w=t+\tfrac{1}{6}(x+k_2)^2,\qquad z=(x+k_2)\chi(w),\qquad \varphi=k(w),$$

and the reduced system is

$$\chi\zeta^2-\chi''-9a\chi^3=0,\qquad \chi\zeta'+2\chi'\zeta+9c\chi^3=0.$$

This system coincides with (3.7) if $b=0$.

Reduction 2.1. $\eta=0,\ \delta(z)=\log z$. From (2.16) and (3.3), it follows that the symmetry reduction is given by

$$z=\exp(k_1\,\mathrm{e}^{at}x)\chi(t),\qquad \varphi=\left[\frac{ck_1}{a}(\mathrm{e}^{at}-1)+k_2\right]x+k(t).$$

The reduced system is a nonhomogeneous linear system for $\log\chi$ and k whose solutions are

$$\log\chi = -\frac{c^2k_1^2}{a^3}(\mathrm{e}^{2at} - 2at\,\mathrm{e}^{at} - 1) - \frac{2ck_1k_2}{a^2}(at\,\mathrm{e}^{at} - \mathrm{e}^{at} + 1) + \frac{k_1^2}{a}\mathrm{e}^{at}(\mathrm{e}^{at} - 1) + \frac{b - k_2^2}{a}(\mathrm{e}^{at} - 1) + k_3\,\mathrm{e}^{at},$$

$$k = \frac{c^3k_1^2}{2a^4}(\mathrm{e}^{2at} - 4at\,\mathrm{e}^{at} + 4\,\mathrm{e}^{at} - 2at - 5) - \frac{c^2k_1k_2^2}{a^3}(2at\,\mathrm{e}^{at} - 4\,\mathrm{e}^{at} + 2at + 4)$$
$$+\frac{3ck_1^2}{2a^2}(\mathrm{e}^{2at} - 2\,\mathrm{e}^{at} + 1) + \frac{bc - ck_2^2}{a^2}(\mathrm{e}^{at} - 1 - at) + \frac{2k_1k_2 + ck_3}{a}(\mathrm{e}^{at} - 1) + k_4.$$

We finally remark that if $a = 0$ similarity variables and solutions can be obtained from the above expressions by taking limit as $a \to 0$.

4. The case $\delta(z) = z^2$: new reductions and solutions

As it was pointed out in Section 1, the λ–ω systems that have drawn the most interest are those that, in absence of diffusion, possess a stable limit cycle. In order to find the functional forms of λ and ω corresponding to these systems we write the diffusionless dynamical system associated to (1.2) in terms of the polar coordinates in the phase space (z, ϕ):

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z\lambda(z), \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega(z).$$

It is clear that this system exhibits a stable limit cycle if and only if there exists $z_0 > 0$ such that $\lambda(z_0) = 0$ and $(\mathrm{d}/\mathrm{d}z)(z\lambda(z))|_{z=z_0} < 0$. One of the simplest examples verifying these conditions is $\lambda(z) = az^2 + b$ with $a < 0$, $b > 0$. On the other hand, in the previous sections we have found that the λ–ω system corresponding to $\lambda(z) = az^2 + b$, $\omega(z) = cz^2 + d$ admits a nonclassical symmetry and we have reduced the system by means of this symmetry to the ordinary system (3.7). Since this system presents the behaviour we are interested in, we focus our attention on this reduction, taking $a < 0$ and $b > 0$. We also observe that this system is a generalization of the one studied by Steeb and Strampp ([21], $\lambda = c - z^2$, $\omega = 2$). In this section, we obtain a family of solutions of the corresponding λ–ω system by means of classical symmetries of the reduced system (3.7).

System (3.7) has two fixed points: $(-\bar\chi, \bar\zeta)$ and $(\bar\chi, \bar\zeta)$, where

$$\bar\chi = -\sqrt{\frac{3}{2}}\frac{b}{2|c|}\sqrt{3a + \sqrt{9a^2 + 8c^2}}, \qquad \bar\zeta = \frac{3b}{4c}(3a + \sqrt{9a^2 + 8c^2}), \tag{4.1}$$

if $c \neq 0$ and

$$\bar\chi = -\frac{b}{\sqrt{-2a}}, \qquad \bar\zeta = 0, \tag{4.2}$$

if $c = 0$. By standard methods it can be checked that two of the three eigenvalues of the linear associated system have positive real part, consequently, these solutions are not linearly stable. The corresponding solutions of the λ–ω system will be considered later.

4.1. Reduction to a first-order autonomous system

In order to get classical symmetries for system (3.7), we look for an infinitesimal generator of the form

$$v = \xi_1(w, \chi, \zeta)\frac{\partial}{\partial w} + \phi_1(w, \chi, \zeta)\frac{\partial}{\partial \chi} + \phi_2(w, \chi, \zeta)\frac{\partial}{\partial \zeta},$$

that leaves invariant (3.7). The infinitesimals ξ_1, ϕ_1 and ϕ_2 must satisfy a linear system of six determining equations. Since looking for the general solution of this system is too difficult, we restrict ourselves to solutions of the form

$$\begin{aligned}
&\xi_1(w,\chi,\zeta)=\xi_1(w),\\
&\phi_1(w,\chi,\zeta)=\phi_{11}(w)\chi+\phi_{12}(w)\zeta+\phi_{13}(w),\\
&\phi_2(w,\chi,\zeta)=\phi_{21}(w)\chi+\phi_{22}(w)\zeta+\phi_{23}(w).
\end{aligned}$$

Introducing these last expressions in the system of determining equations, it is easy to find that

$$\xi_1(w,\chi,\zeta)=k_1+k_2\,\mathrm{e}^{-(3b/2)w},\qquad \phi_1(w,\chi,\zeta)=\tfrac{3}{2}bk_2\,\mathrm{e}^{-(3b/2)w}\chi,\qquad \phi_2(w,\chi,\zeta)=\tfrac{3}{2}bk_2\,\mathrm{e}^{-(3b/2)w}\zeta,$$

where k_1 and k_2 are arbitrary constants. Hence, two infinitesimal generators are

$$v_1=\frac{\partial}{\partial w},\qquad v_2=\mathrm{e}^{-(3b/2)w}\left[\frac{\partial}{\partial w}+\frac{3b}{2}\chi\frac{\partial}{\partial \chi}+\frac{3b}{2}\zeta\frac{\partial}{\partial \zeta}\right].$$

It can be checked that $[v_1,v_2]=-\frac{3}{2}bv_2$. If we reduce (3.7) by using v_2, we get that the similarity variables are

$$w_1=\chi\,\mathrm{e}^{-(3b/2)w},\qquad \chi_1=\frac{\chi'}{\chi^2}-\frac{3b}{2\chi},\qquad \zeta_1=\zeta\,\mathrm{e}^{-(3b/2)w}. \tag{4.3}$$

In terms of these variables, (3.7) is reduced to the first-order system

$$w_1^3\chi_1\frac{\mathrm{d}\chi_1}{\mathrm{d}w_1}+2w_1^2\chi_1^2-\zeta_1^2+9aw_1^2=0,\qquad w_1\chi_1\frac{\mathrm{d}\zeta_1}{\mathrm{d}w_1}+2\chi_1\zeta_1+9cw_1=0. \tag{4.4}$$

We can use symmetry v_1 to transform (4.4) into an autonomous system. By writing v_1 in terms of variables (4.3), we obtain

$$\tilde{v}_1=-\frac{3b}{2}w_1\frac{\partial}{\partial w_1}+\frac{3b}{2}\zeta_1\frac{\partial}{\partial \zeta_1}.$$

It can be checked that $\tilde{v}_1$ is a classical symmetry of (4.4). The corresponding canonical coordinates are given by

$$w_2=-\frac{2}{3b}\log w_1,\qquad \chi_2=\chi_1,\qquad \zeta_2=\frac{\zeta_1}{w_1}. \tag{4.5}$$

In terms of these variables, (4.4) takes the form

$$\frac{\mathrm{d}\chi_2}{\mathrm{d}w_2}=\frac{3b}{2}\frac{2\chi_2^2-\zeta_2^2+9a}{\chi_2},\qquad \frac{\mathrm{d}\zeta_2}{\mathrm{d}w_2}=\frac{9b}{2}\frac{\chi_2\zeta_2+3c}{\chi_2}. \tag{4.6}$$

4.2. Sources and blow-up solutions

Using the previous reductions, we are going to obtain some families of solutions of the λ–ω system we are considering.

System (4.6) has two stationary solutions of the form

$$(\chi_2,\zeta_2)=\pm(\bar{\chi}_2,\bar{\zeta}_2)$$

with

$$\bar{\chi}_2=\frac{1}{2}\sqrt{3\sqrt{9a^2+8c^2}-9a},\qquad \bar{\zeta}_2=-\mathrm{sgn}(c)\sqrt{\frac{3\sqrt{9a^2+8c^2}+9a}{2}},$$

where the sign function is defined as usual: $\mathrm{sgn}(c) = 1$ if $c > 0$, $\mathrm{sgn}(c) = 0$ if $c = 0$ and $\mathrm{sgn}(c) = -1$ if $c < 0$. Before proceeding further, we remark that solutions corresponding to both signs $\pm$ vary only in the signs of u and v. Therefore, it is not worth getting on our discussion with the double sign. From now on we take the solution $(\chi_2, \zeta_2) = (\bar{\chi}_2, \bar{\zeta}_2)$. This solution is transformed by (4.5) into a solution of (4.4):

$$\chi_1 = \bar{\chi}_2, \qquad \zeta_1 = \bar{\zeta}_2 w_1.$$

In order to get the corresponding solution of (3.7), we need to solve the second equation in (4.3) for χ, i.e.

$$\frac{\chi'}{\chi^2} - \frac{3b}{2\chi} = \bar{\chi}_2 \tag{4.7}$$

with an initial condition that we take as

$$\chi(0) = \chi_0 \quad \text{with} \ \ \chi_0 \neq 0. \tag{4.8}$$

By standard methods we obtain that the solution of (4.7) and (4.8) is given by

$$\chi(w) = \left[\left(\frac{1}{\chi_0} - \frac{1}{\bar{\chi}} \right) \mathrm{e}^{-(3b/2)w} + \frac{1}{\bar{\chi}} \right]^{-1}, \tag{4.9}$$

which exhibits different behaviours depending on χ_0. It is easily found that

1. If $\chi_0 = \bar{\chi}$, (4.9) is the constant function
$$\chi(w) = \bar{\chi}$$
corresponding to (4.1) or (4.2).
2. If $\chi_0 > 0$, (4.9) becomes infinity at
$$w = \bar{w} = \frac{2}{3b} \log\left(1 - \frac{\bar{\chi}}{\chi_0} \right) > 0.$$
3. If $\bar{\chi} < \chi_0 < 0$, (4.9) is a bounded solution satisfying
$$\lim_{w\to\infty} \chi(w) = \bar{\chi}. \tag{4.10}$$
4. If $\chi_0 < \bar{\chi}$, (4.9) remains bounded in $[0, \infty)$ and verifies (4.10). On the other hand, by (4.3), we also have
$$\zeta = \bar{\zeta}_2 \left[\left(\frac{1}{\chi_0} - \frac{1}{\bar{\chi}} \right) \mathrm{e}^{-(3b/2)w} + \frac{1}{\bar{\chi}} \right]^{-1},$$
and since $\zeta = k'$
$$k = \frac{2}{3b} \bar{\zeta} \log \left| \left(\frac{1}{\chi_0} - \frac{1}{\bar{\chi}} \right) \mathrm{e}^{-(3b/2)w} + \frac{1}{\bar{\chi}} \right| + \bar{\zeta} w + C, \tag{4.11}$$
where C is an arbitrary constant.

Now, we will analyse the associated solutions of the λ–ω system we are studying. Let us observe that condition (4.8) means that we have fixed the solution of (1.2) on a curve:

$$z = \sqrt{\frac{2}{b}} \tanh\left[\sqrt{\frac{b}{2}}(x - x_0) \right] \chi_0 \quad \text{when} \ \ t + \frac{2}{3b} \log\left[\cosh\left[\sqrt{\frac{b}{2}}(x - x_0) \right] \right] = 0,$$

where we have taken $k_2 = -x_0$ in (3.6). Hence, the solutions we are studying are solutions of (1.2) with boundary conditions.

We will consider four cases, depending on the values of χ_0:

1. $\chi_0 = \bar{\chi}$, in this case from (4.9) and (4.11), we have

$$\chi(w) = \bar{\chi}, \qquad k(w) = \bar{\zeta} w + D \tag{4.12}$$

with $D = C - (2\bar{\zeta}/3b)\log|\bar{\chi}|$. Coming back to (3.6) with $k_2 = -x_0$ and taking into account that $\phi = \varphi + dt$, from (4.12) we get

$$z = \sqrt{\frac{2}{b}} \tanh\left[\sqrt{\frac{b}{2}}(x - x_0)\right]\bar{\chi}, \tag{4.13}$$

$$\phi = \frac{2}{3b}\bar{\zeta}\log\left[\cosh\left[\sqrt{\frac{b}{2}}(x - x_0)\right]\right] + (\bar{\zeta} + d)t + D. \tag{4.14}$$

System (3.2), with $\delta(z) = z^2$, can be written in complex form as

$$A_t = \Delta_x A + (a + c\mathrm{i})|A|^2 A + bA, \tag{4.15}$$

where $A = u + \mathrm{i}v$. This is a particular case of the complex Ginzburg–Landau equation. For this equation, van Saarloos and Hohenberg [23] have studied several classes of uniformly translating solutions, i.e. solutions of the form

$$A = \alpha(\xi)\,\mathrm{e}^{\mathrm{i}(\psi(\xi) - \omega t)}, \tag{4.16}$$

where $\xi = x - Vt$ and V, ω are arbitrary constants. By making the specific ansatz

$$A = a_2\left(\frac{1 + z\,\mathrm{e}^{-2k_0\xi}}{1 + \mathrm{e}^{-2k_0\xi}}\right)\exp\left(\mathrm{i}\left(\frac{q_+}{2}\xi + \frac{q_-}{2k_0}\log\cosh k_0\xi - \omega t + E\right)\right), \tag{4.17}$$

where a_2, k_0, q_+, q_- are real and z is complex, they have proved that, for some values of the constants, (4.17) is a solution of (4.15). Our solution (4.13) and (4.14), with $x_0 = 0$, is a particular case of (4.17): it corresponds to $a_2 = \sqrt{2/b}\,\bar{\chi},\ z = -1,\ V = 0,\ q_+ = 0,\ k_0 = \sqrt{b/2},\ q_- = \sqrt{8/9b}\,\bar{\zeta}$ and $w = -\bar{\zeta}$.

In order to classify our solution (4.13) and (4.14), among the different classes of coherent structures studied in [23], we must put it in the corresponding settings. If we define the variables $q(\xi) = \mathrm{d}\phi/\mathrm{d}\xi$ and $k(\xi) = (1/\alpha)\,\mathrm{d}\alpha/\mathrm{d}\xi$, insertion of (4.16) into (4.15) leads to the system

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\xi} = k\alpha, \qquad \frac{\mathrm{d}k}{\mathrm{d}\xi} = -k^2 - Vk + q^2 - a\alpha^2 - b, \qquad \frac{\mathrm{d}q}{\mathrm{d}\xi} = -2qk - Vq - \omega - c\alpha^2. \tag{4.18}$$

The classification of coherent structures: pulses, fronts and domain boundaries (sources and sinks) [23] can be established in terms of the fixed points of system (4.18). Since $V = 0$, it is easy to check that there are four fixed points of (4.18) whose first coordinate is not null. These are denoted by N_1, N_2, N_3 and N_4 and their coordinates appear in the second column of Table 1. These nonlinear fixed points $N_i = (\alpha_i, q_i, k_i)$, correspond to travelling wave solutions of (4.15) of the form $A(x,t) = \alpha_i\,\mathrm{e}^{-\omega t\mathrm{i} + q_i x t}$. The linear stability character of N_i, for $1 \le i \le 4$, can be obtained by linearization of (4.18) about N_i and by determining the sign of the real parts of the roots of the corresponding secular equations $\lambda^3 + a_{i1}\lambda^2 + a_{i2}\lambda + a_{i3} = 0$; these signs appear in Table 1.

We will now analyse the relationship between these fixed points and our solution (4.13) and (4.14). For this solution

$$\alpha(x) = \sqrt{\frac{2}{b}}\tanh\left(\sqrt{\frac{b}{2}}x\right)\bar{\chi}, \qquad q(x) = \frac{\sqrt{2}}{3\sqrt{b}}\bar{\zeta}\tanh\left(\sqrt{\frac{b}{2}}x\right), \tag{4.19}$$

Table 1

Fixed points	Coordinates	Real parts of roots	
		$c > 0$	$c < 0$
N_1	$\left(+\sqrt{-\frac{\omega}{c}}, +\sqrt{(b - \frac{a}{c}\omega)}, 0\right)$	$(+, +, -)$	$(+, -, -)$
N_2	$\left(+\sqrt{-\frac{\omega}{c}}, -\sqrt{(b - \frac{a}{c}\omega)}, 0\right)$	$(+, -, -)$	$(+, +, -)$
N_3	$\left(-\sqrt{-\frac{\omega}{c}}, +\sqrt{(b - \frac{a}{c}\omega)}, 0\right)$	$(+, +, -)$	$(+, -, -)$
N_4	$\left(-\sqrt{-\frac{\omega}{c}}, -\sqrt{(b - \frac{a}{c}\omega)}, 0\right)$	$(+, -, -)$	$(+, +, -)$

and $k(x) = (1/\alpha(x))\alpha'(x)$. Since $\sqrt{2/b}\,\bar{\chi} = \sqrt{-\omega/c}$ and $\frac{1}{3}\sqrt{2/b}\,\bar{\zeta} = \sqrt{b - (a/c)\omega}$, it follows that

$$\begin{aligned}
&\lim_{x\to-\infty}(a(x), q(x), k(x)) = N_4, && \lim_{x\to+\infty}(a(x), q(x), k(x)) = N_1, && \bar{\chi} > 0,\ c > 0,\\
&\lim_{x\to-\infty}(a(x), q(x), k(x)) = N_2, && \lim_{x\to+\infty}(a(x), q(x), k(x)) = N_3, && \bar{\chi} < 0,\ c > 0,\\
&\lim_{x\to-\infty}(a(x), q(x), k(x)) = N_3, && \lim_{x\to+\infty}(a(x), q(x), k(x)) = N_2, && \bar{\chi} > 0,\ c < 0,\\
&\lim_{x\to-\infty}(a(x), q(x), k(x)) = N_1, && \lim_{x\to+\infty}(a(x), q(x), k(x)) = N_4, && \bar{\chi} < 0,\ c < 0.
\end{aligned}$$

Therefore, our solutions (4.13) and (4.14) correspond to heteroclinic trajectories of (4.18) that connect two nonlinear fixed points and, hence, they correspond to domain boundaries, either if $c > 0$ or $c < 0$. The group velocities in the corresponding frames moving with velocity $V = 0$ are given, according with formula (2.51) in [23], by $\tilde{v}_{\mathrm{g}} = 4q_i c\alpha_i^2$. In every case, these domain boundaries have outgoing waves: the fixed points in the left column ($x \to -\infty$) have negative v_{g} and those in the right column ($x \to +\infty$) have positive v_{g} for the corresponding signs of $\bar{\chi}$ and c. Therefore [23], these heteroclinic trajectories are sources (target). [1]

Now, we will analyse some asymptotic aspects of solution (4.13) and (4.14). Let us observe that

$$z(x) \sim (x - x_0)\bar{\chi} \quad \text{as } x \to x_0, \qquad \phi_x(x_0) = 0,$$

and that $z(x)$ and ϕ_x are bounded functions as $(x - x_0) \to \infty$. This type of solutions corresponds to a wave which is emitted on alternating sides of the core $x = x_0$ and periodically. This type of solutions is, for two-dimensional λ–ω systems, the analogue to spiral waves.

Let us analyse the behaviour of solutions given by (4.13) and (4.14) nearby and faraway the core. From (4.14), it is clear that

$$\text{if} \quad \sqrt{\frac{b}{2}}|x - x_0| \gg 1 \quad \text{then } \phi \sim \frac{1}{3}\sqrt{\frac{2}{b}}|x - x_0|\bar{\zeta} + (\bar{\zeta} + d)t + D - \frac{2\bar{\zeta}}{3b}\log 2,$$

while

$$\text{if} \quad \sqrt{\frac{b}{2}}|x - x_0| \gg 1 \quad \text{then } \phi \sim \frac{1}{6}(x - x_0)^2\bar{\zeta} + (\bar{\zeta} + d)t + D.$$

[1] It should be mentioned that the nomenclature for coherent structures is not uniform in the literature. We have adopted the terminology in Ref. [23].

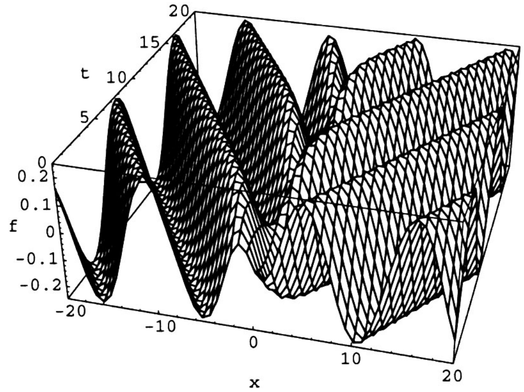

Fig. 1. Source solution corresponding to $a=-1,\ b=\frac{2}{3},\ c=\sqrt{2},\ d=\sqrt{2},\ x_0=0,\ \chi_0=1/\sqrt{6}$.

Then, the solution of (1.2) determined by (4.13) and (4.14) can be considered as a wave with constant amplitude and velocity when $|x-x_0| \gg 1$. This wave has opposed velocities as $x \to \pm\infty$. If we denote these velocities by $v_\pm$ then

$$v_\pm = \mp 3\sqrt{\frac{b}{2}}\left(1+\frac{d}{\bar{\zeta}}\right),$$

and then the role of x_0 varies depending on the sign of the parameters. This solution, for $d/\bar{\zeta} < -1$, is plotted in Fig. 1.

If $\chi_0 \neq \bar{\chi}$, using again (3.6), we have

$$z(x,t) = \sqrt{\frac{2}{b}}\frac{\sinh[\sqrt{b/2}(x-x_0)]}{(1/\chi_0 - 1/\bar{\chi})\mathrm{e}^{-(3b/2)t} + (1/\bar{\chi})\cosh[\sqrt{b/2}(x-x_0)]},$$

$$\phi(x,t) = \frac{2}{3b}\bar{\zeta}\log\left|\left(\frac{1}{\chi_0}-\frac{1}{\bar{\chi}}\right)\mathrm{e}^{-(3b/2)t} + \frac{1}{\bar{\chi}}\cosh\left[\sqrt{\frac{b}{2}}(x-x_0)\right]\right| + (\bar{\zeta}+d)t + C.$$

2. If $\chi_0 > 0$, the solution blows up at any fixed point at a finite time given by

$$t = \frac{2}{3b}\log\frac{1-\bar{\chi}/\chi_0}{\cosh[\sqrt{b/2}(x-x_0)]}, \tag{4.20}$$

thus, the blow-up propagates through x_0 with a non-constant velocity

$$v = -\frac{3\sqrt{b/2}(1-\bar{\chi}/\chi_0)\mathrm{e}^{-(3b/2)t}}{[1+(1-\bar{\chi}/\chi_0)^2\mathrm{e}^{-3bt}]^{1/2}}.$$

We plot the amplitude of this solution in Fig. 2, the blow-up on the curve (4.20) can be clearly appreciated.

Let us observe that if we take $t=\mathrm{e}^{\mathrm{i}\psi\tau}$ in Eq. (3.60) of [23] and assume τ is now the temporal variable, a family of solutions describing blow-up processes can be found. However, this behaviour is not explicitly mentioned in [23] and the relationship between this family of solutions and our blow-up solutions is not clear.

3. If $\bar{\chi} < \chi_0 < 0$, the solution is bounded in x and t and as $t \to \infty$ behaves as the source (4.13) and (4.14). This solution is plotted in Fig. 3.
4. If $\chi_0 < \bar{\chi}$, for $t > 0$, the solution remains bounded and it also approaches to the source as $t \to \infty$.

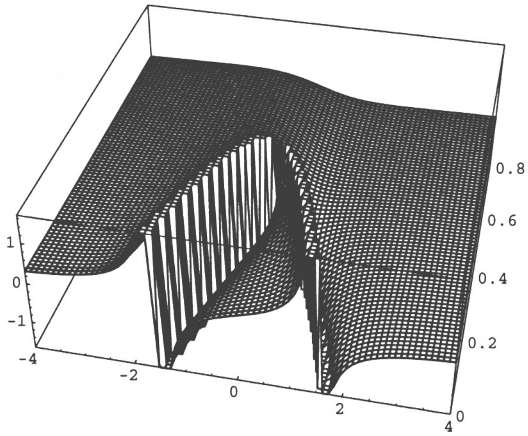

Fig. 2. Amplitude of a solution exhibiting a blow-up process. The corresponding parameters are $a=-1$, $b=\frac{2}{3}$, $c=\sqrt{2}$, $d=\sqrt{2}$, $x_0=0$, $\chi_0=1$.

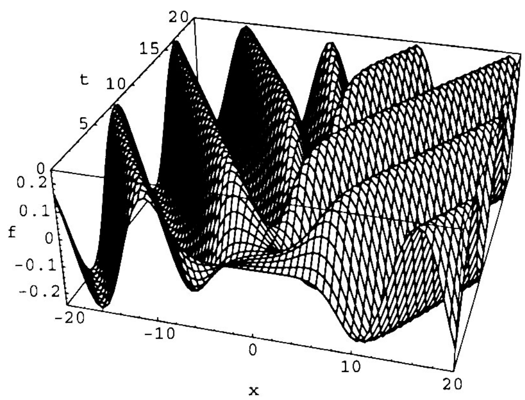

Fig. 3. Bounded solution behaving as a source as $t\to\infty$. The corresponding parameters are $a=-1$, $b=\frac{2}{3}$, $c=\sqrt{2}$, $d=\sqrt{2}$, $x_0=0$, $\chi_0=-\frac{1}{100}$.

5. Conclusions

In this work, we have classified the one-dimensional λ–ω systems which admit classical and nonclassical symmetries and we have reduced these systems to ODE systems by using some of these symmetries. Besides, we have been able to reduce the order of one of these systems (the one obtained by using a nonclassical symmetry) and in this way we have got some solutions of the system. Consequently, some new solutions of the corresponding λ–ω system have been found. As far as we know, some of these theoretical results are new.

Finally, we have devoted ourselves to get the dynamical interpretation of the solutions obtained in this symmetry reduction context. In this way, we have found

1. Special solutions consisting in sources with core in an arbitrary point $x=x_0$. Depending on the sign of the asymptotic velocities the behaviour of the core varies, and both cases are not equivalent due to that (1.2) is not invariant under temporal inversion. The asymptotic behaviour nearby and faraway the core is analysed.
2. A family of solutions which present a finite time blow-up process propagating through a fixed point $x=x_0$.
3. A family of solutions which behaves asymptotically (as $t\to\infty$) as the sources.

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