SYMMETRY ANALYSIS AND SOLUTIONS FOR A FAMILY OF CAHN–HILLIARD EQUATIONS

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In this paper we find some new classes of solutions for a family of Cahn–Hilliard equations. For some equations of this family several solutions have already been obtained by using several methods: the Lie method, the direct method and the singular manifold method. We make full analysis of the symmetry reductions of the family of Cahn–Hilliard equations by using the classical Lie method of infinitesimals and the nonclassical method. New classes of nonlocal symmetries for the family of Cahn–Hilliard equations are obtained. These *nonclassical potential* symmetries are realized as local nonclassical symmetries of a related integrated equation. For an equation of the Cahn–Hilliard family with the conditional Painlevé condition, we also compare symmetry reductions by using the nonclassical method with those obtained elsewhere by the singular manifold method. For this equation, we obtain nonclassical symmetries that reduce the original equation to ordinary differential equations with the Painlevé property. Such symmetries have not been derived elsewhere neither by the direct method nor by the singular manifold method.

1. Introduction

The Cahn–Hilliard equation was introduced to study phase separation in binary alloy glasses and polymers [4] and it is a good approach to spinodal decomposition. Based on the numerical version of the Fourier transformation approach to the nonlinear Cahn–Hilliard diffusion equation, computer simulations of the spinodal decomposition for a model alloy were carried out by Liu and Haasen [14]. The Cahn–Hilliard diffusion equation is also an equation that serves as a model for many problems in physical chemistry, developmental biology [2], and population movement [8]. The existence of a weak solution for the Cahn–Hilliard equation with degenerate mobility was proved in [9].

The Cahn–Hilliard flux equation describing diffusion for the decomposition of a one-dimensional binary solution can be written as

$$u_t + ku_{xxxx} - (f(u)u_x)_x = 0\,, \tag{1}$$

which is appropiate for the cases in which the motion is isotropic. Here u is the solute concentration at point x, t is the time, $f'(u)$ is the interdiffusion coefficient of solute, which is concentration dependent, $k/2$ is the gradient energy coefficient describing the contribution of the diffuse interface to the decomposition. In [8] bifurcations of the equilibrium to nonuniform states have been discussed for $f(u) = D_0 + D_2 u^2$ and in [15] several nonlinear results were derived.

Although the direct method is found to be more powerful than the Lie classical method [5], similarity reductions for the Cahn–Hilliard equation (1), with $f(u) = u$ and $f(u) = u^2$, have been obtained in [17] by using classical Lie symmetries as well as the direct method. The direct method did not yield any reduction that could not be obtained by Lie classical symmetries.

In this paper we solve a complete group classification problem for Eq. (1) by studying those diffusion coefficients $f(u)$ which admit the classical symmetry group. Both the symmetry group and the diffusion coefficients will be found through consistent application of the Lie-group formalism. The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists.

Motivated by the fact that symmetry reductions for many partial differential equations (PDE's) are known that are not obtained by using the classical Lie group method, there have been several generalizations of the classical Lie group method for symmetry reductions. Clarkson and Kruskal [5] have developed a direct method for deriving similarity reductions of PDE's; this method does not employ group theory. Bluman and Cole [3] developed a nonclassical method to study the symmetry reductions of the heat equation. Recently, the family of Cahn–Hillard equations (1) has raised a great interest because of an apparent contradiction between the scope of the singular manifold method (SMM) and the nonclassical symmetry reductions.

In [10] Estévez and Gordoa developed a method for identifying the nonclassical symmetries of PDE's using the SMM, based on the Painlevé property (PP), as a tool. They studied six different equations, of which four were equations with the PP while the other two were equations with only the conditional PP. The obtained results allowed them to propose the following conjecture: "The singular manifold method allows us to identify nonclassical symmetries that reduce the original equation to an ODE with the Painlevé property".

The combination of this statement with the Ablowitz, Ramani and Segur conjecture [1] means that for equations with the PP, the SMM should identify all nonclassical symmetries. Nevertheless, for equations with the conditional PP, the SMM is only able to identify the symmetries for which the associated reduced ODE's are of the Painlevé type. Recently Tanriver and Choudhury [18] have applied this method to a family of Cahn–Hilliard equations and their results were in apparent contradiction with the conjecture proposed in [10]. However, Estévez and Gordoa proved in [11] that the results of [18] were incomplete.

In [11] the authors claim that, for (1) with $f(u) = u$, the SMM allows them to determine two different symmetries and that these symmetries are the *only ones* in which the associated similarity reduction leads to an ODE of Painlevé type. Nevertheless, for

(1) with $f(u) = u$, apart from the symmetries derived in [11], we have derived *two new* nonclassical symmetries for which the corresponding associated similarity reductions lead to two different ODE's of the Painlevé type.

2. Lie symmetries and optimal systems

To apply the classical method to (1) we require that the infinitesimal generator

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u \tag{2}$$

leaves invariant the set of solutions of (1). This yields an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi u_x + \tau u_t - \phi = 0\,. \tag{3}$$

We consider the classical Lie group symmetry analysis of Eq. (1). The invariance of Eq. (1) under a Lie group of point transformations with infinitesimal generator (2) leads to a set of forty determining equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The solutions of this system depend on $f(u)$. For $f(u)$ arbitrary, the only symmetries admitted by (1) are the group of space and time translations, which are defined by the infinitesimal generators

$$V_1 = \partial_x\,, \qquad V_2 = \partial_t\,.$$

In this case, we obtain travelling wave reductions

$$z = x - \lambda t, \qquad u = h(z),$$

where $h(z)$, after integrating once with respect to z, satisfies

$$kh''' - f(h)h' - \lambda h = k_1\,. \tag{4}$$

Eq. (4) is invariant under translations and this allows us to reduce the order by one. The only functional forms of $f(u)$, with $f(u) \neq$ const., for which Eq. (1) has extra symmetries, are $f(u) = (au+b)^n$ and $f(u) = de^{au}$, and these symmetries are defined respectively by the following infinitesimal generators:

$$V_3^1 = x\partial x + 4t\partial_t - \frac{2}{an}(au+b)\partial_u, \qquad V_3^2 = x\partial x + 4t\partial_t - \frac{2}{a}\partial_u.$$

For the sake of completeness, we provide the generators of the nontrivial one-dimensional optimal system which are:
– for $f(u) = (au+b)^n$ the set $\{\langle V_1\rangle, \langle V_2\rangle, \langle V_1 + V_2\rangle, \langle V_3^1\rangle\}$;
– for $f(u) = de^{au}$, the set $\{\langle V_1\rangle, \langle V_2\rangle, \langle V_1 + V_2\rangle, \langle V_3^2\rangle\}$.
Since Eq. (1), with $f(u) = (au+b)^n$ and $f(u) = de^{au}$, has additional symmetries, and the reductions that correspond to V_1 and V_2 have already been derived, we must only

determine the similarity variables and similarity solutions corresponding to V_3^1 and V_3^2, which are:
— for V_3^1:

$$z = xt^{-\frac{1}{4}}, \qquad u = t^{-\frac{1}{2n}}h(z) - \frac{b}{a},$$

where $h(z)$ satisfies the ODE

$$kh'''' - a^n h^n h'' - \frac{z}{4}h' - na^n h^{n-1}(h')^2 - \frac{1}{2n}h = 0. \tag{5}$$

Eq. (5) does not admit Lie symmetries. Nevertheless, for $n = 2$ this equation can be easily integrated with respect to z, yielding

$$kh''' - a^2h^2h' - \frac{z}{4}h = k_1;$$

— for V_3^2:

$$z = xt^{-\frac{1}{4}}, \qquad u = -\frac{1}{a}\ln\left(t^{\frac{1}{2}}h(z)\right),$$

where $h(z)$ satisfies the ODE

$$4kh^3h'''' - 4d\left[h^2h'' - 2h(h')^2\right] - 4kh^2\left[4h'h''' + 3(h'')^2\right]$$
$$-zh^3h' + 48kh(h')^2h'' - 24k(h')^4 + 2h^4 = 0. \tag{6}$$

Eq. (6) does not admit Lie symmetries.

3. Nonclassical symmetries

The basic idea of the method is that the PDE (1) is augmented with the invariance surface condition (3) which is associated with the vector field (2). By requiring that both (1) and (3) are invariant under the transformation with infinitesimal generator (2), one obtains an overdetermined nonlinear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. To obtain nonclassical symmetries of (1), we apply the algorithm described in [7] for calculating the determining equations. We can distinguish two different cases:
— In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,u) = 1$. The nonclassical method applied to (1) gives rise only to the classical symmetries.
— In the case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and the determining equation for the infinitesimal ϕ is

$$k\phi_{xxxx} + 4k\phi\phi_{uu}\phi_{xx} + 4k\phi_{ux}\phi_{xx} - f\phi_{xx} + 3k\phi_{uu}(\phi_x)^2 + 6k\phi^2\phi_{uuu}\phi_x + 12k\phi\phi_{uux}\phi_x$$
$$+10k\phi\phi_u\phi_{uu}\phi_x + 6k\phi_{uxx}\phi_x + 4k\phi_u\phi_{ux}\phi_x - 3f'\phi\phi_x + k\phi^4\phi_{uuuu} + 4k\phi^3\phi_{uuux}$$
$$+6k\phi^3\phi_u\phi_{uuu} + 6k\phi^2\phi_{uuxx} + 12k\phi^2\phi_u\phi_{uux} + 4k\phi^3(\phi_{uu})^2 + 12k\phi^2\phi_{ux}\phi_{uu}$$
$$+7k\phi^2(\phi_u)^2\phi_{uu} - f\phi^2\phi_{uu} + 4k\phi\phi_{uxxx} + 6k\phi\phi_u\phi_{uxx} + 8k\phi(\phi_{ux})^2$$
$$+4k\phi(\phi_u)^2\phi_{ux} - 2f\phi\phi_{ux} - 2f'\phi^2\phi_u + \phi_t - f''\phi^3 = 0. \tag{7}$$

We cannot solve (7) in general because of its complexity. Thus we proceed by making an ansatz on the form of $\phi(x,t,u)$. Due to the invariance under temporal and spatial translations, we can take $t_0 = 0$ and $x_0 = 0$ without loss of generality, and in this way we have found many new solutions. In Table 1 we list the functional forms of $f(u)$ for which we obtain nonclassical symmetries, the corresponding similarity solutions and the corresponding ODE's.

Table 1: Each row shows the functions, infinitesimals, similarity solutions and ODE_i.

i	$f_i(u)$	$\phi_i(x,t,u)$	u_i	ODE_i
1	$u^n, \quad n \neq -1, -2$	$-\left(\frac{2}{k\alpha}\right)^{\frac{1}{2}} u^{\frac{n+2}{2}}$	$\left(\frac{2k\alpha}{n^2(x+w)^2}\right)^{\frac{1}{n}}$	$w' = 0$
2	u	$-\frac{x}{3t}$	$-\frac{x^2}{6t} + w(t)$	$3tw' + w = 0$
3	$k_1u^2 + k_2u$	$-\frac{i}{2\sqrt{k_1 t}}$	$w(t) - \frac{ix}{2\sqrt{k_1 t}}$	$4k_1tw' + 2k_1w + k_2 = 0$
4	$k_1u + \frac{k_2}{\sqrt{u}}$	$\frac{2u}{x}$	$x^2w(t)$	$w' - 6k_1w^2 = 0$
5	$k_1u^{\frac{2}{3}} + k_2u^{-\frac{2}{3}}$	$\frac{3u}{x}$	$x^3\,w(t)$	$w' - 12k_1w^{\frac{5}{3}} = 0$
6	$-k_1f_1(u) + \frac{k_2}{u} + k_3$	$-\frac{u}{\sqrt{2k_1t}}$	$w(t)\exp\left(-\frac{x}{\sqrt{2k_1t}}\right)$	$4k_1^2t^2w' + 2k_1twf_2(w) + kw = 0$

In Table 1 $\alpha = (n+1)(n+2)$, $f_1(u) = \log u - 1$ and $f_2(w) = k_1 \log w - k_3$.
Solutions of all these similarity equations can be obtained by using elementary techniques in ODE's. Consequently, we obtain exact solutions of Eq. (1),

$$u_1 = \left(\frac{2k\alpha}{n^2x^2}\right)^{\frac{1}{n}}, \qquad u_2 = -\frac{x^2}{6t} + \frac{k_1}{t^{\frac{1}{3}}}, \qquad u_3 = -\frac{ix}{2\sqrt{k_1t}} - \frac{k_2}{2k_1},$$

$$u_4 = -\frac{x^2}{6k_1t}, \qquad u_5 = \frac{x^3}{16\sqrt{2}(-k_1t)^{\frac{3}{2}}}, \qquad u_6 = \exp\left(-\frac{x}{\sqrt{2t}} + \frac{k}{2t} + k_3\right).$$

We must point out that:
– For $i = 1$, we obtain travelling waves.
– For $i = 2$, u_2 *is not* a travelling wave reduction and *it is not invariant* under the scaling group.
– For $i = 3, 4, 5$, when $f(u)$ takes the functional forms of f_3, f_4 and f_5, Eq. (1) does not admit any classical symmetry but translations. Consequently, these solution cannot be obtained by Lie classical symmetries. The scaling reduction can be used to reduce (1) to a system of ODE's.
– For $i = 6$, when $f(u)$ takes the functional form f_6, Eq. (1) does not admit any

classical symmetries but translations, and consequently this solution is unobtainable by Lie classical symmetries. We point out that u_6 is a noncharacteristic solution. Most of the solutions obtained are characteristic solutions, and consequently, they do not feel the influence of the diffuse interface.

The complexity of the determining equation (7) appears for many $\tau = 0$ symmetries [6] and one advantage of the SMM is that it provides nontrivial solutions of (7). In a recent paper [11], Estévez and Gordoa have studied the Cahn–Hilliard equations (1) with $f(u) = u$ and $f(u) = u^2$ by using the SMM method. Below we compare these results with ours by using the nonclassical method.

In [11] the authors claim that besides the trivial generator

$$\xi = 0, \qquad \tau = 1, \qquad \phi = 0 \tag{8}$$

(which corresponds to a classical symmetry). For $f(u) = u^2$ the *only* nontrivial infinitesimal generators of the nonclassical symmetries that reduce (1) to an ODE with the PP are

$$\xi = 1, \qquad \tau = 0, \qquad \phi = -\frac{1}{\sqrt{6k}} u^2. \tag{9}$$

(9) is a particular case $i = 1$, with $n = 2$. This last infinitesimal generator yields the similarity reduction u_1, where $w(t)$ satisfies ODE_1 which satisfies the PP.

For $f(u) = u$ they got the following symmetry

$$\xi = 1, \qquad \tau = 0, \qquad \phi = \frac{2u}{x + x_0},$$

which corresponds to case $i = 4$ with $k_1 = 1$ and $k_2 = 0$. In [11] the authors claim that the SMM allows to determine two different symmetries, and that these symmetries are the *only ones* in which the associated similarity reduction leads to an ODE of the Painlevé type.

Nevertheless, it is easy to check that the following symmetry

$$\xi = 1, \qquad \tau = 0, \qquad \phi = -\frac{u^{3/2}}{\sqrt{3k}},$$

which is a particular case of $i = 1$ with $n = 1$, satisfies Eq. (7) for the nonclassical symmetries with $\tau = 0$ and yields the similarity reduction u_1, where $w(t)$ satisfies ODE_1, which also satisfies the PP.

It is also easy to check that the following symmetry, corresponding to the case $i = 2$,

$$\xi = 1, \qquad \tau = 0, \qquad \phi = -\frac{x}{3t}$$

satisfies Eq.(7) for the nonclassical symmetries with $\tau = 0$ and yields the similarity reduction

$$u = -\frac{x^2}{6t} + w(t),$$

where $w(t)$ satisfies the linear ODE

$$3tw' + w = 0$$

which satisfies the PP.

Therefore, for the Cahn–Hilliard equation (1) with $f(u) = u$, we have obtained symmetry reductions by using the nonclassical method that were not obtained in [17] by the direct method, nor in [11] by the SMM.

4. Nonclassical potential symmetries

In order to find the potential symmetries of (1), we write the equation in a conserved form, and the associated auxiliary system is given by

$$\begin{cases} v_x = u\,, \\ v_t = f(u)u_x - ku_{xxx}\,. \end{cases} \tag{10}$$

If $(u(x), v(x))$ satisfies (10), then $u(x)$ solves the Cahn–Hilliard equation and $v(x)$ solves an integrated Cahn–Hilliard equation

$$v_t = f(v_x)v_{xx} - kv_{xxxx}\,. \tag{11}$$

The basic idea for obtaining *nonclassical potential* symmetries is that (10) or (11) are augmented with the invariance surface condition

$$\xi v_x + \tau v_t - \varphi = 0\,. \tag{12}$$

By requiring that both (11) and (12) are invariant under the transformations with infinitesimal generator

$$X = \xi(x,t,v)\partial_x + \tau(x,t,v)\partial_t + \varphi(x,t,v)\partial_v$$

one obtains an overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,v)$, $\tau(x,t,v)$ and $\varphi(x,t,v)$. We obtain nonclassical potential symmetries if any of the following conditions

$$(\varphi)_{vv} \neq 0\,, \qquad (\varphi)_{xv} \neq 0 \tag{13}$$

is satisfied.

In the case $\tau \neq 0$, without loss of generality we may set $\tau(x,t,v) = 1$. The nonclassical method applied to (11) gives rise only to the classical symmetries.

In the case $\tau = 0$, without loss of generality we may set $\xi = 1$ and we obtain the determining equation for the infinitesimal φ which is solved by making an ansatz on the form of $\varphi(x,t,v)$. In this way we have found some new solutions of (11).

In Table 2 we list the functions f_i, infinitesimals φ_i, similarity solutions v_i and reduced ODE's corresponding to each $i = 1, \ldots, 4$.

Table 2: Each row shows the functions, infinitesimals, similarity solutions and ODE_i.

i	$f_i(v_x)$	$\varphi_i(x,t,v)$	v_i	ODE_i
1	$-\log(v_x)$	$\dfrac{v}{\sqrt{2t}}$	$w(t)\exp\left(\dfrac{x}{\sqrt{2t}}\right)$	$4t^2w'+tw\log\left(\dfrac{w^2}{2t}\right)+kw=0$
2	$\dfrac{k_1}{\sqrt{v_x}}-\dfrac{v_x}{6}$	$\dfrac{x^2}{t}$	$\dfrac{x^3}{3t}+w(t)$	$\sqrt{t}w'-2k_1=0$
3	$\frac{3}{2}v_x$	$-\dfrac{v^{\frac{2}{3}}}{t^{\frac{1}{3}}}$	$\left(-\dfrac{x}{3t^{\frac{1}{3}}}-\dfrac{w(t)}{3}\right)^3$	$3tw'+w=0$
4	$-v_x^2+k_1v_x$	$\dfrac{v^{\frac{1}{2}}}{t^{\frac{1}{4}}}$	$\left(\frac{x}{2t^{\frac{1}{4}}}+w(t)\right)^2$	$4tw'+w-k_1t^{\frac{1}{4}}=0$

By using elementary techniques in ODE's, we obtain the following exact solutions of Eq. (11):

$$v_1=\sqrt{2t}\exp\left(\frac{x}{\sqrt{2t}}+\frac{k}{2t}-1\right),\qquad v_2=\frac{x^3}{3t}+4k_1t^{\frac{1}{2}},$$

$$v_3=\left(-\frac{x}{3t^{\frac{1}{3}}}-\frac{k_1}{3t^{\frac{1}{3}}}\right)^3,\qquad v_4=\left(\frac{x}{2t^{\frac{1}{4}}}+\frac{k_1t^{\frac{1}{4}}}{2}\right)^2.$$

Although by (10) we get exact solutions of (1), these solutions have been obtained in Section 3 by the nonclassical method. We must point out that for $i=1,2$ the symmetries obtained do not satisfy (13), and consequently, these symmetries are not nonclassical potential symmetries, while for $i=3,4$ the symmetries obtained do satisfy (13) and this means that we have nonclassical potential symmetries.

For $i=1,2,4$, when $f(v_x)$ takes the functional form f_1, f_2 and f_4, solutions v_1, v_2 and v_4 are not group-invariant; consequently, they cannot be obtained by Lie classical symmetries.

5. Concluding remarks

In this paper we have seen a classification of symmetry reductions of a family of Cahn–Hilliard equations (1) using the classical Lie method of infinitesimals. We have found that the nonclassical method yields symmetry reductions which are unobtainable by using the classical Lie method. For $f(u)=u$ and $f(u)=u^2$, the nonclassical method with $\tau=0$ leads to symmetry reductions that were not obtained in [17] by the direct method.

We have also derived a new class of nonlocal symmetries: these *nonclassical potential* symmetries are realized as local nonclassical symmetries of (11) and yield solutions of (11) which are not group-invariant.

It is known [16] that the Cahn–Hilliard equation with $f(u) = u$ has the conditional Painlevé property. We have compared the symmetry reductions of this equation by using the nonclassical method with those derived in [11] by the singular manifold method. For this Cahn–Hilliard equation we have obtained nonclassical symmetries that reduce the equation to ODE's with the Painlevé property and were not obtained in [11] by the singular manifold method.

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