Acta Math. Hungar.
87 (1–2) (2000), 135–146.

SPACES OF $\mathcal{S}$-BOUNDED MULTIPLIER CONVERGENT SERIES

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Abstract. We prove that some results on uniform convergence of sequences of unconditionally convergent series, in Banach spaces, can be generalized to sequences of weakly unconditionally Cauchy series.

1. Introduction

The normed spaces of bounded sequences, convergent sequences, null sequences and eventually null sequences of real numbers, endowed with the sup norm, will be denoted, as usual, by ℓ_∞, c, c_0 and c_{00}, respectively.

C. Swartz [7] studied a version of Schur Lemma in metric linear spaces for bounded multiplier convergent series (BM-convergent series): i.e. series $\sum_{i=1}^{\infty} x_i$ such that $\sum_{i=1}^{\infty} a_i x_i$ is convergent for every $(a_i)_{i\in \mathbf{N}} \in \ell_\infty$. The Schur Lemma for Banach spaces can also be obtained as a consequence of the result of Swartz.

If X denotes a Banach space, Qingying Bu and Congxin Wu [5] studied the space $BMC(X)$ of the sequences $\overline{x} = (x_i)_{i\in\mathbf{N}}$ such that the corresponding series are BM-convergent. This space was endowed with the norm

$$\|\overline{x}\|_{\mathrm{bmc}} = \sup\left\{\left\|\sum_{i=1}^{\infty} t_i x_i\right\| \,:\, (t_i)_{i\in\mathbf{N}} \in B_{\ell_\infty}\right\}.$$

Let us recall that a series $\sum_{i=1}^{\infty} x_i$ is called unconditionally convergent if $\sum_{i=1}^{\infty} x_{\pi(i)}$ converges for every permutation π of $\mathbf{N}$. It is well known that, in Banach spaces, BM-convergence is equivalent to unconditional convergence.

The results of Swartz [7] can be reformulated in Banach spaces throughout the space $BMC(X)$.

A series $\sum_{i=1}^{\infty} x_i$ is called weakly unconditionally Cauchy if for every permutation π of the natural numbers $\left(\sum_{k=1}^{i} x_{\pi(k)}\right)_{i\in\mathbf{N}}$ is a weakly Cauchy sequence; alternatively, $\sum_{i=1}^{\infty} x_i$ is a weakly unconditionally Cauchy series if and only if for every $x^* \in X^*$ (where X^* is the dual space of X), $\sum_{i=1}^{\infty} |x^*(x_i)| < \infty$.

In this paper we prove that some results on uniform convergence of sequences of unconditionally convergent series, in Banach spaces, can be generalized to sequences of weakly unconditionally Cauchy series throughout the spaces

$$X(\mathcal{S}) = \left\{ \overline{x} = (x_i)_{i\in\mathbf{N}} \in X^{\mathbf{N}} : \sum_{i=1}^{\infty} a_i x_i \text{ is convergent for every } (a_i)_{i\in\mathbf{N}} \in \mathcal{S} \right\},$$

where $\mathcal{S}$ is a subspace of ℓ_∞ such that $c_0 \subseteqq \mathcal{S}$ and whose norm is given by

$$\|\overline{x}\|_{\mathcal{S}} = \sup \left\{ \left\| \sum_{i=1}^{\infty} a_i x_i \right\| : (a_i)_{i\in\mathbf{N}} \in B_{\mathcal{S}} \right\}.$$

These spaces will be called *spaces of $\mathcal{S}$-bounded multiplier convergent series.*

If $\overline{x} = (x_i)_{i\in\mathbf{N}} \in X(\mathcal{S})$ then it is clear that $\sum_{i=1}^{\infty} x_i$ is a weakly unconditionally Cauchy series, because (cf. [1]) in Banach spaces weakly unconditionally Cauchy series can be characterized as the series $\sum_{i=1}^{\infty} x_i$ such that $\sum_{i=1}^{\infty} t_i x_i$ *is convergent for every null sequence* $(t_i)_{i\in\mathbf{N}}$.

It is also well known (cf. [1] and [4]) that if X is a Banach space then the following conditions are equivalent:

1. There exists a weakly unconditionally Cauchy series which is convergent –but not unconditionally– in X.
2. There exists a weakly unconditionally Cauchy series which is weakly convergent, but does not converge.
3. There exists a weakly unconditionally Cauchy series which is not weakly convergent.
4. X has a copy of c_0.

Hence, it is meaningful to consider weakly unconditionally Cauchy series that are not unconditionally convergent.

Our study contains, as special cases, several results of Swartz (cf. [7, Theorem 3, Corollary 4 and Proposition 5]) and of Qingying Bu and Congxin Wu (cf. [5, Proposition 1, Proposition 2, Proposition 3 and Theorem 5]). In particular, we will prove that if X is a Banach space, these results on uniform convergence, that are true in $X(\ell_\infty)$, are also valid in other cases. This happens, for instance, when $\mathcal{S}$ is a Grothendieck space; we point out that a space of this type does not necessarily contain a copy of ℓ_∞.

2. Completeness of $X(\mathcal{S})$

It is well known that in a normed space X, $\sum_{i=1}^{\infty} x_i$ is a weakly unconditionally Cauchy series if and only if the set

$$E = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbf{N},\ |\alpha_i| \leqq 1,\ i \in \{1, \ldots, n\} \right\} \tag{2.1}$$

is bounded (cf. [2]).

For any given series $\sum_{i=1}^{\infty} x_i$ in X, let us consider the set $\mathcal{S}\left(\sum_{i=1}^{\infty} x_i\right)$ of the sequences $(a_i)_{i\in\mathbf{N}} \in \ell_\infty$ such that $\sum_{i=1}^{\infty} a_i x_i$ converges. This set, endowed with the sup norm, will be called *the space of convergence* of the series $\sum_{i=1}^{\infty} x_i$.

PROPOSITION 2.1. *Let X be a real normed space and let $\sum_{i=1}^{\infty} x_i$ be a series in X. Then $\sum_{i=1}^{\infty} x_i$ is a weakly unconditionally Cauchy series if and only if the linear mapping $\sigma : \mathcal{S}\left(\sum_{i=1}^{\infty} x_i\right) \to X$, defined by $\sigma\left((a_i)_{i\in\mathbf{N}}\right) = \sum_{i=1}^{\infty} a_i x_i$ is continuous.*

PROOF. Let us consider a non-zero sequence $(a_i)_{i\in\mathbf{N}} \in \mathcal{S}\left(\sum_{i=1}^{\infty} x_i\right)$ and the sequence $(s_i)_{i\in\mathbf{N}}$ of the partial sums of the series $\sum_{i=1}^{\infty} a_i x_i$. We have

$$\frac{1}{\left\|(a_i)_{i\in\mathbf{N}}\right\|} s_n \in E = \left\{ \sum_{k=1}^{m} \alpha_k x_k \,:\, m \in \mathbf{N},\ |\alpha_k| \leqq 1,\ i \in \{1,\dots,m\} \right\},$$

for every $n \in \mathbf{N}$. Therefore, there exists $M > 0$ such that $\|s_n\| \leqq M\left\|(a_i)_{i\in\mathbf{N}}\right\|$, for every $n \in \mathbf{N}$. Hence $\left\|\sigma\left((a_i)_{i\in\mathbf{N}}\right)\right\| \leqq M\left\|(a_i)_{i\in\mathbf{N}}\right\|$.

Conversely, let $A = \left\{(\alpha_i)_{i\in\mathbf{N}} \in c_{00} : |\alpha_i| \leqq 1\right\} \subseteqq \mathcal{S}\left(\sum_{i=1}^{\infty} x_i\right)$. It is clear that A is bounded, therefore $\sigma(A) = E$ is also bounded. $\square$

In what follows, X will denote a real Banach space and $\mathcal{S}$ will be a subspace of ℓ_∞ such that $c_0 \subseteqq \mathcal{S}$.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subspaces of ℓ_∞ such that $c_0 \subseteqq \mathcal{S}_1 \subseteqq \mathcal{S}_2$ then $X(\ell_\infty) \subseteqq X(\mathcal{S}_2) \subseteqq X(\mathcal{S}_1) \subseteqq X(c_0)$. We also have that $X(c_0)$ is the space of the $\overline{x} = (x_i)_{i\in\mathbf{N}} \in X^{\mathbf{N}}$ such that $\sum_{i=1}^{\infty} x_i$ is a weakly unconditionally Cauchy series and $X(\ell_\infty)$ is the space of the $\overline{x} = (x_i)_{i\in\mathbf{N}} \in X^{\mathbf{N}}$ such that $\sum_{i=1}^{\infty} x_i$ is an unconditionally convergent series.

It is clear that X does not have a copy of c_0 if and only if $X(\mathcal{S}) = X(\ell_\infty)$ for every subspace $\mathcal{S}$ of ℓ_∞ such that $c_0 \subseteqq \mathcal{S}$.

Let $\overline{x} = (x_i)_{i\in\mathbf{N}} \in X(\mathcal{S})$. By Proposition 2.1,

$$\|\overline{x}\|_{\mathcal{S}} = \sup\left\{ \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \,:\, (a_i)_{i\in\mathbf{N}} \in B_{\mathcal{S}} \right\}$$

defines a norm in $X(\mathcal{S})$ and

$$\|\overline{x}\|_{\mathcal{S}} = \sup\left\{ \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \,:\, (a_i)_{i\in\mathbf{N}} \in B_{c_0} \right\}$$

$$= \sup\left\{ \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \,:\, (a_i)_{i\in\mathbf{N}} \in B_{c_{00}} \right\}$$

$$= \sup\left\{ \sum_{i=1}^{\infty} \left| f(x_i) \right| \ : \ f \in B_{X^*} \right\}$$

$$= \sup\left\{ \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \ : \ |\varepsilon_i| = 1 \text{ if } i \in \{1, \dots, n\}, \ n \in \mathbf{N} \right\}.$$

Let us consider the map $\varphi : X(\mathcal{S}) \to \mathcal{CL}(\mathcal{S}, X)$ defined by $\varphi(\overline{x}) = \sigma_{\overline{x}}$, where $\sigma_{\overline{x}}((a_i)_{i\in\mathbf{N}}) = \sum_{i=1}^{\infty} a_i x_i$, and $\mathcal{CL}(\mathcal{S}, X)$ denotes the usual space of continuous linear maps from $\mathcal{S}$ to X. It is clear that φ is a linear isometry. Let $\varphi' : X(\mathcal{S}) \to \mathcal{CL}(c_0, X)$ denote the map defined by $\varphi(\overline{x}) = \sigma'_{\overline{x}} = \sigma_{\overline{x}|_{c_0}}$. It is also clear that φ' is a linear isometry.

THEOREM 2.2. *$X(\mathcal{S})$ is a Banach space.*

PROOF. Let $(\overline{x}^n)_{n\in\mathbf{N}}$ be a Cauchy sequence in $X(\mathcal{S})$, where, for every $n \in \mathbf{N}$, $\overline{x}^n = (x_i^n)_{i\in\mathbf{N}}$. Let us denote, for every $n \in \mathbf{N}$, $\sigma_n = \sigma_{\overline{x}^n}$ and $\sigma'_n = \sigma'_{\overline{x}^n}$. There exists $\varrho_0 \in \mathcal{CL}(\mathcal{S}, X)$ such that $\lim_{n\to\infty} \sigma_n = \varrho_0$. Let $\varrho'_0 = \varrho_{0|_{c_0}}$. It follows that $\lim_{n\to\infty} \sigma'_n = \varrho'_0$ in $\mathcal{CL}(c_0, X)$.

For every $i \in \mathbf{N}$, we have that $\|x_i^p - x_i^q\| \leqq \|\overline{x}^p - \overline{x}^q\|_{\mathcal{S}}$ if $p, q \in \mathbf{N}$ and, therefore, there exists x_i^0 such that $\lim_{n\to\infty} x_i^n = x_i^0$. Let $\overline{x}^0 = (x_i^0)_{i\in\mathbf{N}}$ and $\overline{b} = (b_1, \dots, b_p, 0, \dots) \in c_{00}$. Then

$$\varrho'_0(\overline{b}) = \lim_{n\to\infty} \sigma'_n(\overline{b}) = \lim_{n\to\infty} (b_1 x_1^n + \dots + b_p x_p^n) = b_1 x_1^0 + \dots + b_p x_p^0.$$

It is clear now that if $(b_i)_{i\in\mathbf{N}} \in c_0$ then $\varrho'_0\big((b_i)_{i\in\mathbf{N}}\big) = \sum_{i=1}^{\infty} b_i x_i^0$.

Let $(a_i)_{i\in\mathbf{N}} \in B_{\mathcal{S}}$. We will prove that $\sum_{i=1}^{\infty} a_i x_i^0$ is convergent and therefore, $\overline{x}^0 = (x_i^0)_{i\in\mathbf{N}} \in X(\mathcal{S})$.

Let $\varepsilon > 0$ and $m \in \mathbf{N}$ be such that $\|\varrho_0 - \sigma_m\| < \frac{\varepsilon}{2}$. There exists an $n_0 \in \mathbf{N}$ such that if $q > p \geqq n_0$ then $\left\| \sum_p^q a_i x_i^m \right\| < \frac{\varepsilon}{2}$. Hence, if $\overline{a} = (0, \dots, 0, a_p, \dots, a_q, 0, \dots)$ then

$$\left\| \sum_{p}^{q} a_i x_i^0 \right\| = \left\| \varrho_0(\overline{a}) \right\| \leqq \left\| \varrho_0(\overline{a}) - \sigma_m(\overline{a}) \right\| + \left\| \sigma_m(\overline{a}) \right\| < \varepsilon.$$

Let $\sigma'_o = \sigma'_{\overline{x}^0}$. It is clear that $\sigma'_0 = \varrho'_0$. Since $\|\overline{x}^n - \overline{x}^0\|_{\mathcal{S}} = \|\sigma'_n - \varrho'_0\|$ for every $n \in \mathbf{N}$, we have $\lim_{n\to\infty} \overline{x}^n = \overline{x}^0$. □

REMARKS 2.3. 1. In [5], the space $X(\ell_\infty)$ is denoted by $BMC(X)$ and it is proved that this space is a Banach space with the norm

$$\|\overline{x}\|_{\text{bmc}} = \sup\left\{ \left\| \sum_{i=1}^{\infty} t_i x_i \right\| \ : \ (t_i)_{i\in\mathbf{N}} \in B_{\ell_\infty} \right\},$$

where $\overline{x} = (x_i)_{i\in\mathbf{N}}$. It is clear that $\|\overline{x}\|_{\text{bmc}} = \|\overline{x}\|_{\mathcal{S}}$ when $\mathcal{S} = \ell_\infty$.

2. In [6] the space $X(c_0)$ is denoted by $CMC(X)$. It is also clear that if $\mathcal{S} = c_0$ then $\|\overline{x}\|_{\text{bmc}} = \|\overline{x}\|_{\mathcal{S}}$. If we take $\mathcal{S} = c_0$ then $CMC(X)$ is a Banach space.

3. Let $\overline{x} = (x_i)_{i\in\mathbf{N}} \in X(\mathcal{S})$. For every $n \in \mathbf{N}$, let us denote

$$\overline{x}(j \geqq n) = (0, \dots, 0, x_n, x_{n+1}, \dots).$$

If $\sum_{i=1}^{\infty} x_i$ is an unconditionally convergent series then $\sum_{i=1}^{\infty} x_i$ is uniformly convergent in B_{ℓ_∞}. Hence, $\lim_{n\to\infty} \left\|\overline{x}(j \geqq n)\right\|_{\mathcal{S}} = 0$.

Conversely, let us suppose that $\lim_{n\to\infty} \left\|\overline{x}(j \geqq n)\right\|_{\mathcal{S}} = 0$. We will prove that $\sum_{i=1}^{\infty} x_i$ is an unconditionally convergent series. If the series $\sum_{i=1}^{\infty} x_i$ is not unconditionally convergent then there exists $(a_j)_{j\in\mathbf{N}} \in B_{\ell_\infty}$ such that $\sum_{j=1}^{\infty} a_j x_j$ does not converge. Hence, there exists $\delta > 0$ and a strictly increasing sequence of natural numbers $p_1 < p_2 < \ldots < p_k \ldots$ such that $\left\| \sum_{j=p_i+1}^{p_{i+1}} a_j x_j \right\| > \delta$, for every $i \in \mathbf{N}$. Then, for every i, we have

$$\left\|\overline{x}(j \geqq p_i)\right\|_{\mathcal{S}} > \delta.$$

4. Let $\mathcal{S}$ be a subspace of ℓ_∞ such that $c_0 \subseteqq \mathcal{S}$. Let us prove that ℓ_∞ can be isometrically identified with a subspace of $\mathcal{S}^{**}$ in such way that $\mathcal{S} \subseteqq \ell_\infty \subseteqq \mathcal{S}^{**}$. If $(a_j)_{j\in\mathbf{N}} \in \ell_\infty$, let us consider the map $h : \mathcal{S}^* \to \mathbf{R}$ defined, for every $g \in \mathcal{S}^*$, by $h(g) = \sum_{j=1}^{\infty} a_j g\big((e_j)_{j\in\mathbf{N}}\big)$, where $(e_j)_{j\in\mathbf{N}}$ is the c_0-basis. It is clear that $\|h\| = \left\|(a_j)_{j\in\mathbf{N}}\right\|$. □

3. Main results

Let X be a Banach space. We will say that X is a Grothendieck space if every weak* convergent sequence $(x_n^*)_{n\in\mathbf{N}}$ in X^* is weakly convergent.

Let $\mathcal{M}$ be a subspace of X^{**} such that $X \subseteqq \mathcal{M} \subseteqq X^{**}$. We will say that X is $\mathcal{M}$-Grothendieck if every $\sigma(X^*, X)$-convergent sequence $(x_n^*)_{n\in\mathbf{N}}$ in X^* is $\sigma(X^*, \mathcal{M})$-convergent. Let us observe that X is a Grothendieck space if and only if X is a X^{**}-Grothendieck space.

If we substitute, in the next theorem, $\mathcal{S}$ by ℓ_∞ we recover Theorem 3 of [7] for Banach spaces.

THEOREM 3.1. *Let X be a Banach space and let $\mathcal{S}$ be a subspace of ℓ_∞ such that $c_0 \subseteqq \mathcal{S}$ and $\mathcal{S}$ is an ℓ_∞-Grothendieck space. Let $(\overline{x}^n)_{n\in\mathbf{N}}$ be a sequence in $X(\mathcal{S})$, where $\overline{x}^n = (x_i^n)_{i\in\mathbf{N}}$ for every $n \in \mathbf{N}$. The sequence $(\overline{x}^n)_{n\in\mathbf{N}}$ is convergent in $X(\mathcal{S})$ if and only if, for every $(a_j)_{j\in\mathbf{N}} \in \mathcal{S}$,*

$\lim_{i\to\infty}\left(\sum_{j=1}^{\infty} a_j x_j^i\right)$ exists in X. In this case, $\lim_{n\to\infty}\overline{x}^n=\overline{x}^0\in X(\mathcal{S})$, where $\overline{x}^0=(x_i^0)_{i\in\mathbf{N}}$ is such that $x_i^0=\lim_{j\to\infty}x_i^j$ for every $i\in\mathbf{N}$.

PROOF. The necessity of the condition is obvious. Let us prove that the condition is sufficient. We will prove that $(\overline{x}^n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $X(\mathcal{S})$. Proceeding towards a contradiction, assume that there exist $\delta>0$ and a sequence $(n_k)_{k\in\mathbf{N}}$ of natural numbers such that $\left\|\overline{x}^{n_k}-\overline{x}^{n_{k+1}}\right\|_{\mathcal{S}}>\delta$, for every $k\in\mathbf{N}$.

For every $k\in\mathbf{N}$, let $\overline{z}^k=(z_i^k)_{i\in\mathbf{N}}=\left(x_i^{n_k}-x_i^{n_{k+1}}\right)_{i\in\mathbf{N}}$. We have $\overline{z}^k\in X(\mathcal{S})$ and $\lim_{i\to\infty}\left(\sum_{j=1}^{\infty}a_j z_j^i\right)=0$, for every $(a_j)_{j\in\mathbf{N}}\in\mathcal{S}$. We also have that $\|\overline{z}^k\|_{\mathcal{S}}>\delta$, for every $k\in\mathbf{N}$. Let us denote $\sigma_k=\sigma_{\overline{z}^k}\in\mathcal{CL}(\mathcal{S},X)$.

For every $k\in\mathbf{N}$ let $f_k\in B_{X^*}$ be such that

$$\sum_{j=1}^{\infty}\left|f_k(z_j^k)\right|>\delta. \tag{3.2}$$

If $(a_j)_{j\in\mathbf{N}}\in\mathcal{S}$ then $\left|f_k\left(\sigma_k\left((a_j)_{j\in\mathbf{N}}\right)\right)\right|\leqq\left\|\sigma_k\left((a_j)_{j\in\mathbf{N}}\right)\right\|$ and, therefore, $(f_k\circ\sigma_k)_{k\in\mathbf{N}}$ is a weak* convergent sequence in $\mathcal{S}^*$ that converges to 0. Hence, if $h=(a_j)_{j\in\mathbf{N}}\in\ell_\infty$ then

$$\lim_{k\to\infty}h\left(f_k\circ\sigma_{k|_{c_0}}\right)=\lim_{k\to\infty}\sum_{j=1}^{\infty}a_j f_k(z_j^k)=0.$$

This means that $\left\{\left(f_k(z_j^k)\right)_{j\in\mathbf{N}}\right\}_{k\in\mathbf{N}}$ is a weakly convergent sequence that converges to 0 in ℓ_1 and, hence, converges to 0 in the norm topology. This contradicts (3.2).

It is clear that in $X(\mathcal{S})$ we have $\lim_{n\to\infty}\overline{x}^n=\overline{x}^0$, where $\overline{x}^0=(x_i^0)_{i\in\mathbf{N}}$ is such that $x_i^0=\lim_{j\to\infty}x_i^j$ for every $i\in\mathbf{N}$. □

REMARKS 3.2. 1. There exists a closed subspace $\mathcal{S}$ of ℓ_∞ such that $\mathcal{S}\neq\ell_\infty$, $\mathcal{S}$ is a Grothendieck space and $\mathcal{S}$ does not have a copy of ℓ_∞. To prove this result, let us recall that Haydon [3] constructed, by transfinite induction, a Boolean algebra $\mathcal{F}$ with the following characteristics:

1) $\mathcal{F}$ is a subalgebra of $\mathcal{P}(\mathbf{N})$ such that $\left\{\{i\}:i\in\mathbf{N}\right\}\subsetneqq\mathcal{F}$. If T is the Stone space of $\mathcal{F}$ and $\mathcal{C}(T)$ is the corresponding space of continuous functions, we can isometrically identify $\mathcal{C}(T)$ with a closed subspace $\mathcal{S}$ of ℓ_∞ such that $c_0\subsetneqq\mathcal{S}$. 2) $\mathcal{C}(T)$ is a Grothendieck space that does not have a copy of ℓ_∞.

2. Let $\overline{x}=(x_i)_{i\in\mathbf{N}}\in X(\mathcal{S})$. Then $\left(\left\|\overline{x}(j\geqq m)\right\|\right)_{m\in\mathbf{N}}$ is a decreasing sequence. Let us denote $\alpha_{\overline{x}}=\lim_{n\to\infty}\left\|\overline{x}(j\geqq m)\right\|_{\mathcal{S}}$. The number $\alpha_{\overline{x}}$ will

be called the *control number* of the series $\sum_{i=1}^{\infty} x_i$. We have that $\alpha_{\overline{x}} = 0$ if and only if $\overline{x}$ is unconditionally convergent.

THEOREM 3.3. *Let $(\overline{x}^n)_{n\in\mathbf{N}}$ be a sequence in $X(\mathcal{S})$ and let $\overline{x}^0 \in X(\mathcal{S})$. We set $\alpha_0 = \alpha_{\overline{x}^0}$ and, for $n \in \mathbf{N}$, $\alpha_n = \alpha_{\overline{x}^n}$. Then:*

1) *If $\lim_{n\to\infty} \overline{x}^n = \overline{x}^0$ then $\lim_{n\to\infty} \alpha_n = \alpha_0$ and $\lim_{n\to\infty} \left\|\overline{x}^i(j \geqq n)\right\|_{\mathcal{S}} = \alpha_i$ uniformly in $i \in \mathbf{N}$.*

2) *If* i) $\lim_{n\to\infty} \left\|\overline{x}^i(j \geqq n)\right\|_{\mathcal{S}} = \alpha_i$ *uniformly in* $i \in \mathbf{N}$, ii) $\lim_{n\to\infty} \alpha_n = 0$, iii) *for every $j \in \mathbf{N}$ $\lim_{i\to\infty} x_j^i = x_j^0$ exists in X; then $(\overline{x}^n)_{n\in\mathbf{N}}$ converges in $X(\mathcal{S})$ to $\overline{x}^0 = (x_j^0)_{j\in\mathbf{N}}$ and $\overline{x}^0 \in X(\ell_\infty)$.*

PROOF. 1) We will first prove that $\lim_{n\to\infty} \alpha_n = \alpha_0$. Let $\varepsilon > 0$. There exists an $N \in \mathbf{N}$ such that $\|\overline{x}^i - \overline{x}^0\|_{\mathcal{S}} < \frac{\varepsilon}{3}$, for every $i \geqq N$. Let us fix $i \geqq N$. For α_0 there exists an m_0 such that $\left\|\overline{x}^0(j \geqq m_0)\right\|_{\mathcal{S}} < \alpha_0 + \frac{\varepsilon}{3}$. For α_i there exists $(a_j)_{j\in\mathbf{N}} \in B_{\mathcal{S}}$ such that

$$\alpha_i \leqq \left\|\overline{x}^i(j \geqq m_0)\right\|_{\mathcal{S}} < \left\|\sum_{j=m_0}^{\infty} a_j x_j^i\right\| + \frac{\varepsilon}{3}$$

$$\leqq \left\|\sum_{j=m_0}^{\infty} a_j(x_j^i - x_j^0)\right\| + \left\|\sum_{j=m_0}^{\infty} a_j x_j^0\right\| + \frac{\varepsilon}{3}$$

$$\leqq \frac{\varepsilon}{3} + \left\|\overline{x}^0(j \geqq m_0)\right\|_{\mathcal{S}} + \frac{\varepsilon}{3} < \alpha_0 + \varepsilon.$$

There exist an $m_i \in \mathbf{N}$ such that $\left\|\overline{x}^i(j \geqq m_i)\right\| < \alpha_i + \frac{\varepsilon}{3}$ and a sequence $(a_j)_{j\in\mathbf{N}} \in B_{\mathcal{S}}$ such that $\left\|\overline{x}^0(j \geqq m_i)\right\|_{\mathcal{S}} < \left\|\sum_{j=m_i}^{\infty} a_j x_j^0\right\| + \frac{\varepsilon}{3}$. We have that

$$\alpha_0 \leqq \left\|\overline{x}^0(j \geqq m_i)\right\|_{\mathcal{S}} < \left\|\sum_{j=m_i}^{\infty} a_j x_j^0\right\| + \frac{\varepsilon}{3}$$

$$\leqq \left\|\sum_{j=m_i}^{\infty} a_j(x_j^0 - x_j^i)\right\| + \left\|\sum_{j=m_i}^{\infty} a_j x_j^i\right\| + \frac{\varepsilon}{3}$$

$$\leqq \frac{\varepsilon}{3} + \left\|\overline{x}^i(j \geqq m_i)\right\|_{\mathcal{S}} + \frac{\varepsilon}{3} < \alpha_i + \varepsilon.$$

Hence, if $i \geqq N$ then $\alpha_0 - \varepsilon < \alpha_i < \alpha_0 + \varepsilon$. This proves that $\lim_{n\to\infty} \alpha_n = \alpha_0$.

Let us prove that $\lim_{n\to\infty} \left\|\overline{x}^i(j \geqq n)\right\|_{\mathcal{S}} = \alpha_i$ uniformly in $i \in \mathbf{N}$. Let $\varepsilon > 0$ and $N \in \mathbf{N}$ be such that for every $i \geqq N$ we have $\|\overline{x}^i - \overline{x}^0\|_{\mathcal{S}} < \frac{\varepsilon}{4}$. On the other hand, there exists an $n_0 \in \mathbf{N}$ such that $\left| \left\| \overline{x}^k(j \geqq n) \right\|_{\mathcal{S}} - \alpha_k \right| < \frac{\varepsilon}{4}$ for every $n \geqq n_0$, where $k \in \{1, \dots, N-1\}$ and $\left| \left\| \overline{x}^0(j \geqq n) \right\|_{\mathcal{S}} - \alpha_0 \right| < \frac{\varepsilon}{4}$.

Let $i \geqq N$ and $n \geqq n_0$. There exists a sequence $(a_j)_{j\in\mathbf{N}} \in B_{\mathcal{S}}$ such that

$$\alpha_i \leqq \left\| \overline{x}^i(j \geqq n) \right\|_{\mathcal{S}} < \left\| \sum_{j=n}^{\infty} a_j x_j^i \right\| + \frac{\varepsilon}{4}$$

$$\leqq \left\| \sum_{j=n}^{\infty} a_j (x_j^i - x_j^0) \right\| + \left\| \sum_{j=n}^{\infty} a_j x_j^0 \right\| + \frac{\varepsilon}{4}$$

$$\leqq \| \overline{x}^i - \overline{x}^0 \|_{\mathcal{S}} + \left\| \overline{x}^0(j \geqq n) \right\|_{\mathcal{S}} + \frac{\varepsilon}{4} \leqq \alpha_i + \varepsilon.$$

Hence, for every $i \in \mathbf{N}$, if $n \geqq n_0$ then $\alpha_i \leqq \left\| \overline{x}^i(j \geqq n) \right\|_{\mathcal{S}} \leqq \alpha_i + \varepsilon$.

2) Let us prove that $(\overline{x}^n)_{n\in\mathbf{N}}$ *is a Cauchy sequence. Let* $\varepsilon > 0$. *There* exists a $p_0 \in \mathbf{N}$ such that $\alpha_n < \frac{\varepsilon}{5}$, for $n \geqq p_0$. There exists an $m \in \mathbf{N}$ such that $\left| \left\| \overline{x}^i(j \geqq n) \right\|_{\mathcal{S}} - \alpha_i \right| < \frac{\varepsilon}{5}$, for $n \geqq m$ and $i \in \mathbf{N}$.

Since $\lim_{j\to\infty} x_k^j = x_k^0$, for every $k \in \{1, \dots, m-1\}$, there exists an integer $n_k \in \mathbf{N}$ such that $\|x_k^p - x_k^q\| < \frac{\varepsilon}{5(m-1)}$, for $p, q \geqq n_k$.

Let $p, q \geqq n_0 = \max\{p_0, n_1, n_2, \dots, n_{m-1}\}$. It is clear that if $(a_j)_{j\in\mathbf{N}} \in B_{\mathcal{S}}$ then

$$\left\| \sum_{j=1}^{\infty} a_j (x_j^p - x_j^q) \right\| \leqq \left\| \sum_{j=1}^{m-1} a_j (x_j^p - x_j^q) \right\| + \left\| \sum_{j=m}^{\infty} a_j (x_j^p - x_j^q) \right\|$$

$$\leqq \frac{\varepsilon}{5} + \left\| \sum_{j=m}^{\infty} a_j x_j^p \right\| + \left\| \sum_{j=m}^{\infty} a_j x_j^q \right\|.$$

By taking supremum in $B_{\mathcal{S}}$, we have

$$\| \overline{x}^p - \overline{x}^q \|_{\mathcal{S}} \leqq \frac{\varepsilon}{5} + \left\| \overline{x}^p(j \geqq m) \right\|_{\mathcal{S}} + \left\| \overline{x}^q(j \geqq m) \right\|_{\mathcal{S}}$$

$$\leqq \frac{\varepsilon}{5} + \alpha_p + \frac{\varepsilon}{5} + \alpha_q + \frac{\varepsilon}{5} < \varepsilon.$$

Hence $(\overline{x}^n)_{n\in\mathbf{N}}$ converges to some $\overline{y}^0 \in X(\mathcal{S})$. It is easy to check that $\overline{y}^0 = \overline{x}^0$. From 1) we deduce that $\alpha_{\overline{x}^0} = \lim_{n\to\infty} \alpha_n = 0$. Therefore $\overline{x}^0 \in X(\ell_\infty)$. $\square$

REMARKS 3.4. 1. Let $(\overline{x}^n)_{n\in\mathbf{N}}$ be a sequence in $X(\mathcal{S})$ such that $\overline{x}^n = (x_i^n)_{i\in\mathbf{N}}$ for every $n \in \mathbf{N}$. Let us suppose that $\left(\sum_{j=1}^{\infty} a_j x_j^i\right)_{i\in\mathbf{N}}$ converges uniformly in $B_{\mathcal{S}}$. For $\varepsilon > 0$ there exists an n_0 such that if $n \geqq n_0$ then $\left\| \sum_{j=n}^{\infty} a_j x_j^i \right\| < \varepsilon$, for every $(a_j)_{j\in\mathbf{N}} \in B_{\mathcal{S}}$ and every $i \in \mathbf{N}$. Hence $\left\|\overline{x}^i(j \geqq n)\right\|_{\mathcal{S}} < \varepsilon$ for every $i \in \mathbf{N}$; therefore, $\alpha_i = \lim_{n\to\infty} \left\|\overline{x}^i(j \geqq n)\right\|_{\mathcal{S}} = 0$ uniformly in $i \in \mathbf{N}$ and $\{\overline{x}^i : i \in \mathbf{N}\} \subseteqq X(\ell_\infty)$. It is clear that if $\lim_{n\to\infty} \overline{x}^n = \overline{x}^0$ then $\overline{x}^0 \in X(\ell_\infty)$.

Let us suppose that $\{\overline{x}^i : i \in \mathbf{N}\} \subseteqq X(\ell_\infty)$ and $\lim_{n\to\infty} \overline{x}^n = \overline{x}^0$. From Theorem 3.3 we deduce that $\lim_{n\to\infty} \left\|\overline{x}^i(j \geqq n)\right\|_{\mathcal{S}} = 0$ uniformly in $i \in \mathbf{N}$ and, therefore, $\left(\sum_{j=1}^{\infty} a_j x_j^i\right)_{i\in\mathbf{N}}$ converges uniformly in B_{ℓ_∞}. This result coincides with Corollary 4 of Swartz (cf. [7]), in the case when X is a Banach space.

2. Let us consider in $X(c_0)$ the sequence $(\overline{x}^i)_{i\in\mathbf{N}}$ where

$$\overline{x}^i = (e_1, e_2, \dots, e_{i-1}, 0, e_{i+1}, \dots),$$

for every $i \in \mathbf{N}$. We have $\alpha_i = \alpha_{\overline{x}^i} = 1$, for every $i \in \mathbf{N}$, and $\lim_{i\to\infty}(x_j^i) = e_j$, for every $j \in \mathbf{N}$. Since $\left\|\overline{x}^i(j \geqq n)\right\|_{\mathcal{S}} = 1$, for every $i, n \in \mathbf{N}$, then

$$\lim_{n\to\infty} \left\|\overline{x}^i(j \geqq n)\right\|_{\mathcal{S}} = 1$$

uniformly in $i \in \mathbf{N}$. The sequence $(\overline{x}^i)_{i\in\mathbf{N}}$ does not converge in $X(\mathcal{S})$ because $\left\|\overline{x}^i - \overline{x}^{i+1}\right\|_{\mathcal{S}} = 1$ for every $i \in \mathbf{N}$. □

DEFINITION 3.5. Let X be a Banach space and $\mathcal{S}$ a subspace of ℓ_∞, such that $c_0 \subseteqq \mathcal{S}$. If $\overline{x} = (x_i)_{i\in\mathbf{N}} \in X(\mathcal{S})$, let us denote $\overline{x}(i) = x_i$, for every $i \in \mathbf{N}$. Assume that $\mathcal{A} \subseteqq X(\mathcal{S})$.

1. We will say that $\mathcal{A}$ is uniformly convergent if $\lim_{n\to\infty} \left\|\overline{x}(j \geqq n)\right\|_{\mathcal{S}} = \alpha_{\overline{x}}$ uniformly in $\overline{x} \in \mathcal{A}$.

2. We will say that $\mathcal{A}$ is weakly uniformly convergent if

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} \left| f\left(\overline{x}(j)\right) \right| = 0,$$

for every $f \in B_{X^*}$, uniformly in $\overline{x} \in \mathcal{A}$.

REMARKS 3.6. 1. If $\mathcal{S} = \ell_\infty$ these definitions coincide with Definition 4 of [5].

2. If $(\overline{x}^n)_{n\in\mathbf{N}}$ is a convergent sequence in $X(\mathcal{S})$ then $\{\overline{x}^n : n \in \mathbf{N}\}$ is uniformly and weakly uniformly convergent.

By Theorem 3.3, $\{\overline{x}^n : n \in \mathbf{N}\}$ is uniformly convergent. We will prove that it is also weakly uniformly convergent. Proceeding towards a contradiction, assume that there exist $f \in B_{X^*}$ and $\delta > 0$ such that there exists a subsequence of $(\overline{x}^n)_{n\in\mathbf{N}}$, denoted also by $(\overline{x}^n)_{n\in\mathbf{N}}$, such that $\sum_{j=n}^{\infty}\left|f(x_j^n)\right| > \delta$, for every $n\in\mathbf{N}$. Let $n_0\in\mathbf{N}$ be such that $\|\overline{x}^n-\overline{x}^0\|_{\mathcal{S}}<\frac{\delta}{3}$, for every $n\geqq n_0$. Let $n>n_0$ be such that $\sum_{j=n}^{\infty}\left|f(x_j^0)\right|<\frac{\delta}{3}$. Then

$$\sum_{j=n}^{\infty}\left|f(x_j^n)\right| \leqq \sum_{j=n}^{\infty}\left|f(x_j^n-x_j^0)\right| + \sum_{j=n}^{\infty}\left|f(x_j^0)\right| \leqq \|\overline{x}^n-\overline{x}^0\|_{\mathcal{S}}+\frac{\delta}{3}<\frac{2\delta}{3}.$$

This is a contradiction.

3. It is easy to check that if $\mathcal{S}=\ell_\infty$ and $\mathcal{A}\subseteqq X$ is uniformly convergent then $\mathcal{A}$ is weakly uniformly convergent. It has been proved, in the remark following Theorem 7 of [5], that the converse is false.

THEOREM 3.7. *Let X be a Banach space and let $\mathcal{S}$ be a subspace of ℓ_∞ such that $c_0\subseteqq\mathcal{S}$. Let $\mathcal{A}\subseteqq X(\mathcal{S})$. If $\mathcal{A}$ is relatively compact then*

i) *$\mathcal{A}(i)=\left\{\overline{x}(i)\,:\,\overline{x}\in\mathcal{A}\right\}$ is relatively compact in X, for every $i\in\mathbf{N}$.*

ii) *$\mathcal{A}$ is uniformly convergent and weakly uniformly convergent.*

iii) *$H=\{\alpha_{\overline{x}}\,:\,\overline{x}\in\mathcal{A}\}$ is relatively compact in $\mathbf{R}$.*

PROOF. i) Let $i\in\mathbf{N}$ and let $\left(\overline{x}^n(i)\right)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{A}(i)$. Since $\mathcal{A}$ is relatively compact, there exists a subsequence $(\overline{x}^{n_k})_{k\in\mathbf{N}}$ of $(\overline{x}^n)_{n\in\mathbf{N}}$ that converges to some $\overline{x}^0$. Hence $\lim_{k\to\infty}\overline{x}^{n_k}(i)=\overline{x}^0(i)$.

ii) If $\mathcal{A}$ is not uniformly convergent then there exist $\delta>0$ and a sequence $(\overline{x}^n)_{n\in\mathbf{N}}$ in $\mathcal{A}$ such that $\alpha_{\overline{x}^n}+\delta<\|\overline{x}^n(j\geqq n)\|_{\mathcal{S}}$. Then, there exists a subsequence $(\overline{x}^{n_k})_{k\in\mathbf{N}}$ of $(\overline{x}^n)_{n\in\mathbf{N}}$ that converges to some $\overline{x}^0$. Then, by Theorem 3.3, $\lim_{n\to\infty}\left|\left\|\overline{x}^{n_k}(j\geqq n)\right\|_{\mathcal{S}}-\alpha_{\overline{x}^{n_k}}\right|=0$ uniformly in $k\in\mathbf{N}$. This contradicts $\alpha_{\overline{x}^{n_k}}+\delta<\left\|\overline{x}^{n_k}(j\geqq n_k)\right\|_{\mathcal{S}}$ for every $k\in\mathbf{N}$.

If $\mathcal{A}$ is not weakly uniformly convergent then there exist $f\in B_{X^*}$, $\delta>0$ and a sequence $(\overline{x}^n)_{n\in\mathbf{N}}$ in $\mathcal{A}$ such that $\sum_{j=n}^{\infty}\left|f\left(\overline{x}^n(j)\right)\right|>\delta$, but there exists a subsequence $(\overline{x}^{n_k})_{k\in\mathbf{N}}$ of $(\overline{x}^n)_{n\in\mathbf{N}}$ that converges to some $\overline{x}^0$. By Remark 3.6-2, $\lim_{n\to\infty}\sum_{j=n}^{\infty}\left|f\left(\overline{x}^{n_k}(j)\right)\right|=0$ uniformly in $k\in\mathbf{N}$, which contradicts $\sum_{j=n_k}^{\infty}\left|f\left(\overline{x}^{n_k}(j)\right)\right|>\delta$, for every $k\in\mathbf{N}$.

iii) Let $(\alpha_{\overline{x}^n})_{n\in\mathbf{N}}$ be a sequence in H. There exists a subsequence $(\overline{x}^{n_k})_{k\in\mathbf{N}}$ of $(\overline{x}^n)_{n\in\mathbf{N}}$ that converges to some $\overline{x}^0$. By Theorem 3.3-1, $\lim_{k\to\infty}\alpha_{\overline{x}^{n_k}}=\alpha_{\overline{x}^0}$. □

REMARKS 3.8. 1. The set $\mathcal{A}=\{\overline{x}^i\,:\,i\in\mathbf{N}\}\subseteqq X(c_0)$, where $\overline{x}^i$ is defined as in Remark 3.4-2, satisfies conditions i), ii) and iii) of Theorem 3.7, but is not relatively compact.

2. It has been proved in [5] that if $\mathcal{A}\subseteqq X(\ell_\infty)$ then $\mathcal{A}$ is relatively compact if and only if $\mathcal{A}$ is uniformly convergent and $\mathcal{A}(i)$ is relatively compact in X, for every $i\in\mathbf{N}$. □

THEOREM 3.9. *Let X be a Banach space and let $\mathcal{S}$ be a subspace of ℓ_∞ such that $c_0\subseteqq\mathcal{S}$. Let $\mathcal{S}'$ be the subspace of ℓ_∞ of the sequences $(b_n)_{n\in\mathbf{N}}$*

such that $(b_n a_n)_{n\in\mathbf{N}} \in \mathcal{S}$ for every $(a_n)_{n\in\mathbf{N}} \in \mathcal{S}$. Assume that $\mathcal{S}'$ is an ℓ_∞-Grothendieck subspace of ℓ_∞. Let $(\overline{x}^n)_{n\in\mathbf{N}}$ be a sequence in $X(\mathcal{S})$ such that $\lim_{i\to\infty}\sum_{j=1}^\infty a_j x_j^i$ exists for every $(a_j)_{j\in\mathbf{N}} \in \mathcal{S}$. Let us denote, for every $j \in \mathbf{N}$, $x_j^0 = \lim_{n\to\infty} x_j^n$. Then $\overline{x}^0 = (x_j^0)_{j\in\mathbf{N}} \in X(\mathcal{S})$ and

$$\lim_{i\to\infty}\left(\sum_{j=1}^\infty a_j x_j^i\right) = \sum_{j=1}^\infty a_j x_j^0,$$

for every $(a_j)_{j\in\mathbf{N}} \in \mathcal{S}$.

PROOF. Let us observe that $c_0 \subsetneqq \mathcal{S}'$. If $\mathcal{S} = c$, it is clear that $\mathcal{S}' = c$ and c is not an ℓ_∞-Grothendieck space. If $\mathcal{S} = c_0$ then $\mathcal{S}' = \ell_\infty$, which is a Grothendieck space. Let $(a_j)_{j\in\mathbf{N}} \in \mathcal{S}$ and let us consider, for every $n \in \mathbf{N}$, $\overline{y}^n = (a_j x_j^n)_{j\in\mathbf{N}}$. Then, for every $(b_j)_{j\in\mathbf{N}} \in \mathcal{S}'$ we have that $(b_j a_j)_{j\in\mathbf{N}} \in \mathcal{S}$. Therefore, $\lim_{i\to\infty}\left(\sum_{j=1}^\infty b_j y_j^n\right)$ exists. By Theorem 3.1, $(\overline{y}^n)_{n\in\mathbf{N}}$ is convergent in $X(\mathcal{S}')$ to some $\overline{y}^0 = (y_j^0)_{j\in\mathbf{N}}$, where $y_j^0 = \lim_{n\to\infty} y_j^n = a_j x_j^0$, for every $j \in \mathbf{N}$. Also

$$\lim_{i\to\infty}\left(\sum_{j=1}^\infty b_j y_j^i\right) = \sum_{j=1}^\infty b_j y_j^0, \quad \text{for every} \quad (b_j)_{j\in\mathbf{N}} \in \mathcal{S}'.$$

Hence, if $(b_j)_{j\in\mathbf{N}}$ is the constant sequence $b_j = 1$, then $\sum_{j=1}^\infty a_j x_j^0$ converges and $\lim_{i\to\infty}\left(\sum_{j=1}^\infty a_j x_j^i\right) = \sum_{j=1}^\infty a_j x_j^0$. □

REMARK 3.10. When $\mathcal{S} = c_0$, Theorem 3.9 coincides with Proposition 5 of [7]. With the hypothesis of Theorem 3.9 it is not, in general, true that $\lim_{n\to\infty} \overline{x}^n = \overline{x}^0$ (cf. [7]). □

PROBLEMS 3.11. 1. Are there any subspaces $\mathcal{S}$ of ℓ_∞ such that $c_0 \subsetneqq \mathcal{S}$ and $\mathcal{S}$ is not an ℓ_∞-Grothendieck space but such that Theorem 3.1 remains valid? *Is Theorem 3.9 still true when $\mathcal{S}'$ is not an* ℓ_∞-Grothendieck space?

2. Let $\sum_{i=1}^\infty e_i$ and $\sum_{i=1}^\infty x_i$ be two series in ℓ_∞, where

$$x_i = \begin{cases} e_i, & \text{if } i \text{ odd,} \\ \frac{1}{i} e_i, & \text{if } i \text{ even.} \end{cases}$$

Both series are weakly unconditionally Cauchy and $\alpha_{\sum_{i=1}^\infty e_i} = \alpha_{\sum_{i=1}^\infty x_i}$, but $\mathcal{S}\left(\sum_{i=1}^\infty e_i\right) = c_0$ and $\mathcal{S}\left(\sum_{i=1}^\infty x_i\right) \neq c_0$.

Let $\sum_{i=1}^\infty x_i$ and $\sum_{i=1}^\infty y_i$ be two weakly unconditionally Cauchy series in a Banach space X:

(a) Let us suppose that $\mathcal{S}\big(\sum_{i=1}^{\infty} x_i\big) = \mathcal{S}\big(\sum_{i=1}^{\infty} y_i\big)$. Is $\alpha_{\sum_{i=1}^{\infty} x_i} = \alpha_{\sum_{i=1}^{\infty} y_i}$?

(b) If for every $M \subseteqq \mathbf{N}$, $\alpha_{\sum_{i\in M} x_i} = \alpha_{\sum_{i\in M} y_i}$, is $\mathcal{S}\big(\sum_{i=1}^{\infty} x_i\big) = \mathcal{S}\big(\sum_{i=1}^{\infty} y_i\big)$?

3. Let X be a Banach space and let

$$\mathcal{L} = \big\{\mathcal{S} \text{ subspace of } \ell_\infty : c_0 \subseteqq \mathcal{S} \text{ and } X(\mathcal{S}) = X(\ell_\infty)\big\}.$$

If $\mathcal{F} = \bigcap_{\mathcal{S}\in\mathcal{L}} \mathcal{S}$, is $\mathcal{F}$ the least subspace of ℓ_∞ such that $c_0 \subseteqq \mathcal{F}$ and $X(\mathcal{F}) = X(\ell_\infty)$? Does X have a copy of c_0 if and only if $\mathcal{F} = c_0$? How can the space $\mathcal{F}$ be characterized?

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(Received July 27, 1998; revised September 23, 1998)

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