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# The Structure of Countably Generated Projective Modules Over Regular Rings[1]

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We prove that, for every regular ring $R$, there exists an isomorphism between the monoids of isomorphism classes of finitely generated projective right modules over the rings $\mathrm{End}_R(R_R^{(\omega)})$ and $\mathrm{RCFM}(R)$, where the latter denotes the ring of countably infinite row- and column-finite matrices over $R$. We use this result to give a precise description of the countably generated projective modules over simple regular rings and over regular rings satisfying $s$-comparability. © 2000 Academic Press

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## INTRODUCTION

One of the most relevant topics in the theory of von Neumann regular rings is the study of the finitely generated projective modules. This analysis is usually carried out by using stable and non-stable $K$-theory. The arbitrary projective modules over a regular ring have also been objects of interest, see, for example [13, 15, 18, 16]. A fundamental result for this theory is the fact that a projective module $P$ over a regular ring $R$ is a direct sum of cyclic projective modules [14, Theorem 4]. Moreover, $P$ satisfies the exchange property, see [25] and [31], and so the ring $\mathrm{End}(P_R)$ is an exchange ring in the sense of Warfield [32]. If we concentrate attention on the countably generated projective $R$-modules, then it is natural to consider the ring $E = \mathrm{End}(R_R^{(\omega)})$, since the category of finitely generated projective modules over $E$ is equivalent to the category $\mathrm{CP}(R)$ of countably generated projective $R$-modules. Recent results on the structure and $K$-theory of exchange rings (e.g., [1, 2, 27]) can then be applied.

Our approach to the study of $\mathrm{CP}(R)$ benefits from the use of several techniques related to the structure of multiplier rings of $\sigma$-unital (non-unital) regular rings, for which a detailed study has been performed in [6]. If $R$ is a regular ring and $\mathrm{FM}(R)$ denotes the (non-unital) regular ring of countably infinite matrices over $R$ having only a finite number of nonzero entries, then the multiplier ring of $\mathrm{FM}(R)$ is the ring $B = \mathrm{RCFM}(R)$ of row- and column-finite matrices over $R$ (see, e.g., [6, Proposition 1.1]). The ring $B$ is a subring of the ring $E = \mathrm{End}(R_R^{(\omega)})$, which is the ring of column-finite matrices over $R$. Although important differences are detected between $B$ and $E$, we prove below that their respective monoids of isomorphism classes of finitely generated projective modules are isomorphic (Theorem 1.3). By using this, the results in [6] can be applied to obtain explicit information on the countably generated projective $R$-modules.

More precise results can be obtained for particular classes of regular rings. More precisely, we consider simple regular rings and regular rings satisfying $s$-comparability for some positive integer $s$. It turns out that the structure of the countably generated projective modules over a simple regular ring $R$ depends heavily on the compact convex set of pseudo-rank functions on $R$. Moreover, we study how the known comparison theory of finitely generated projective modules over a regular ring satisfying $s$-comparability extends to the countably generated ones, with respect to the relations $\lesssim$ and $\lesssim^{\oplus}$. We also extend results of Kutami ([16]) concerning the behavior of directly finite projective modules over regular rings satisfying $s$-comparability. Since a regular ring $R$ satisfies $s$-comparability for some $s \geq 1$ if and only if $R$ satisfies 2-comparability ([4, Theorem 2.8]), we will state our results in terms of the 2-comparability condition.

In outline the paper is as follows. In Sect. 1, we recall the necessary definitions, and we prove that for any regular ring $R$ there exists an isomorphism between the monoids of isomorphism classes of finitely generated projective right modules over the rings $\mathrm{End}_R(R_R^{(\omega)})$ and $\mathrm{RCFM}(R)$. Section 2 is devoted to the study of simple regular rings and regular rings with $s$-comparability for some positive integer $s$. In particular, we obtain our main result on the comparison theory for countably generated projective modules over regular rings satisfying $s$-comparability (Theorem 2.6). In Sect. 3, we describe the relations between the ideals of $\mathrm{End}(R_R^{(\omega)})$ and $\mathrm{RCFM}(R)$ for a regular ring $R$, with special emphasis in the case of regular rings with $s$-comparability. Finally, we deal in Sect. 4 with property (DF), which was introduced by Kutami in [16]. Briefly, a regular ring $R$ satisfies property (DF) if the class of directly finite projective modules is closed under finite direct sums. We show that every regular ring with $s$-comparability satisfies property (DF). This extends a result of Kutami, who proved the same result under the additional hypothesis that $R$ is unit-regular.

After a preliminary version of this paper was privately circulated, we were aware of a paper by Kutami [19], which contains a different proof of our Corollary 4.7. Since we believe that our approach is of independent interest, we include our original proof. We have also incorporated some references to [19] in Sect. 4.

## 1. COUNTABLY GENERATED PROJECTIVE MODULES AND INTERVALS

We start by fixing some notation and terminology. Throughout, $R$ will denote a unital von Neumann regular ring (see [8] for definitions and properties on this class of rings). For a ring $T$, let $\mathrm{CP}(T)$ denote the category of countably generated projective right $T$-modules, and let $\mathrm{FP}(T)$ denote the category of finitely generated projective right $T$-modules. We denote by $V(T)$ the monoid of isomorphism classes of objects from $\mathrm{FP}(T)$. For $A \in \mathrm{FP}(T)$, we will denote the class of $A$ in $V(T)$ by $[A]$ or $[A]_T$. In the sequel, for $X, Y$ arbitrary $T$-modules, we will use $X \lesssim Y$ to denote "$X$ is isomorphic to a submodule of $Y$," $X \lesssim^{\oplus} Y$ to denote "$X$ is isomorphic to a direct summand of $Y$," $X \prec Y$ to denote "$X$ is isomorphic to a proper submodule of $Y$," and $X \prec^{\oplus} Y$ to denote "$X$ is isomorphic to a proper direct summand of $Y$." We will use the following important fact on regular rings [8, Theorem 1.11]: If $P$ is a projective right module over a regular ring $R$ and $A \in \mathrm{FP}(R)$, then $A \lesssim P$ if and only if $A \lesssim^{\oplus} P$.

For a cardinal number $\kappa$ and a right $T$-module $X$, we will denote by $\kappa X$ the direct sum of $\kappa$ copies of $X$. Similarly, for an ordinal number $\alpha$, we will denote by $X^{(\alpha)}$ the direct sum of $|\alpha|$ copies of $X$, indexed by $\alpha$. The symbol $\subset$ will always indicate *strict* inclusion.

Let $E = \mathrm{CFM}(R)$ be the ring of $\omega \times \omega$ column-finite matrices over $R$. The ring $E$ can be identified with the ring $\mathrm{End}(R_R^{(\omega)})$. Let $B = \mathrm{RCFM}(R)$ be the subring of $E$ consisting of matrices in $E$ with only finitely many nonzero entries in each row. There are ideals of $B$ and $E$ which play an important role. These are defined as follows:

$$F = \mathrm{FM}(R) = \left\{x \in B \mid x\left(R_R^{(\omega)}\right) \subseteq R_R^{(n)} \text{ for some } n\right\},$$

$$G = \left\{x \in E \mid x\left(R_R^{(\omega)}\right) \subseteq R_R^{(n)} \text{ for some } n\right\}.$$

Note that $F$ consists of the matrices in $B$ with only finitely many nonzero entries, while $G$ consists of the matrices in $E$ with only finitely many nonzero rows. Of course, $\mathrm{FM}(R) = \varinjlim M_n(R)$, where the (non-unital) embeddings $M_n(R) \to M_{n+1}(R)$ are given by

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

We will use repeatedly the following well-known lemma:

LEMMA 1.1. *Let $E$, $B$, $F$, and $G$ be as defined above. Then $F$ is an ideal of $B$, $G$ is an ideal of $E$, and $G = FE$ and $F = EF$.*

The ring $B$ is the multiplier ring of the non-unital ring $F$, that is, $B$ is the biggest unital ring containing $F$ as an essential ideal [6, Proposition 1.1]. Similarly, the ring $E$ is the multiplier ring of $G$. However, there are remarkable differences between the pairs $(F, B)$ and $(G, E)$. First of all, since all the matrix rings $M_n(R)$ over a regular ring $R$ are regular rings [8, Theorem 1.7], we see that $F$ is a (non-unital) regular ring. This is not the case for $G$ in general. For, let $R$ be a regular ring which is not artinian. By [8, Corollary 2.16], there exists a sequence $(e_n)$ of nonzero orthogonal idempotents in $R$. Let $X$ be the matrix in $G$ such that all its rows but the first are equal to zero, and the first row of $X$ is $[e_1, e_2, \dots]$. Then it is easy to see that $X$ is not a von Neumann regular element. A second difference comes from the notion of $\sigma$-unital rings. To define this concept, we first recall the definition of strict convergence in the multiplier ring $\mathcal{M}(I)$ of a semiprime ring $I$. A net $(x_\gamma)_{\gamma \in \Gamma}$ of elements of $\mathcal{M}(I)$ is said to *converge strictly* to $x \in \mathcal{M}(I)$ in case that, given $a_1, \dots, a_k \in I$, there is $\gamma_0 \in \Gamma$ such that $(x - x_\gamma)a_i = a_i(x - x_\gamma) = 0$ for all $\gamma \geq \gamma_0$ and $i = 1, \dots, k$. A *$\sigma$-unit* for a semiprime ring $I$ is a sequence $(a_n)_{n \in \mathbb{N}}$ of elements in $I$ such that $(a_n)$ converges strictly to 1 in the multiplier ring $\mathcal{M}(I)$, and such that $(a_n)$ is an increasing sequence in the sense that $a_n a_{n+1} = a_n = a_{n+1} a_n$ for al $n \in \mathbb{N}$. A semiprime ring $I$ is said to be *$\sigma$-unital* in case there exists a

$\sigma$-unit in $I$, see [6] and [20, p. 14]. The (non-unital) ring $F$ is $\sigma$-unital, since $(\mathrm{diag}(1, \overset{(n}{\ldots}, 1, 0, 0, \ldots))_{n \in \mathbb{N}}$ is a $\sigma$-unit, but $G$ is not $\sigma$-unital. The theory developed in [6] works for multiplier rings of regular $\sigma$-unital rings, hence that theory encompasses the ring $B$ but not the ring $E$. Another important difference between $B$ and $E$ will be pointed out in Sect. 3. In spite of all these differences, the monoids $V(B)$ and $V(E)$ are isomorphic (Theorem 1.3).

If $T$ is a ring and $M_T$ a module, it is well known that there is a categorical equivalence between the category $\mathrm{FP}(\mathrm{End}(M_T))$ and the category $\mathrm{add}(M_T)$ of $T$-modules which are direct summands of $nM_T$ for some $n$. Setting $S = \mathrm{End}(M_T)$, this equivalence is provided by the functors $- \otimes_S M$, from $\mathrm{FP}(S)$ to $\mathrm{add}(M_T)$ and $\mathrm{Hom}(M, -)$, from $\mathrm{add}(M_T)$ to $\mathrm{FP}(S)$, see e.g. [7, Theorem 4.7].

Since $\mathrm{CP}(T) = \mathrm{add}(T_T^{(\omega)})$, we obtain the following.

PROPOSITION 1.2. *Let $T$ be a ring. Then there is a categorical equivalence between the category* $\mathrm{CP}(T)$ *of countably generated projective right $T$-modules and the category of finitely generated projective right modules over the ring* $\mathrm{End}(T_T^{(\omega)})$.

Let $W(T)$ be the set of isomorphism classes of objects from $\mathrm{CP}(T)$. Then $W(T)$ has a natural structure of abelian monoid, induced by the direct sum of projective modules. By Proposition 1.2, we have $W(T) \cong V(\mathrm{End}(T_T^{(\omega)}))$. For $e = e^2 \in \mathrm{End}(T_T^{(\omega)})$, the class $[e\,\mathrm{End}(T_T^{(\omega)})]$ corresponds under this isomorphism to the class of the countably generated projective $T$-module $e(T_T^{(\omega)})$.

Let $R$ be a regular ring. As before, set $E = \mathrm{CFM}(R) = \mathrm{End}(R_R^{(\omega)})$ and $B = \mathrm{RCFM}(R)$. Next, we will prove that $V(B)$ and $V(E)$ are isomorphic monoids. This contrasts with the fact that the categories $\mathrm{FP}(E)$ and $\mathrm{FP}(B)$ are never equivalent. For, if $\mathrm{FP}(E)$ and $\mathrm{FP}(B)$ were equivalent categories, then it is easy to check that $E$ and $B$ would be Morita-equivalent rings, which would contradict [12, Theorem 8].

We shall use the idempotent picture of $V(-)$, see [30], so that, for a ring $T$, the monoid $V(T)$ is identified with the monoid of equivalence classes of idempotents in $\mathrm{FM}(T)$. For idempotents $e$ and $f$ in $T$, we write $e \sim_T f$ in case $e$ and $f$ are equivalent idempotents in $T$, i.e., there exist $x \in eTf$ and $y \in fTe$ such that $e = xy$ and $f = yx$.

THEOREM 1.3. *Let $R$ be a regular ring. Then the natural inclusion $i$: $B \to E$ induces a monoid isomorphism $V(i)$: $V(B) \to V(E)$.*

*Proof.* Since $E_E \cong 2E_E$ and $B_B \cong 2B_B$, we see that every element in $V(E)$ (respectively, $V(B)$) is represented by an idempotent of $E$ (respectively, $B$). Now the map $V(i)$ is defined by $V(i)([p]_B) = [p]_E$ for every idempotent $p \in B$. We first prove that $V(i)$ is surjective. Let $p$ be an idempotent in $E$, and consider $P = p(R_R^{(\omega)})$. Then $P$ is a countably generated projective module over the regular ring $R$, and so, by [14, Theorem 4], we have $P \cong \oplus_{n=1}^{\infty} e_n R$ for some idempotents $e_n \in R$. Now consider the following idempotent in $B$

$$q = \operatorname{diag}(e_1, e_2, e_3, \dots).$$

Since $p(R_R^{(\omega)}) \cong q(R_R^{(\omega)})$, we have $p \sim_E q$, and so $V(i)([q]_B) = [p]_E$, proving the surjectivity of $V(i)$.

Now we will prove injectivity of $V(i)$. Let $p, q$ be two idempotents in $B$ such that $p \sim_E q$. We need to prove that $p \sim_B q$. Let $F$ be the ideal of $B$ consisting of the matrices with only a finite number of nonzero entries. Recall that $B$ is the multiplier ring of $F$, and $F$ is obviously a non-unital regular ring. Similarly, $E$ is the multiplier ring of its ideal $G$, the subring of $E$ consisting in the matrices with just a finite number of nonzero rows. By [6, Lemma 2.1], there exist increasing sequences $(e_n)_{n \geq 1}$, $(f_n)_{n \geq 1}$ of idempotents in $F$, with $e_n \in pFp$ and $f_n \in qFq$, such that $e_n$ converges to $p$ and $f_n$ converges to $q$ in the strict topology. For $n \geq 1$, set $g_n = e_u - e_{u-1}$ and $h_n = f_u - f_{u-1}$ (here $e_0 = f_0 = 0$). Since $p$ and $q$ are equivalent in $E$, there exist $x \in pEq$ and $y \in qEp$ such that $p = xy$ and $q = yx$.

Put $g_1' = yg_1x$, and note that $g_1'$ is an idempotent in $G$. Since $f_n$ converges strictly to $q$ and $yg_1 \in EF = F$ there exists $n \geq 1$ such that $f_n(yg_1) = q(yg_1) = yg_1$. Consequently, $f_n g_1' = f_n(yg_1)x = (yg_1)x = g_1'$. Changing notation, we can assume that $n = 1$, so that $h_1 g_1' = f_1 g_1' = g_1'$. Write $g_1'' = g_1' h_1 \in EF = F$. Then $x_1 := g_1 x g_1'' \in F$ and $y_1 := g_1' y g_1 \in F$, and we have

$$x_1 y_1 = g_1 x g_1'' g_1' y g_1 = g_1 x g_1' y g_1 = g_1,$$

and $y_1 x_1 = g_1' g_1'' = g_1''$. Moreover $p - g_1 = x_1' y_1'$ and $q - g_1'' = y_1' x_1'$, where $x_1' = (1 - g_1)x(1 - g_1')$ and $y_1' = (1 - g_1'')y(1 - g_1)$. So $g_1, g_1'' \in F$, $g_1 \sim_B g_1''$ and $p - g_1 \sim_E q - g_1''$. Observe that $g_1'' \leq h_1$. Write $h_1' = x_1'(h_1 - g_1'')y_1' \leq p - g_1$. Since $e_2 - g_1, e_3 - g_1, e_4 - g_1, \dots$ converges in the strict topology induced by $F$ to $p - g_1$, there exists $n \geq 2$ such that $(e_n - g_1)h_1' = h_1'$. Changing notation, we can assume that $n = 2$, so that $g_2 h_1' = (e_2 - g_1)h_1' = h_1'$. Set $h_1'' = h_1' g_2 \in EF = F$. By using the same argument as before, we have $h_1'', h_1 - g_1'' \in F$, $h_1'' \sim_B h_1 - g_1''$ and $p - (h_1'' + g_1) \sim_E q - h_1$. Observe that $h_1'' \leq g_2$. Continuing this process, after renumbering, we get sequences of idempotents in $F$

$$\begin{array}{lllll} g_1'' \leq h_1, & g_2'' \leq h_2, & \dots, & g_n'' \leq h_n, & \dots, \\ h_1'' \leq g_2, & h_2'' \leq g_3, & \dots, & h_n'' \leq g_{n+1}, & \dots \end{array}$$

such that $g_1 \sim_B g_1''$, and $g_{n+1} - h_n'' \sim_B g_{n+1}''$ and $h_n - g_n'' \sim_B h_n''$ for all $n \geq 1$. Since $p = g_1 + \sum_{n=1}^{\infty}(g_{n+1} - h_n'') + \sum_{n=1}^{\infty} h_n''$ and $q = \sum_{n=1}^{\infty} g_n'' + \sum_{n=1}^{\infty}(h_n - g_n'')$, we obtain from [6, Lemma 1.6] that $p \sim_B q$, as desired.

This proves that $V(i)$ is injective and so we conclude that it is a monoid isomorphism from $V(B)$ onto $V(E)$. ∎

Recall that a ring $R$ is said to be *unit-regular* if for each $x \in R$ there is a unit $u \in R$ such that $x = xux$. The unit-regular rings are exactly the regular rings with stable rank one [8, Proposition 4.12]. Also, we infer from [8, Theorem 4.5] that a regular ring $R$ is unit-regular if and only if $V(R)$ is a cancellative monoid. By combining Theorem 1.3 with the results in [6] we will obtain a description of $W(R)$ in terms of intervals in $V(R)$ for any unit-regular ring $R$. To this end we recall the definition of an interval in a monoid $M$.

DEFINITION. Let $M$ be an abelian monoid. An *interval* in $M$ is a nonempty, hereditary, upward directed subset $I$ of $M$. An interval $I$ in a monoid $M$ is said to be *countably generated* provided that $I$ has a countable cofinal subset.

Intervals have been extensively used in the theory of multiplier $C^*$-algebras, e.g. [10, 11, 29], and recently in the study of multiplier rings of regular rings [6].

Given an abelian monoid $M$ we denote by $\Lambda_\sigma(M)$ the abelian monoid of countably generated intervals in $M$, with the sum defined by

$$X + Y = \{z \in M \mid z \leq x + y \text{ for some } x \in X \text{ and some } y \in Y\},$$

where $X, Y \in \Lambda_\sigma(M)$.

THEOREM 1.4. *Let $R$ be a unit-regular ring and let* CP$(R)$ *be the category of countably generated projective right $R$-modules. Then there is a monoid isomorphism $\Phi$: $W(R) \to \Lambda_\sigma(V(R))$ such that $\Phi([P])$ is the interval in $V(R)$ generated by the increasing sequence $\{[e_1 \oplus e_2 \oplus \cdots \oplus e_n] \mid n = 1, 2, \ldots\}$, for any $P \in$ CP$(R)$ and any decomposition $P \cong \bigoplus_{i=1}^{\infty} e_i R$ with $e_i = e_i^2 \in R$.*

*Proof.* Set $E = \mathrm{CFM}(R) = \mathrm{End}(R_R^{(\omega)})$, and $B = \mathrm{RCFM}(R)$, and recall that $B$ is the multiplier ring of FM$(R)$ [6, Proposition 1.1]. By Proposition 1.2 and Theorem 1.3 we have a monoid isomorphism $\tau$: $W(R) \to V(B)$. This isomorphism sends $[P]$ to the class in $V(B)$ of the idempotent $e = \mathrm{diag}(e_1, e_2, \ldots) \in B$, where $P \cong \bigoplus_{i=1}^{\infty} e_i R$ and $e_i$ are idempotents in $R$. By [6, Theorem 2.7], there is a monoid isomorphism $\mu$: $V(B) \to \Lambda_\sigma(V(R))$ which sends $[p] \in V(B)$ to the interval in $V(R)$ generated by $\{[p_n]\}$, where $(p_n)_{n \geq 1}$ is a $\sigma$-unit for $p\mathrm{FM}(R)p$ consisting of idempotents.

Define $\Phi = \mu \circ \tau$. Then $\Phi$ is a monoid isomorphism from $W(R)$ onto $\Lambda_\sigma(V(R))$. Now, let $P \in \mathrm{CP}(R)$ and let $P \cong \bigoplus_{i=1}^{\infty} e_i R$ with $e_i$ idempotents in $R$, and write $e = \mathrm{diag}(e_1, e_2, \dots)$. Since $(\mathrm{diag}(e_1, e_2, \dots, e_n, 0, 0 \dots))_{n \in \mathbb{N}}$ is a $\sigma$-unit for $e\mathrm{FM}(R)e$ consisting of idempotents, it follows from the above description that $\Phi([P])$ is the interval of $\Lambda_\sigma(V(R))$ generated by the countable set $\{[e_1 \oplus \cdots \oplus e_n] \mid n \geq 1\}$. ∎

PROPOSITION 1.5. *Let $R$ be a unit-regular ring, and let $\Phi \colon W(R) \to \Lambda_\sigma(V(R))$ be the isomorphism of Theorem* 1.4. *Let $P$ and $Q$ be countably generated projective $R$-modules. Then*

(a) *$P \lesssim Q$ if and only if $\Phi([P]) \subseteq \Phi([Q])$.*

(b) *$P \lesssim^{\oplus} Q$ if and only if there is $Z \in \Lambda_\sigma(V(R))$ such that $\Phi([P]) + Z = \Phi([Q])$.*

*Proof.* (a) Write $P \cong \bigoplus_{i=1}^{\infty} P_i$ and $Q \cong \bigoplus_{i=1}^{\infty} Q_i$, for $P_i, Q_i \in \mathrm{FP}(R)$. By [8, Proposition 4.8], we have $P \lesssim Q$ if and only if $P_1 \oplus \cdots \oplus P_n \lesssim Q$ for all $n \geq 1$. Since each $P_i$ is finitely generated, this holds if and only if for each $n \geq 1$ there exists $m \geq 1$ such that $P_1 \oplus \cdots \oplus P_n \lesssim Q_1 \oplus \cdots \oplus Q_m$. By the description of $\Phi$ in Theorem 1.4, the latter statement holds if and only if $\Phi([P]) \subseteq \Phi([Q])$.

(b) This is clear from Theorem 1.4. ∎

*Remark* 1.6. If $M$ is any abelian monoid, the *algebraic pre-order* on $M$ is defined by the rule $x \leq y$ iff there is $z \in M$ such that $x + z = y$. Note that Proposition 1.5(b) says that the algebraic pre-order on $\Lambda_\sigma(V(R))$ corresponds to the pre-order relation $\lesssim^{\oplus}$ on $\mathrm{CP}(R)$. Similarly, Proposition 1.5(a) says that the order induced by the inclusion of intervals corresponds to the relation $\lesssim$ on $\mathrm{CP}(R)$.

Let $M$ be an abelian monoid. Recall that $M$ is a *Riesz monoid* if whenever $p \leq q_1 + q_2$ in $M$, there exist $p_1, p_2 \in M$ such that $p = p_1 + p_2$ and $p_i \leq q_i$ for $i = 1, 2$, where $\leq$ denotes the algebraic pre-order on $M$. It follows from [8, Theorem 2.8] that $V(R)$ is a Riesz monoid for every regular ring $R$.

Let $n$ be a positive integer. Say that an abelian pre-ordered monoid $(M, \leq)$ is *$n$-unperforated* (respectively, *strictly $n$-unperforated*) in case $nx \leq ny$ implies $x \leq y$ (respectively, $nx < ny$ implies $x < y$) for all $x, y \in M$. Similarly, say that $M$ is *$n$-torsionfree* in case $nx = ny$ implies $x = y$ for all $x, y \in M$. The pre-ordered monoid $(M, \leq)$ is said to be *unperforated* (respectively, *strictly unperforated*) in case $(M, \leq)$ is $n$-unperforated (respectively, strictly $n$-unperforated) for all $n \geq 1$; and $M$ is said to be *torsionfree* in case $M$ is $n$-torsionfree for all $n \geq 1$.

The following lemma is essentially contained in [10].

LEMMA 1.7 [cf. 10, Lemma 2.3]. *Let $M$ be a partially ordered cancellative Riesz monoid (with respect to its algebraic order).*

(a) *If $M$ is $n$-unperforated for some $n$, then $\Lambda_\sigma(M)$ is $n$-unperforated for the order given by set inclusion. Moreover, $\Lambda_\sigma(M)$ is $n$-torsionfree.*

(b) *If $M$ is strictly $n$-unperforated for some $n$, and $X, Y$ are intervals in $\Lambda_\sigma(M)$, and $Y$ has no maximum element, then $nX \subseteq nY \Rightarrow X \subseteq Y$, and $nX = nY \Rightarrow X = Y$.*

*Proof.* (a) Assume that $M$ is $n$-unperforated for some $n$. Then $\Lambda_\sigma(M)$ is $n$-unperforated with respect to the order given by set inclusion by [10, Lemma 2.3(a)]. Since set inclusion induces actually a partial ordering on $\Lambda_\sigma(M)$, it follows that $\Lambda_\sigma(M)$ is $n$-torsionfree.

(b) If $x \in X$, then there is $y \in Y$ such that $nx \leq ny$. Since $Y$ has not a maximum element and $Y$ is upward directed, there is $y' \in Y$ such that $y < y'$. It follows that $nx < ny'$ and so $x < y'$ by strict $n$-unperforation, which gives $x \in Y$. We conclude that $X \subseteq Y$. If $nX = nY$, then $X$ does not have a maximum element, and we obtain $X \subseteq Y$ and $Y \subseteq X$, thus $X = Y$. ∎

As an immediate consequence of Proposition 1.5 and Lemma 1.7 we obtain

PROPOSITION 1.8. *Let $R$ be a unit-regular ring.*

(a) *If $V(R)$ is $n$-unperforated for some $n$, then $(W(R), [\lesssim])$ is also $n$-unperforated.*

(b) *Assume that $V(R)$ is strictly $n$-unperforated for some $n$. If $P, Q \in$ CP$(R)$ and $Q$ is not finitely generated, then $nP \lesssim nQ \Rightarrow P \lesssim Q$, and $nP \cong nQ \Rightarrow P \cong Q$.*

We will use Proposition 1.8 to show that both $E$ and $B$ satisfy a weak cancellation property, called separativity, which has been proved to be very useful in the study of some questions on exchange rings (see [2]).

Recall that a ring $T$ is *separative* provided the following cancellation property holds for finitely generated projective right (equivalently, left) $T$-modules $A$ and $B$:

$$A \oplus A \cong A \oplus B \cong B \oplus B \Rightarrow A \cong B.$$

See [2] for the origin of this terminology and for a number of equivalent conditions.

COROLLARY 1.9. *Let $R$ be a unit-regular ring and let $n \geq 2$. If $nP \prec nQ$ implies $P \prec Q$ for all $P, Q \in FP(R)$, then both $E$ and $B$ are separative rings.*

*Proof.* By Proposition 1.2 and Theorem 1.3, it is enough to show that $W(R)$ is separative. By [2, Lemma 2.1(iii)], it suffices to see that, for $A, B \in \mathrm{CP}(R)$, if $nA \cong nB$ and $(n+1)A \cong (n+1)B$ then $A \cong B$. From the above relations we get that $nA \oplus A \cong nA \oplus B$. Therefore, if $A$ is finitely generated, [8, Theorem 4.14] gives us that $A \cong B$. If $A$ is not finitely generated and $nA \cong nB$, then $A \cong B$ by Proposition 1.8(b). ∎

*Remark* 1.10. By using Corollary 1.9, [2, Proposition 2.3], and Proposition 4.2, one could obtain a different approach to a result of Kutami [17, Corollary 1.6], which states that any unit-regular ring $R$ such that $V(R)$ is unperforated satisfies the property special (DF), that is, $nP$ is directly finite for every $n \in \mathbb{N}$ and every projective $R$-module $P$.

Now we shall investigate the $n$-unperforation in $(W(R), [\lesssim^{\oplus}])$ for a unit-regular ring. The following generalization of [9, Proposition 2.19] is due to Goodearl. We thank him for allowing us to include it here.

PROPOSITION 1.11 (Goodearl). *Let $M$ be a cancellative Riesz monoid which is $n!$-unperforated for some $n \in \mathbb{N}$. If $x, y, z \in M$ and $n^k x \le n^k y + z$, then $x = v + w$ for some $v, w \in M$ with $v \le y$ and $n^k w \le z$.*

*Proof.* By an easy induction argument (on $k$), similar to the one in [9, Proposition 2.19], it suffices to consider only the case $k = 1$.

To prove the result in the case $k = 1$, we use induction on $n$. The case $n = 1$ is trivial. Assume that $n > 1$ and $nx \le ny + z$ for some $x, y, z \in M$. Note that $n!$-unperforation is equivalent to $k$-unperforation for all $1 \le k \le n$. We have $nx \le ny + z \le n(y+z)$ and so, by $n$-unperforation, we get $x \le y + z$. Therefore, $x = x_1 + x_2$ with $x_1 \le y$ and $x_2 \le z$. Write $y = x_1 + y_1$; then $nx_1 + nx_2 \le nx_1 + ny_1 + z$ and so $nx_2 \le ny_1 + z$. If $x_2 = v_2 + w$ with $v_2 \le y_1$ and $nw \le z$, then $x = (x_1 + v_2) + w$ with $x_1 + v_2 \le y$ and $nw \le z$. Thus, without loss of generality, we may assume that $x \le z$, say $z = x + z'$.

Now $nx \le ny + x + z'$ and so $(n-1)x \le ny + z' = (n-1)y + (y + z')$. By induction, $x = x_3 + x_4$ with $x_3 \le y$ and $(u-1)x_4 \le y + z'$. Write $y = x_3 + y_3$; then $nx_4 \le x_4 + y + z' = y_3 + x + z' = y_3 + z$. As above, $nx \le ny + z$ implies $nx_4 \le ny_3 + z$, and it suffices to show that $x_4 = v_4 + w$ with $v_4 \le y_3$ and $nw \le z$. Thus, without loss of generality, we may now assume that $nx \le y + z$.

We now have $nx = a + b$ with $a \le y$ and $b \le z$. By [33, Lemma 1.9], we have $x = x_0 + x_1 + \cdots + x_n$ with $x_1 + 2x_2 + \cdots + nx_n = b$ and $nx_0 + (n-1)x_1 + \cdots + x_{n-1} = a$. Set $v = x_0 + \cdots + x_{n-1}$ and $w = x_n$; then $x = v + w$ with $v \le a \le y$ and $nw \le b \le z$.

Therefore the induction works. ∎

THEOREM 1.12. *Let $R$ be a unit-regular ring. If $V(R)$ is $n!$-unperforated for some $n$, then $(W(R), [\lesssim^{\oplus}])$ is also $n!$-unperforated.*

*Proof.* The proof is similar to the one of [10, Corollary 2.4(b)]. We include it for completeness.

By Theorem 1.4 and Proposition 1.5(b), it suffices to see that $\Lambda_\sigma(V(R))$ is $k$-unperforated for all $k \leq n$ (with respect to the algebraic order). Assume that $kX = kY + Z$ for $X, Y, Z \in \Lambda_\sigma(V(R))$. Set $W = \{w \in M \mid kw \in Z\}$.

By using Proposition 1.11, it follows as in [10, Lemma 2.3(c)] that $W$ is an interval in $V(R)$ and $X = Y + W$. It is not clear that $W$ is a countably generated interval, but there is a countably generated interval $W'$ such that $W' \subseteq W$ and $X = Y + W'$. To construct it, consider an increasing cofinal sequence $(x_m)$ for $X$. Since $W$ is upward directed, it is easy to construct by induction increasing sequences $(y_m)$ and $(w_m)$ of elements of $Y$ and $W$, respectively, such that $x_m \leq y_m + w_m$ for all $m$. Take as $W'$ the interval generated by the family $\{w_m\}$. Since $X = Y + W'$ and $W' \in \Lambda_\sigma(V(R))$, we conclude that $\Lambda_\sigma(V(R))$ is $k$-unperforated. ∎

## 2. SIMPLE RINGS AND RINGS WITH $s$-COMPARABILITY

In this section we will give a precise description of the structure of the countably generated projective modules over some special types of regular rings. Our approach is based on the reduction to the simple case.

Let $s$ be a positive integer. A regular ring $R$ is said to satisfy *$s$-comparability* in case for every $x, y \in R$, either $xR \lesssim s(yR)$ or $yR \lesssim s(xR)$. For $s = 1$ we just obtain the comparability axiom of [8, Chap. 8]. By [4, Theorem 2.8], a regular ring $R$ satisfies $s$-comparability for some $s \geq 1$ if and only if $R$ satisfies 2-comparability. Therefore, from now on, we will mainly use the 2-comparability condition instead of the longer condition "$s$-comparability for some $s \geq 1$". Directly finite regular rings with 2-comparability are not always unit-regular [4, Example 3.2], but so are in the simple case, by a result of O'Meara [23, Corollary 2].

The following facts will be used repeatedly, see [4]:

(1) Let $R$ be a nonzero regular ring satisfying 2-comparability. Then the lattice $L(R)$ of two-sided ideals of $R$ is totally ordered. In particular $R$ has a unique maximal ideal $M$.

(2) If $R$ is a regular ring satisfying 2-comparability, then FP($R$) also satisfies 2-comparability, i.e., given two finitely generated projective right $R$-modules $P$ and $Q$, ether $P \lesssim 2Q$ or $Q \lesssim 2P$ ([4, Proposition 2.1]).

(3) If $R$ is a nonzero directly finite regular ring satisfying 2-comparability, and $M$ is its unique maximal ideal, then $R/M$ is a simple directly finite regular ring satisfying 2-comparability, and so it is unit-regular by O'Meara's result, see [4, Corollary 2.7].

Say that $R$ is *strictly unperforated* in case $nP \prec nQ$ implies $P \prec Q$ for all $n \geq 1$ and $P, Q \in \mathrm{FP}(R)$. As we will see below (Remark 2.3(c)), every simple regular ring satisfying 2-comparability is strictly unperforated (in case $R$ is directly finite, this follows from [3, Corollary 4.5]).

Let $\mathbb{P}(R)$ be the compact convex set of pseudo-rank functions defined on $R$ [8, Chap. 16]. For pseudo-rank functions on non-unital regular rings, we will follow the conventions used in [6].

Let $K$ be a compact convex set. We denote by $\mathrm{Aff}(K)$ the ordered Banach space of all the affine continuous real-valued functions on $K$, and we denote by $\mathrm{LAff}(K)$ the monoid of all affine and lower semicontinuous functions on $K$ with values in $\mathbb{R} \cup \{+\infty\}$. Let $\mathrm{LAff}_\sigma(K)$ denote the submonoid of $\mathrm{LAff}(K)$ whose elements are pointwise suprema of increasing sequences of affine real-valued continuous functions on $K$. The semigroup of strictly positive elements in $\mathrm{LAff}_\sigma(K)$ will be denoted by $\mathrm{LAff}_\sigma(K)^{++}$.

Let $R$ be a unital regular ring such that $\mathbb{P}(R) \neq \emptyset$. Notice that, by [8, Proposition 16.8], every $N \in \mathbb{P}(R)$ can be uniquely extended to an unnormalized pseudo-rank function on $\mathrm{FM}(R)$, also denoted by $N$, such that $N(\mathrm{diag}(x_1, \dots, x_n, 0, 0, \dots)) = N(x_1) + \cdots + N(x_n)$. We have a monoid homomorphism $\phi: V(R) \to \mathrm{Aff}(\mathbb{P}(R))^+$ defined by $\phi([e])(N) = N(e)$ for all $e \in \mathrm{FM}(R)$ and all $N \in \mathbb{P}(R)$. By using this map, one can define a semigroup structure on $V(R) \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$ extending the given semigroup operations in $V(R)$ and $\mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$, and by setting $x + f = \phi(x) + f$ for $x \in V(R)$ and $f \in \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$.

THEOREM 2.1. *Let $R$ be a simple, nonartinian, strictly unperforated, unit-regular ring. Then there exists a monoid isomorphism $\mu: W(R) \to V(R) \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$. This isomorphism is the identity on $V(R)$ and it is given by the formula*

$$\mu([P])(N) = \sup\{N(e_1) + \cdots + N(e_n) \mid n \geq 1\} = \sum_{i=1}^{\infty} N(e_i),$$

*where $[P] \in W(R) \setminus V(R)$, $N \in \mathbb{P}(R)$ and $P \cong \bigoplus_{i=1}^{\infty} e_i R$ with $e_i = e_i^2 \in R$.*

*Proof.* By Theorem 1.4, there is a monoid isomorphism $\Phi: W(R) \to \Lambda_\sigma(V(R))$. Write $M = V(\mathrm{FM}(R)) = V(R)$, and note that $M$ is a conical simple refinement monoid. Since $R$ is simple and nonartinian, $M$ has no atoms. Furthermore, $M$ is cancellative and strictly unperforated because $R$

is strictly unperforated and unit-regular. So, it follows from [29, Theorem 3.9] that there is a monoid isomorphism $\varphi\colon \Lambda_\sigma(M) \to M \sqcup \mathrm{LAff}_\sigma(S_u)^{++}$, where $S_u$ is the state space $\mathrm{St}(M, u)$ for a given nonzero element $u \in M$. Here, the semigroup operation in $M \sqcup \mathrm{LAff}_\sigma(S_u)^{++}$ is defined by extending the given ones in $M$ and $\mathrm{LAff}_\sigma(S_u)^{++}$, and by setting $x + f = \phi_u(x) + f$, for $x \in M$ and $f \in \mathrm{LAff}_\sigma(S_u)^{++}$, where $\phi_u\colon M \to \mathrm{Aff}(S_u)$ is the natural representation homomorphism. The isomorphism $\varphi$ sends an interval of the form $[0, x]$, with $x \in M$, to $x$, and sends an interval $X \in \Lambda_\sigma(M)$ without a maximum element to $\sup(X) = \sup\{\phi_u(x_n) \mid n \in \mathbb{N}\} \in \mathrm{LAff}_\sigma(S_u)^{++}$, where $(x_n)_{n \in \mathbb{N}}$ is a cofinal increasing sequence in $X$.

Now fix $u = [R_R] \in M$, and note that $u$ is also represented by the idempotent $e := \mathrm{diag}(1, 0, \dots)$ of $\mathrm{FM}(R)$.

As we have already observed, it follows from [8, Proposition 16.8] that every $N \in \mathbb{P}(R)$ can be uniquely extended to an unnormalized pseudo-rank function $\mathrm{FM}(R)$, also denoted by $N$, such that $N(\mathrm{diag}(x_1, \dots, x_n, 0, 0, \dots)) = N(x_1) + \cdots + N(x_n)$. As in [6], we denote by $\mathbb{P}(\mathrm{FM}(R))_e$ the compact convex set of all the pseudo-rank functions $N$ on $\mathrm{FM}(R)$ such that $N(e) = 1$. By the previous observation, we can identify $\mathbb{P}(\mathrm{FM}(R))_e$ with $\mathbb{P}(R)$. Hence, by [6, Proposition 3.2], there is an affine homeomorphism $\alpha\colon \mathbb{P}(R) \to S_u$ such that $\alpha(N)([f]) = N(f)$ for every $N \in \mathbb{P}(\mathrm{FM}(R))_e$ and every idempotent $f \in \mathrm{FM}(R)$. This affine homeomorphism induces a monoid isomorphism $M \sqcup \mathrm{LAff}_\sigma(S_u)^{++} \to M \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$. Composing $\varphi \circ \Phi$ with this isomorphism we get a monoid isomorphism $\mu\colon W(R) \to M \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$. From the description of $\Phi$ given in Theorem 1.4 and the description of the above maps we get the desired properties of the map $\mu$. ∎

Note that if $R$ is a simple artinian ring then $W(R) \cong V(R) \sqcup \{\infty\} = \mathbb{Z}^+ \sqcup \{\infty\}$.

COROLLARY 2.2. *Let $R$ be a simple regular ring satisfying 2-comparability.*

(a) *If $R$ is either artinian or directly infinite then $W(R) \cong V(R) \sqcup \{\infty\}$, that is, there is a unique $P \in \mathrm{CP}(R) \setminus \mathrm{FP}(R)$ up to isomorphism, namely $P = \aleph_0 R_R$.*

(b) *If $R$ is nonartinian and directly finite, then $W(R) \cong V(R) \sqcup \mathbb{R}^{++} \sqcup \{\infty\}$.*

(c) *If $R$ is nonartinian and directly finite, there exists a unique countably additive map $t\colon W(R) \to \mathbb{R}^+ \sqcup \{\infty\}$ such that $t([eR]) = N(e)$ for idempotents $e \in R$, where $N$ is the unique pseudo-rank function on $R$. Thus, if $P \cong \bigoplus_{i=1}^{\infty} e_i R$ with $e_i = e_i^2 \in R$, then $t([P]) = \sum_{i=1}^{\infty} N(e_i)$.*

*Proof.* (a) The simple artinian case is clear, so assume that $R$ is a simple directly infinite regular ring with 2-comparability. By [27, Proposition 1.7(3)], $R$ is purely infinite, that is, $P \prec Q$ for every two nonzero finitely generated projective $R$-modules. Therefore, we conclude from Proposition 1.2, Theorem 1.3 and [6, Proposition 2.12] that $W(R) \cong V(R) \sqcup \{\infty\}$.

(b) Assume now that $R$ is a nonartinian, directly finite, simple regular ring satisfying 2-comparability. By [23, Corollary 2], $R$ is unit-regular. By [3, Corollary 4.5], $R$ has a unique pseudo-rank function and is strictly unperforated. Since $\mathbb{P}(R)$ is a singleton, we have $\mathrm{LAff}_\sigma(\mathbb{P}(R))^{++} = \mathbb{R}^{++} \sqcup \{\infty\}$. So, Theorem 2.1 gives $W(R) \cong V(R) \sqcup \mathbb{R}^{++} \sqcup \{\infty\}$.

(c) As observed in (b), $R$ has a unique pseudo-rank function $N$, and therefore there is a unique state $s$ on $(V(R), [R])$ by [8, Proposition 17.12]. By (b), there is a semigroup isomorphism $\lambda\colon W(R) \setminus V(R) \to \mathbb{R}^{++} \sqcup \{\infty\}$. Define $t\colon W(R) \to \mathbb{R}^{+} \sqcup \{\infty\}$ by putting $t = s$ on $V(R)$ and $t = \lambda$ on $W(R) \setminus V(R)$. It is straightforward to see, by using Theorem 2.1, that $t$ is a countably additive map. ∎

*Remark* 2.3. (a) Note that in Corollary 2.2(b), the semigroup operation in $V(R) \sqcup \mathbb{R}^{++} \sqcup \{\infty\}$ is given by the semigroup operations in $V(R)$ and in $\mathbb{R}^{++} \sqcup \{\infty\}$ (where, of course, $\alpha + \infty = \infty$ for all $\alpha \in \mathbb{R}$), and by the rule $[e] + f = N(e) + f$ for every idempotent $e \in \mathrm{FM}(R)$ and every $f \in \mathbb{R}^{++} \sqcup \{\infty\}$, where $N$ is the unique pseudo-rank function on $R$. Clearly, $V(R) \sqcup \mathbb{R}^{++}$ is a submonoid of $V(R) \sqcup \mathbb{R}^{++} \sqcup \{\infty\}$.

(b) Note that in Corollary 2.2(b) the unique (up to isomorphism) directly infinite module that appears is that corresponding to $\aleph_0 R_R$.

(c) The proof of Corollary 2.2 shows that every simple regular ring satisfying 2-comparability is strictly unperforated.

For a right $R$-module $X$ define $\mathrm{tr}(X) = \sum_{f \in X^*} f(X)$, where $X^* = \mathrm{Hom}(X_R, R_R)$. The set $\mathrm{tr}(X)$ is always a two-sided ideal of $R$, called the *trace ideal* of $X$, and it is a principal two-sided ideal in case $X \in \mathrm{FP}(R)$ and $R$ is a regular ring. If $R$ is a regular ring with 2-comparability, then it is not true in general that $R$ satisfies full comparability, see for example [8, Example 18.19]. However, by [4, Proposition 2.3(b)], two finitely generated projectives $P$ and $Q$ are always comparable provided their trace ideals are different.

In order to prove our main result on comparison for countably generated projective modules, we need some preliminaries.

DEFINITION. Let $R$ be a regular ring satisfying 2-comparability. Let $M$ be the unique maximal ideal of $R$. For $A, B \in \mathrm{FP}(R)$, write $A \prec_M B$ in case $A/AM \prec B/BM$.

LEMMA 2.4. *Let $R$ be a regular ring satisfying 2-comparability. Let $M$ be the unique maximal two-sided ideal of $R$.*

(a) *If $A, B \in \mathrm{FP}(R)$ and $A \prec_M B$, then there is $C \in \mathrm{FP}(R)$ such that $C \neq CM$ and $B \cong A \oplus C$.*

(b) *Set $P = \bigoplus_{i=1}^{\infty} P_i$ and $Q = \bigoplus_{i=1}^{\infty} Q_i$, for $P_i, Q_i \in \mathrm{FP}(R)$. Assume that $P_1 \prec_M Q_1 \prec_M P_1 \oplus P_2 \prec_M Q_1 \oplus Q_2 \prec_M \cdots$ and that $P_i \neq P_i M$ and $Q_i \neq Q_i M$ for all $i \geq 1$. Then $P \cong Q$.*

*Proof.* (a) The proof is the same as that of [4, Proposition 2.3(c)].

(b) Notice first that the condition $P \neq PM$ for $P \in \mathrm{FP}(R)$ means that $P$ is a generator for the category Mod $- R$.

Since $P_1 \prec_M Q_1$ there exists by (a) a decomposition $Q_1 = Q_1' \oplus Q_1''$ such that $P_1 \cong Q_1'$ and $Q_1'' \neq Q_1'' M$. Further, $Q_1 \prec_M P_1 \oplus P_2$ and so we have an isomorphism $Q_1' \oplus Q_1'' \oplus T_1 \cong P_1 \oplus P_2 \cong Q_1' \oplus P_2$ for some $T_1$ such that $T_1 \neq T_1 M$. Since both $Q_1'' \oplus T_1$ and $P_2$ are generators, and $R$ is separative by [27, Theorem 2.2], we conclude that $Q_1'' \oplus T_1 \cong P_2$. So we can write $P_2 = P_2' \oplus P_2''$ with $P_2' \cong Q_1''$ and $P_2'' \neq P_2'' M$. Since $P_1 \oplus P_2 \prec_M Q_1 \oplus Q_2$, we can find by (a) a decomposition $Q_1 \oplus Q_2 \cong P_1 \oplus P_2 \oplus T_2$ with $T_2 \neq T_2 M$. Now note that

$$(Q_1' \oplus Q_1'') \oplus Q_2 \cong Q_1 \oplus Q_2 \cong P_1 \oplus P_2' \oplus P_2'' \oplus T_2 \cong (Q_1' \oplus Q_1'') \oplus P_2'' \oplus T_2.$$

By using again that $R$ is separative, we get $Q_2 \cong P_2'' \oplus T_2$. So we obtain a decomposition $Q_2 = Q_2' \oplus Q_2''$ such that $Q_2' \cong P_2''$ and $Q_2'' \neq Q_2'' M$. Continuing in this way we obtain decompositions $P_n = P_n' \oplus P_n''$ and $Q_n = Q_n' \oplus Q_n''$ for all $n \geq 1$ such that $P_1' = 0$ and $Q_n' \cong P_n''$ and $Q_n'' \cong P_{n+1}'$ for all $n \geq 1$. Finally we get

$$P = \bigoplus_{n=1}^{\infty} P_n = \bigoplus_{n=1}^{\infty} (P_n' \oplus P_n'') \cong \bigoplus_{n=1}^{\infty} (Q_n' \oplus Q_n'') = Q,$$

as desired. ∎

For an ideal $I$ of a ring $T$, let $\mathrm{FP}(I)$ (respectively, $\mathrm{CP}(I)$) denote the full subcategory of $\mathrm{FP}(T)$ (respectively, $\mathrm{CP}(T)$) whose objects are the finitely generated (respectively, countably generated) projective modules $A$ such that $A = AI$.

LEMMA 2.5. *Let $R$ be a regular ring and let $e$ be an idempotent in $R$. Set $I = ReR$, and let $M$ be an ideal of $R$ such that $M \subseteq I$. Then there are equivalences of categories $\mathrm{CP}(I) \to \mathrm{CP}(eRe)$ and $\mathrm{CP}(I/M) \to \mathrm{CP}(eRe/eMe)$ such that the following diagram commutes, where the vertical* *arrows are the functors reduction modulo M and modulo eMe, respectively*:

$$\begin{array}{ccc} \mathrm{CP}(I) & \longrightarrow & \mathrm{CP}(eRe) \\ \downarrow & & \downarrow \\ \mathrm{CP}(I/M) & \longrightarrow & \mathrm{CP}(eRe/eMe). \end{array}$$

*Proof.* It is well known that there is a categorical equivalence between $\mathrm{FP}(eRe)$ and $\mathrm{FP}(I)$, see for example [2, Lemma 1.5(c)]. Indeed, the equivalence is given by the functors $(-) \otimes_{eRe} eR$ from $\mathrm{FP}(eRe)$ to $\mathrm{FP}(I)$ and $(-) \otimes_R Re$ from $\mathrm{FP}(I)$ to $\mathrm{FP}(eRe)$. Since every projective $R$-module (respectively, projective $eRe$-module) is a direct sum of modules in $\mathrm{FP}(R)$ (respectively, $\mathrm{FP}(eRe)$), the above equivalence extends to an equivalence between the category $\mathrm{CP}(I)$ and the category $\mathrm{CP}(eRe)$. For $P \in \mathrm{CP}(I)$, we have $(P \otimes_R Re)/(P \otimes_R Re)(eMe) \cong P/PM \otimes_{R/M} (R/M)(e+M)$, so that the stated diagram is commutative. ∎

THEOREM 2.6. *Let $R$ be a regular ring satisfying 2-comparability, and let $P$ and $Q$ be countably generated projective right $R$-modules.*

(a) *If* $\mathrm{tr}(P) \subset \mathrm{tr}(Q)$ *then* $P \prec Q$.

(b) *If* $\mathrm{tr}(P) = \mathrm{tr}(Q)$ *and* $\mathrm{tr}(P)$ *is not a principal two-sided ideal, then* $P \cong Q$.

(c) *Assume that* $\mathrm{tr}(P) = \mathrm{tr}(Q)$ *is a nonzero principal two-sided ideal, and let $M$ be the unique maximal ideal of* $\mathrm{tr}(P)$. *Then we have*:

(c1) *If either $P/PM$ or $Q/QM$ is not finitely generated, then $P$ and $Q$ are comparable with respect to* $\lesssim$.

(c2) *If both $P/PM$ and $Q/QM$ are not finitely generated and $P/PM \cong Q/QM$, then $P \cong Q$.*

(c3) *If $P/PM$ and $Q/QM$ are both finitely generated and $P/PM \prec Q/QM$, then $P \prec Q$.*

*Proof.* Write $P \cong \bigoplus_{i=1}^{\infty} P_i$ and $Q \cong \bigoplus_{i=1}^{\infty} Q_i$, where $P_i, Q_i \in \mathrm{FP}(R)$.

(a) If $\mathrm{tr}(Q_i) \subseteq \mathrm{tr}(P)$ for all $i$, then $\mathrm{tr}(Q) = \sum \mathrm{tr}(Q_i) \subseteq \mathrm{tr}(P)$, a contradiction. So there is $i \geq 1$ such that $\mathrm{tr}(P) \subset \mathrm{tr}(Q_i)$. Now by using the technique in [4, Proposition 2.5(1)(2)] we get $\bigoplus_{n=1}^{\infty} P_n \prec Q_i$, so that $P \prec Q$.

(b) By using repeatedly the hypothesis that $\mathrm{tr}(P)$ is not principal, we can arrange the decompositions of $P$ and $Q$ in such a way that

$$(*) \quad \mathrm{tr}(P_1 \oplus \cdots \oplus P_n) \subset \mathrm{tr}(Q_1 \oplus \cdots \oplus Q_n) \subset \mathrm{tr}(P_1 \oplus \cdots \oplus P_{n+1})$$

for all $n \geq 1$. Now we will define inductively a sequence of homomorphisms $\varphi_n$: $P_1 \oplus \cdots \oplus P_n \to Q_1 \oplus \cdots \oplus Q_n$ and $\psi_n$: $Q_1 \oplus \cdots \oplus Q_n \to P_1 \oplus \cdots \oplus P_{n+1}$ such that $\psi_n \circ \varphi_n = \iota_n$ and $\varphi_{n+1} \circ \psi_n = \varsigma_n$, where $\iota_n$: $P_1 \oplus \cdots \oplus P_n \to P_1 \oplus \cdots \oplus P_{n+1}$ and $\varsigma_n$: $Q_1 \oplus \cdots \oplus Q_n \to Q_1 \oplus \cdots \oplus Q_{n+1}$ are the natural inclusion maps. Set $P_0 = Q_0 = 0$ and $\varphi_0 = \psi_0 = 0$. Let $n \geq 0$, and assume we have constructed $\varphi_k$ and $\psi_k$ for $0 \leq k \leq n$. Write $P_1 \oplus \cdots \oplus P_{n+1} = \psi_n(Q_0 \oplus Q_1 \oplus \cdots Q_n) \oplus P'_{n+1}$. Taking into account (*), we get $\operatorname{tr}(P'_{n+1}) \subseteq \operatorname{tr}(P_1 \oplus \cdots \oplus P_{n+1}) \subset \operatorname{tr}(Q_1 \oplus \cdots \oplus Q_{n+1}) = \operatorname{tr}(Q_{n+1})$. (The latter equality follows from the relation $\operatorname{tr}(Q_1 \oplus \cdots \oplus Q_n) \subset \operatorname{tr}(Q_1 \oplus \cdots \oplus Q_{n+1})$ and comparability of ideals.) Therefore there exists an injective homomorphism $\theta$: $P'_{n+1} \to Q_{n+1}$. Let $\varphi_{n+1}$: $P_1 \oplus \cdots \oplus P_{n+1} \to Q_1 \oplus \cdots \oplus Q_{n+1}$ be defined as $\varphi_{n+1}(\psi_n(x_1 + \cdots + x_n) + x'_{n+1}) = x_1 + \cdots + x_n + \theta(x'_{n+1})$, for $x_i \in Q_i$ and $x'_{n+1} \in P'_{n+1}$. Clearly $\varphi_{n+1} \circ \psi_n = \varsigma_n$. The map $\psi_{n+1}$ is defined similarly.

Note that we have $\varphi_{n+1} \circ \iota_n = \varphi_{n+1} \circ \psi_n \circ \varphi_n = \varsigma_n \circ \varphi_n$ and similarly $\psi_{n+1} \circ \varsigma_n = \psi_{n+1} \circ \varphi_{n+1} \circ \psi_n = \iota_{n+1} \circ \psi_n$. We conclude that we can define homomorphisms $\varphi$: $P \to Q$ and $\psi$: $Q \to P$ such that $\psi \circ \varphi = \mathrm{id}_P$ and $\varphi \circ \psi = \mathrm{id}_Q$. So $P \cong Q$, as desired.

(c) Assume that $I := \operatorname{tr}(P) = \operatorname{tr}(Q)$ is a principal two-sided ideal of $R$, and let $M$ be the unique maximal ideal of $I$. Let $e = e^2 \in I \setminus M$, and note that $I = ReR$. By using Lemma 2.5, we can reduce to the case where $I = R$ and $M$ is the unique maximal ideal of $R$. Set $S = R/M$, and note that $S$ is a simple regular ring satisfying 2-comparability.

(c1) Assume that either $P/PM$ or $Q/QM$ is not finitely generated. Since $S$ is a simple regular ring with 2-comparability, we get from Corollary 2.2 that $P/PM$ and $Q/QM$ are comparable. Without loss of generality, we can assume $P/PM \lesssim Q/QM$.

There are now two cases to be considered. Assume first that $P/PM$ is not finitely generated. Clearly $P_1 \oplus \cdots \oplus P_n \prec_M Q$ for all $n \geq 1$, and we can assume that $P_i \neq P_i M$ for all $i$. By Lemma 2.4(a), there is a decomposition $Q = A_1 \oplus B_1$ such that $A_1 \cong P_1$ and $B_1 \neq B_1 M$. Since $P_1 \oplus P_2 \prec_M Q$, there is by Lemma 2.4(a) an isomorphism $P_1 \oplus P_2 \oplus X_2 \cong Q$, where $X_2 \neq X_2 M$. So we obtain $A_1 \oplus (P_2 \oplus X_2) \cong A_1 \oplus B_1$. Since both $P_2 \oplus X_2$ and $B_1$ are generators, we can apply separative cancellation [27, Theorem 2.2], to get $P_2 \oplus X_2 \cong B_1$. Therefore, we obtain a decomposition $B_1 = A_2 \oplus B_2$ such that $A_2 \cong P_2$ and $B_2 \cong X_2$. Note that in particular $B_2 \neq B_2 M$. Continuing in this way, we obtain submodules $A_n$ and $B_n$ of $Q$ such that $Q = A_1 \oplus \cdots \oplus A_n \oplus B_n$ and $A_n \cong P_n$ for all $n \geq 1$. So we get $P \cong \bigoplus_{i=1}^{\infty} P_i \cong \bigoplus_{i=1}^{\infty} A_i \leq Q$. This shows that $P \lesssim Q$, as desired.

Now, assume that $P/PM$ is finitely generated and that $Q/QM$ is infinitely generated. Then there is some $n \geq 1$ such that $P/PM \cong P_1/P_1 M \oplus \cdots \oplus P_n/P_n M$, and, since $Q/QM$ is not finitely generated, there is $m \geq 1$ such that $P_1 \oplus \cdots \oplus P_n \prec_M Q_1 \oplus \cdots \oplus Q_m$. By Lemma 2.4(a), we have $Q_1 \oplus \cdots \oplus Q_m \cong P_1 \oplus \cdots \oplus P_n \oplus C$ for some $C$ with $C \neq CM$. Since $\operatorname{tr}(P_{n+1} \oplus \cdots) \subseteq M \subset \operatorname{tr}(C)$ we conclude from (a) that $\bigoplus_{i=n+1}^{\infty} P_i \prec C$, and therefore $P = (P_1 \oplus \cdots \oplus P_n) \oplus (\bigoplus_{i=n+1}^{\infty} P_i) \prec (P_1 \oplus \cdots \oplus P_n) \oplus C \cong Q_1 \oplus \cdots \oplus Q_m \lesssim Q$, showing $P \prec Q$.

(c2) Since $P/PM$ and $Q/QM$ are both infinitely generated and $P/PM \cong Q/QM$, for each $n \geq 1$ there exist positive integers $t(n)$ and $s(n)$ such that $P_1 \oplus \cdots \oplus P_n \prec_M Q_1 \oplus \cdots \oplus Q_{t(n)}$ and $Q_1 \oplus \cdots \oplus Q_n \prec_M P_1 \oplus \cdots \oplus P_{s(n)}$. Therefore, changing notation we can assume that $P_1 \prec_M Q_1 \prec_M P_1 \oplus P_2 \prec_M Q_1 \oplus Q_2 \prec_M \dots$, and that $P_i \neq P_i M$ and $Q_i \neq Q_i M$ for all $i \geq 1$. By Lemma 2.4(b) we get $P \cong Q$.

(c3) The proof is similar to the last case in (c1). ∎

COROLLARY 2.7. (a) *Let $R$ be a regular ring satisfying 2-comparability, and let $P$ and $Q$ be countably generated projective modules. Then either $P \lesssim 2Q$ or $Q \lesssim 2P$.*

(b) *Let $R$ be a regular ring satisfying comparability, and let $P$ and $Q$ be countably generated projective modules. Then either $P \lesssim Q$ or $Q \lesssim P$.*

*Proof.* (a) If $\operatorname{tr}(P) \neq \operatorname{tr}(Q)$, then the result follows from Theorem 2.6(a) (by using comparability of ideals). So, assume that $\operatorname{tr}(P) = \operatorname{tr}(Q)$. If $\operatorname{tr}(P)$ is not a principal ideal, then the result follows from Theorem 2.6(b). So, we have reduced the problem to the case where $\operatorname{tr}(P) = \operatorname{tr}(Q)$ and $\operatorname{tr}(P)$ is a principal two-sided ideal of $R$. Let $M$ be the unique maximal ideal of $\operatorname{tr}(P)$, let $e = e^2 \in \operatorname{tr}(P) \setminus M$, and set $S = eRe/(eMe)$, a simple regular ring satisfying 2-comparability.

If either $P/PM$ or $Q/QM$ is not finitely generated, then $P$ and $Q$ are comparable by Theorem 2.6(c1), so clearly either $P \lesssim 2Q$ or $Q \lesssim 2P$.

If $P/PM$ and $Q/QM$ are both finitely generated, then $P/PM$ and $Q/QM$ are comparable in case $S$ is either artinian or purely infinite, and $P/PM$ and $Q/QM$ are almost comparable in case $S$ is directly finite and nonartinian, see [3, Corollary 4.5]. In either case we obtain that either $P/PM \prec (2Q)/(2Q)M$ or $Q/QM \prec (2P)/(2P)M$. By Theorem 2.6(c3), we get that either $P \prec 2Q$ or $Q \prec 2P$.

(b) We give a direct proof, which is a slight modification of the proof for the directly finite case, given in [18, Theorem 2.1(a)]. Assume that $P$ and $Q$ are countably generated projective modules over a regular ring with comparability. First consider the case where $P$ is finitely generated and $Q$ is infinitely generated. We can assume $P \not\lesssim Q$. Write $Q = \bigoplus_{i=1}^{\infty} Q_i$, where $Q_i \in \mathrm{FP}(R)$ for all $i$. Proceeding by induction, assume we have for some $n \geq 1$ submodules $P_1, \dots, P_n$ of $P$ such that $P = P_1 \oplus \cdots \oplus P_n \oplus P'_n$ with $P_i \cong Q_i$ for $i = 1, \dots, n$. If $P'_n \lesssim Q_{n+1}$, then $P \lesssim Q$, a contradiction, so that $Q_{n+1} \lesssim P'_n$. Write $P'_n = P_{n+1} \oplus P'_{n+1}$ with $P_{n+1} \cong Q_{n+1}$ and note that $P = P_1 \oplus \cdots \oplus P_{n+1} \oplus P'_{n+1}$. This completes the induction argument.

Assume now that both $P$ and $Q$ are infinitely generated, and assume that $Q \not\lesssim P$. Write $P = \bigoplus_{i=1}^{\infty} A_i$ with $A_i \in \mathrm{FP}(R)$ for all $i$. We have seen before that $A_1$ and $Q$ are comparable, but we cannot have $Q \lesssim A_1$, so $A_1 \lesssim Q$. Write $Q = B_1 \oplus C_1$ with $A_1 \cong B_1$. We cannot have $C_1 \lesssim A_2$, so that we have $A_1 \lesssim C_1$. Continuing in this way, we obtain submodules $\{B_n\}$ of $Q$ such that $Q = B_1 \oplus \cdots \oplus B_n \oplus C_n$ for some $C_n$, and $A_n \cong B_n$ for all $n \geq 1$. We conclude that $P \lesssim Q$, as desired. ∎

## 3. THE LATTICES OF IDEALS

Let $R$ be a regular ring. Recall from Sect. 1 that we denote by $E$ the ring $\mathrm{End}(R_R^{(\omega)}) = \mathrm{CFM}(R)$ and by $B$ the ring $\mathrm{RCFM}(R)$. For a ring $T$, we denote by $L(T)$ the lattice of (two-sided) ideals of $T$. In this section we will obtain some general information on the lattices $L(B)$ and $L(E)$, and then we will carefully study the special situation in which $R$ is a regular ring satisfying 2-comparability.

The ring $B$ is the multiplier ring of the $\sigma$-unital regular ring $F = \mathrm{FM}(R)$, and so every ideal in $B$ is generated by idempotents [6, Theorem 2.5]. Moreover, the ideals of $B$ correspond to certain subsets of $V(B)$, called order-ideals. To define them, let us consider an abelian monoid $M$, endowed with the algebraic pre-order (see Remark 1.6). An *order-ideal* of $M$ is a submonoid $S$ of $M$ such that $S$ is hereditary with respect to the algebraic pre-ordering, i.e., $y \leq x$ for $y \in M$ and $x \in S$ implies $y \in S$. We denote by $L(M)$ the lattice of order-ideals of a monoid $M$. By [6, Theorem 2.7] there is a lattice isomorphism $L(B) \cong L(V(B))$. The situation with the ring $E$ is somewhat different, since the ideals of $E$ need not be generated by idempotents. For example, let $R$ be a commutative nonartinian regular ring, so that $R$ has a sequence $(e_n)$ of nonzero orthogonal idempotents. Consider the matrix $X$ having all rows but the first one equal to zero, and with first row $[e_1, e_2, \dots]$. Then the ideal generated by $X$ in $E$ cannot be generated by idempotents. However we can exploit the fact that $E$ is an exchange ring to obtain some useful information on the ideals of $E$.

THEOREM 3.1. *Let $R$ be a regular ring. Consider the maps $\alpha\colon L(B) \to L(E)$ and $\beta\colon L(E) \to L(B)$ defined by $\alpha(I) = EIE$ and $\beta(I') = I' \cap B$ for $I \in L(B)$ and $I' \in L(E)$. Then $\beta \circ \alpha = \mathrm{Id}_{L(B)}$, so that $\alpha$ is injective and $\beta$ is surjective. Moreover, for $I \in L(B)$, we have $\beta^{-1}(I) = [\alpha(I), \pi_{\alpha(I)}^{-1}(J(E/\alpha(I)))]$, where $\pi_{\alpha(I)}\colon E \to E/\alpha(I)$ stands for the canonical projection.*

*Proof.* Let $i$: $B \to E$ be the canonical inclusion. By Theorem 1.3, the induced map $V(i)$: $V(B) \to V(E)$ is a monoid isomorphism. Hence, we get a lattice isomorphism $L(V(i))$: $L(V(B)) \to L(V(E))$. On the one hand, by [6, Theorem 2.7] we have a lattice isomorphism $L(B) \to L(V(B))$ which sends an ideal $I$ of $B$ to the order-ideal $V(I)$ of $V(B)$. On the other hand, since $E$ is an exchange ring, we obtain from [26, Teorema 4.1.7(i)] a surjective lattice homomorphism $L(E) \to L(V(E))$ sending $I' \in L(E)$ to the order ideal $V(I')$ of $V(E)$. The composition $L(E) \to L(V(E)) \to L(V(B)) \to L(B)$ gives a surjective lattice homomorphism from $L(E)$ onto $L(B)$, which is easily seen to agree with $\beta$. Now, we infer from [26, Teorema 4.1.7(ii)] that, for $I \in L(B)$, we have $\beta^{-1}(I) = [\alpha(I), \pi_{\alpha(I)}^{-1}(J(E/\alpha(I)))]$. In particular we get $\beta \circ \alpha = \mathrm{Id}_{L(B)}$. ∎

Following [27], we define $s$-comparability for a general ring $T$ in terms of its monoid $V(T)$, as follows. First, say that the monoid $M$ satisfies *$s$-comparability* if for any $p, q \in M$ either $p \leq sq$ or $q \leq sp$, where $\leq$ is the algebraic pre-order on $M$. If $T$ is any ring, say that $T$ satisfies *$s$-comparability* provided $V(T)$ satisfies $s$-comparability. Of course, *comparability* stands for 1-comparability.

Let $R$ be a regular ring satisfying 2-comparability. In view of Corollary 2.7, it seems reasonable to ask whether the rings $E$ and $B$ satisfy $s$-comparability for some $s \geq 1$. However this is not true, as can be seen from easy examples. Indeed, let $R$ be any nonsimple regular ring satisfying 2-comparability and let $I$ be a nontrivial ideal of $R$. Set $F = \mathrm{FM}(R)$ and $K = \mathrm{RCFM}(I)$, and note that $F$ and $K$ are incomparable ideals of the ring $B = \mathrm{RCFM}(R)$. Take idempotents $e \in K \setminus F$ and $f \in F \setminus K$. Then $eB$ is not isomorphic to a direct summand of $s(fB)$, and $fB$ is not isomorphic to a direct summand of $s(eB)$. So it follows that $B$ does not satisfy $s$-comparability for any positive integer $s$. Similarly, the ring $E = \mathrm{CFM}(R)$ does not satisfy $s$-comparability for any positive integer $s$. Our next result shows that things become better when we consider the rings $B/F$ and $E/G$.

THEOREM 3.2. *Let $R$ be a regular ring satisfying 2-comparability. Then $B/F$ and $E/G$ are exchange rings satisfying comparability.*

*Proof.* Since $E$ is an exchange ring, we have that $E/G$ is an exchange ring by [22, Proposition 1.4], and $V(E/G) \cong V(E)/V(G)$ by [2, Proposition 1.4]. By a recent result of O'Meara [24], $B$ is also an exchange ring, and so also $B/F$ is an exchange ring and $V(B/F) \cong V(B)/V(F)$. It follows from the above observations and Proposition 1.2 and Theorem 1.3 that $V(E/G) \cong V(B/F) \cong W(R)/V(R)$. Therefore in order to prove that $B/F$ and $E/G$ satisfy comparability, it is enough to prove that, given two countably generated projective $R$-modules $P$ and $Q$, there are finitely generated projective $R$-modules $A$ and $B$ such that either $P \oplus A \lesssim^{\oplus} Q$

$\oplus B$ or $Q \oplus B \lesssim^{\oplus} P \oplus A$. Obviously, we can assume that both $P$ and $Q$ are not finitely generated.

Instead of working in the monoid $W(R)/V(R)$ we will work with countably generated projective modules "modulo FP$(R)$".

It is easy to see that $P$ (respectively, $Q$), falls modulo FP$(R)$ into exactly one the following classes:

(a) The class of those $A \in \mathrm{CP}(R)$ such that $\mathrm{tr}(A)$ is not a principal two-sided ideal.

(b) The class of those $A \in \mathrm{CP}(R)$ which admit a decomposition $A = \bigoplus_{i=1}^{\infty} A_i$, where $A_i \in \mathrm{FP}(R)$ for all $i$ and $\mathrm{tr}(A_i) = \mathrm{tr}(A_j)$ for all $i, j$.

(c) The class of those $A \in \mathrm{CP}(R)$ admitting a decomposition $A = \bigoplus_{i=1}^{\infty} A_i$, where $A_i \in \mathrm{FP}(R)$ for all $i$ and $\mathrm{tr}(A_n) \supset \mathrm{tr}(A_{n+1})$ for all $n \geq 1$.

Assume first that $P$ and $Q$ fall in class (a). We can assume $\mathrm{tr}(P) \subseteq \mathrm{tr}(Q)$. Then there are decompositions $P = \bigoplus_{i=1}^{\infty} P_i$ and $Q = \bigoplus_{i=1}^{\infty} Q_i$, with $P_i, Q_i \in \mathrm{FP}(R)$, such that $\mathrm{tr}(P_n) \subset \mathrm{tr}(P_{n+1})$ and $\mathrm{tr}(Q_n) \subset \mathrm{tr}(Q_{n+1})$ for all $n \geq 1$, and $\mathrm{tr}(P_n) \subset \mathrm{tr}(Q_n)$ for all $n$. By [4, Proposition 2.5(b)], we have $P_n \lesssim^{\oplus} Q_n$ for all $n$, and so $P \lesssim^{\oplus} Q$. Assume that $P$ falls in class (a) and $Q$ falls in class (b). Then $\mathrm{tr}(P) \neq \mathrm{tr}(Q)$. By the same trick as in the above paragraph one obtains that either $P \lesssim^{\oplus} Q$ or $Q \lesssim^{\oplus} P$, depending on whether $\mathrm{tr}(P) \subset \mathrm{tr}(Q)$ or $\mathrm{tr}(Q) \subset \mathrm{tr}(P)$.

The case where $P$ falls in class (b) and $Q$ falls in class (c) is similar to the above case.

Assume now that $P$ falls in class (a) and $Q$ falls in class (c). Write $P = \bigoplus_{i=1}^{\infty} P_i$ and $Q = \bigoplus_{i=1}^{\infty} Q_i$, where $P_i, Q_i \in \mathrm{FP}(R)$ and $\mathrm{tr}(P_n) \subset \mathrm{tr}(P_{n+1})$ and $\mathrm{tr}(Q_n) \supset \mathrm{tr}(Q_{n+1})$ for all $n$. If $\mathrm{tr}(P_n) \supseteq \mathrm{tr}(Q_n)$ for some $n$, then $Q \lesssim^{\oplus} P$ modulo FP$(R)$. So we can assume that $\mathrm{tr}(P_n) \subset \mathrm{tr}(Q_n)$ for all $n$ and so $P_n \lesssim^{\oplus} Q_n$ for all $n$, which gives $P \lesssim^{\oplus} Q$.

Consider now the case where $P$ and $Q$ fall both in class (b). Clearly we can assume that $\mathrm{tr}(P) = \mathrm{tr}(Q)$. Let $M$ be the unique maximal ideal of $\mathrm{tr}(P)$. Since $P/PM$ and $Q/QM$ are both infinitely generated, it follows from Corollary 2.2 that $P/PM$ and $Q/QM$ are comparable with respect to $\lesssim^{\oplus}$. Assume that $P/PM \lesssim^{\oplus} Q/QM$. Then there is $T \in \mathrm{CP}(R)$ such that $(P \oplus T)/(P \oplus T)M \cong Q/QM$. By Theorem 2.6(c2), we get $P \oplus T \cong Q$ and so $P \lesssim^{\oplus} Q$.

Finally consider the case where both $P$ and $Q$ fall in class (c). Write $P = \bigoplus_{i=1}^{\infty} P_i$ and $Q = \bigoplus_{i=1}^{\infty} Q_i$, with $\mathrm{tr}(P_n) \supset \mathrm{tr}(P_{n+1})$ and $\mathrm{tr}(Q_n) \supset \mathrm{tr}(Q_{n+1})$ for all $n \geq 1$. We can assume that $\bigcap_{i=1}^{\infty} \mathrm{tr}(P_i) \subseteq \bigcap_{i=1}^{\infty} \mathrm{tr}(Q_i)$. If $\mathrm{tr}(Q_1) \subseteq \mathrm{tr}(P_i)$ for all $i$, then $\mathrm{tr}(Q_1) \subseteq \bigcap_{i=1}^{\infty} \mathrm{tr}(P_i) \subseteq \bigcap_{i=1}^{\infty} \mathrm{tr}(Q_i)$, a contradiction. So there is $i_1$ such that $\mathrm{tr}(P_{i_1}) \subset \mathrm{tr}(Q_1)$. Now by the same argument, there is $i_2 > i_1$ such that $\mathrm{tr}(P_{i_2}) \subset \mathrm{tr}(Q_2)$. In this way we obtain a strictly increasing sequence $i_1 < i_2 < \cdots$ such that $\mathrm{tr}(P_{i_n}) \subset \mathrm{tr}(Q_n)$ for all $n$. Now note that since $\operatorname{tr}(P_{i_n} \oplus \cdots \oplus P_{i_{n+1}-1}) \subset \operatorname{tr}(Q_n)$ we get $P_{i_n} \oplus \cdots \oplus P_{i_{n+1}-1} \lesssim^{\oplus} Q_n$ and so we get $\bigoplus_{j=i_1}^{\infty} P_j \lesssim^{\oplus} \bigoplus_{j=1}^{\infty} Q_j$, showing that $P \lesssim^{\oplus} Q$ modulo $\operatorname{FP}(R)$. ∎

For a ring $T$, denote by $L_0(T)$ (respectively, $L_1(T)$) the subset of $L(T)$ consisting of the ideals of $T$ which are generated by idempotents (respectively, the semiprimitive ideals of $T$).

COROLLARY 3.3. *Let $R$ be a regular ring satisfying 2-comparability. Then $L(B/F)$ is a totally ordered lattice, and $L_0(E/G)$ and $L_1(E/G)$ are totally ordered subsets of $L(E/G)$.*

*Proof.* By Theorem 3.2, $B/F$ satisfies comparability, so by [27, Lemma 1.5] the lattice $L(V(B/F))$ is totally ordered by inclusion. Now we have observed in the proof of Theorem 3.2 that $V(B/F) \cong V(B)/V(F)$. Therefore, by using [6, Theorem 2.7], we obtain a lattice isomorphism $L(B/F) \to L(V(B/F))$ which sends $I/F$ to $V(I)/V(F)$. We conclude that $L(B/F)$ is totally ordered by inclusion.

Since $E$ is an exchange ring, the idempotents of $E/G$ lift to idempotents of $E$, so that $L_0(E/G) = \{I/G \mid I \in L_0(E) \text{ and } G \subseteq I\}$. Consider the map $\alpha\colon L(B) \to L(E)$ defined in Theorem 3.1. Since $\alpha(F) = G$, we see from Theorem 3.1 that $\alpha$ induces an order-preserving bijection from $L(B/F)$ onto $L_0(E/G)$. Now, it follows from Theorem 3.1 that the map $L_0(E/G) \to L_1(E/G)$ given by the rule $I/G \mapsto \pi_I^{-1}(J(E/I))/G$, where $\pi_I\colon E \to E/I$, is an order-preserving bijection from $L_0(E/G)$ onto $L_1(E/G)$. It follows that $L(B/F) \cong L_0(E/G) \cong L_1(E/G)$ as posets, which gives the desired result. ∎

COROLLARY 3.4. *Let $R$ be a regular ring satisfying 2-comparability. Then $E$ and $B$ are both separative exchange rings.*

*Proof.* By Theorem 1.3 we have $V(B) \cong V(E)$, so that it suffices to prove that $E$ is a separative ring (since by definition, a ring $T$ is separative if and only if $V(T)$ is a separative monoid, see [2, Sect. 2]). Since $R$ is a regular ring satisfying 2-comparability, we see from [27, Theorem 2.2] that $R$ is separative and so $G$ is a separative ideal of $E$ (because $V(G) \cong V(R)$). By Theorem 3.2, $E/G$ is an exchange ring satisfying comparability, and so $E/G$ is separative as well by [27, Theorem 2.2]. It follows from the Extension Theorem [2, Theorem 4.2] that $E$ is a separative ring. ∎

## 4. PROPERTY (DF)

Following Kutami [16], we say that a ring $S$ satisfies property (DF) provided $P \oplus Q$ is directly finite for all directly finite projective right

$S$-modules $P$ and $Q$. In this section we will prove that the strictly unperforated, simple regular rings which satisfy property (DF) are exactly the simple regular rings satisfying 2-comparability (Theorem 4.4). We will also determine the structure of the directly finite projective modules over regular rings with 2-comparability, and we will apply this result to show that these rings satisfy property (DF) (Corollary 4.7).

We start with some general stuff. Kaplansky's classical result [14], stating that every projective module is a direct sum of countably generated ones, suggests that property (DF) could be equivalent to the statement that finite direct sums of countably generated directly finite projectives are again directly finite. This is indeed the case, as we prove below.

LEMMA 4.1. *Let $S$ be any ring and let $P$ be a projective $S$-module. If $P$ is directly infinite, then there is a countably generated direct summand of $P$ which is also directly infinite.*

*Proof.* Let $P$ be a directly infinite projective module. By Kaplansky's theorem, there exist nonzero submodules $X$ and $P_1$ of $P$ such that $P = X \oplus P_1$, $X$ is countably generated, and there is an injective homomorphism $\varphi: P \to P_1$ such that $\varphi(P)$ is a direct summand of $P_1$.

By Kaplansky's theorem, we have $P_1 = \bigoplus_{i \in I} C_i$, where $C_i$ are countably generated submodules of $P_1$. For any subset $L \subseteq I$, put $C_L = \bigoplus_{i \in L} C_i$. Since $X$ is countably generated, there is a countable subset $I_0$ of $I$ such that $\varphi(X) \subseteq C_{I_0}$. Since $X \oplus C_{I_0}$ is countably generated, there is a countable subset $I_1$ of $I$ such that $I_0 \subseteq I_1$ and $\varphi(X \oplus C_{I_0}) \subseteq C_{I_1}$. Continuing in this way, we obtain a sequence $(I_n)$ of countable subsets of $I$ such that $I_n \subseteq I_{n+1}$ and $\varphi(X \oplus C_{I_n}) \subseteq C_{I_{n+1}}$ for all $n \geq 0$. Set $J = \bigcup_{n=0}^{\infty} I_n$, a countable subset of $I$. Set $P' = X \oplus C_J$, and note that $P'$ is a countably generated direct summand of $P$. It remains to prove that $P'$ is directly infinite. Clearly $\varphi(X \oplus C_J) \subseteq C_J$. Since $\varphi$ is an injective homomorphism from $P$ onto a direct summand of $P_1$ and $X \oplus C_J$ is a direct summand of $P$, we conclude that $\varphi(X \oplus C_J)$ is a direct summand of $P_1$. Therefore, we see from the modular law that $\varphi(X \oplus C_J)$ is a direct summand of $C_J$. This proves that $X \oplus C_J$ is directly infinite, as desired. ∎

PROPOSITION 4.2. *Let $S$ be any ring. Then $S$ satisfies property* (DF) *if and only if, for every directly finite countably generated projective modules $P$ and $Q$, the direct sum $P \oplus Q$ is also directly finite.*

*Proof.* Assume that the class of directly finite countably generated projectives is closed under finite direct sums. Let $P$ and $Q$ be projective right $R$-modules such that $P \oplus Q$ is directly infinite. By Lemma 4.1, there is a countably generated direct summand $A$ of $P \oplus Q$ such that $A$ is directly infinite. By Kaplansky's theorem, there are countably generated direct summands $P_1$ and $Q_1$ of $P$ and $Q$, respectively, such that $P_1 \oplus Q_1$

$= A \oplus B$ for some $B$. Since $A$ is directly infinite, $A \oplus B$ is directly infinite and so, either $P_1$ or $Q_1$ is directly infinite by hypothesis. Therefore, either $P$ or $Q$ is directly infinite, and $S$ satisfies property (DF). ∎

Our next goal in this section is to characterize the simple, strictly unperforated, regular rings which satisfy property (DF). We remark that there are no known examples of simple regular rings which do not satisfy strict unperforation. We need a technical lemma.

LEMMA 4.3. *Let $K$ be a Choquet simplex and let $s$ and $t$ be two distinct extreme points of $K$. Then there exist $f_1, f_2 \in \mathrm{LAff}_\sigma(K)^{++}$ such that $f_1(s) = f_2(t) = 1$ and $f_1 + f_2 = \infty$.*

*Proof.* Consider the discrete compact subset $\{s, t\}$ of $\partial_e(K)$, the extreme boundary of $K$. Define a continuous function $g_0\colon \{s, t\} \to \mathbb{R}$ by $g_0(s) = 0$ and $g_0(t) = 1$. By [9, Theorem 11.14] there exists $g \in \mathrm{Aff}(K)$ such that $0 \le g \le 1$ and $g(s) = g_0(s) = 0$ and $g(t) = g_0(t) = 1$.

Write $F_1 = g^{-1}(\{0\})$. By [9, Lemma 5.16], $F_1$ is a closed face of $K$. Note that $s \in F_1$ and $t \notin F_1$. Set $F_2 = \{t\}$. Then $F_1$ and $F_2$ are disjoint closed faces of $K$. Define $g_i \in \mathrm{Aff}(F_i)$ for $i = 1, 2$ by setting $g_1 = 1$ and $g_2 = 0$. By [9, Theorem 11.22] there exists $h \in \mathrm{Aff}(K)$ such that $0 \le h \le 1$ and $h_{|F_i} = g_i$ for $i = 1, 2$. Note that $h(t) = 0$ and $h(s) = 1$.

Now define $\bar{g} = \sup_{n \in \mathbb{N}} ng$ and $\bar{h} = \sup_{n \in \mathbb{N}} nh$. Since $\bar{g}$ and $\bar{h}$ are pointwise suprema of sequences of continuous affine functions on $K$, we have $\bar{g}, \bar{h} \in \mathrm{LAff}_\sigma(K)$. Note that $\bar{g}$ and $\bar{h}$ only take the values 0 and $\infty$, and $\bar{g}(x) = 0$ (respectively, $h(x) = 0$) if and only if $g(x) = 0$ (respectively, $h(x) = 0$). Let us see that $\bar{g} + \bar{h} = \infty$. Take first $x \in F_1$. Then $h(x) = 1$ and so $\bar{h}(x) = \infty$. Take now $x \notin F_1 (= g^{-1}(\{0\}))$. Then $g(x) > 0$ and so $\bar{g}(x) = \infty$. This shows that $\bar{g} + \bar{h} = \infty$.

Finally, set $f_1 = \bar{g} + 1$ and $f_2 = \bar{h} + 1$. Then $f_1, f_2 \in \mathrm{LAff}_\sigma(K)^{++}$ and $f_1(s) = f_2(t) = 1$ and $f_1 + f_2 = \infty$, are required. ∎

THEOREM 4.4. *Let $R$ be a simple, strictly unperforated, regular ring. Then $R$ satisfies property* (DF) *if and only if $R$ satisfies 2-comparability.*

*Proof.* Assume first that $R$ is directly infinite. Then $R$ satisfies comparability. In fact, $R$ satisfies the following property, which clearly implies comparability: Given two nonzero elements $x, y \in R$ then $xR \prec yR$. To see this, let $x$ and $y$ be two nonzero elements of $R$. By simplicity, there is $n \ge 1$ such that $R_R \lesssim n(yR_R)$, so that $n(yR_R)$ is directly infinite. Hence, there is a nonzero $z \in R$ such that $m(zR_R) \lesssim n(yR_R)$ for all $m \ge 1$, and again by simplicity of $R$, we obtain that $A \prec n(yR_R)$ for every $A \in \mathrm{FP}(R)$. In particular, we have $n(xR_R) \prec n(yR_R)$ and so, since $R$ is strictly unperforated, we get $xR \prec yR$. It follows from this strong form of comparability that every nonzero cyclic projective module is directly infinite, and so, by

[14, Theorem 4], every nonzero projective module is directly infinite. Therefore $R$ satisfies property (DF) in this case.

If $R$ is artinian, then $W(R) \cong \mathbb{Z}^+ \sqcup \{\infty\}$, so property (DF) is clear in this case, as is comparability.

Finally, assume that $R$ is nonartinian and directly finite. By [3, Theorem 4.3] and [23, Theorem 1], $R$ is unit-regular. By Theorem 2.1, there exists a monoid isomorphism $\mu$: $W(R) \to V(R) \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$. By [3, Corollary 4.5] and Proposition 4.2, it suffices to see that the directly finite elements of the monoid $M := V(R) \sqcup \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$ form a submonoid if and only if $\mathbb{P}(R)$ is a singleton. If $\mathbb{P}(R)$ is a singleton, then $M = V(R) \sqcup \mathbb{R}^{++} \sqcup \{\infty\}$, so the set of directly finite elements is $V(R) \sqcup \mathbb{R}^{++}$, which is a submonoid of $M$. Assume now that $\mathbb{P}(R)$ is not a singleton. Let $\phi$: $V(R) \to \mathrm{Aff}(\mathbb{P}(R))^+$ be the natural map, defined before Theorem 2.1. Since $R$ is a simple ring, $V(R)$ is a simple monoid and so $\phi(x) \gg 0$ for all nonzero $x \in V(R)$. It follows from this fact and the definition of the semigroup operation in $M$ (see Sect. 2) that the only directly infinite element of $M$ is the constant function $\infty$ on $\mathbb{P}(R)$. By the Krein–Milman theorem there are two different extreme points in $\mathbb{P}(R)$, say $N_1$ and $N_2$. Now $\mathbb{P}(R)$ is a Choquet simplex by [8, Theorem 17.5], and so using Lemma 4.3 we get functions $f_1, f_2 \in \mathrm{LAff}_\sigma(\mathbb{P}(R))^{++}$ such that $f_1(N_1) = f_2(N_2) = 1$ and $f_1 + f_2 = \infty$.

Therefore $f_1$ and $f_2$ are directly finite elements of $M$, but $f_1 + f_2 = \infty$, which is directly infinite. ∎

In [16], Kutami showed that a unit-regular ring $R$ with 2-comparability satisfies property (DF). By using this, he completely characterized the directly finite projective $R$-modules. We will extend Kutami's results to the general case of regular rings with 2-comparability. Note that directly finite regular rings with 2-comparability are not necessarily unit-regular by [4, Example 3.2], and therefore our extension is proper even in the directly finite case.

Although we will use some of the ideas of Kutami [16], we will proceed in a self-contained manner, characterizing in the first place the directly finite projective modules over a regular ring with 2-comparability.

Let $R$ be a regular ring satisfying 2-comparability having a minimal ideal $I_0$, and fix a nonzero idempotent $e_0$ in $I_0$. Suppose that $e_0 R e_0$ is nonartinian and directly finite. Since $e_0 R e_0$ has 2-comparability, there is a unique pseudo-rank function $N$ on $e_0 R e_0$ (see the proof of Corollary 2.2(b)), and so by Corollary 2.2(c) there is a unique countably additive map $t$: $W(e_0 R e_0) \to \mathbb{R}^+ \sqcup \{\infty\}$ such that $t([g(e_0 R e_0)]) = N(g)$ for idempotents $g \in e_0 R e_0$. By Lemma 2.5, there is a categorical equivalence $\mathrm{CP}(I_0) \to \mathrm{CP}(e_0 R e_0)$. Therefore we get a function $D$: $\mathrm{CP}(I_0) \to \mathbb{R}^+ \sqcup \{\infty\}$ inducing a countably additive map $W(I_0) \to \mathbb{R}^+ \sqcup \{\infty\}$, and with $D(e_0 R) = 1$. We call

$D$ the *dimension function* on $I_0$ normalized at $e_0R$. For related material, see [13].

LEMMA 4.5. *Let $R$ be a regular ring satisfying 2-comparability. Let $P$ and $Q$ be finitely generated projective right $R$-modules.*

(a) *If $P$ and $Q$ are directly finite, then so is $P \oplus Q$.*

(b) *If $I$ is a two-sided ideal of $R$, and $P$ is directly finite, then so is $P/PI$.*

(c) *Assume that $R$ has a minimal ideal $I_0$. Assume that $P$ is directly finite, and let $X$ be a directly finite countably generated projective module such that $X = XI_0$. Then $P \oplus X$ is directly finite.*

*Proof.* (a) Consider the ideal $J = \mathrm{tr}(P \oplus Q)$. Then, the arguments used in the proof of Theorem 2.6(c) allow us to assume that $R = J$. Note that either $R = \mathrm{tr}(P)$ or $R = \mathrm{tr}(Q)$ by comparability of ideals. We can assume $R = \mathrm{tr}(P)$. Then $\mathrm{End}(P_R)$ is directly finite and $R$ is Morita equivalent to it. By [4, Corollary 4.7] $R$ is stably finite. So $P \oplus Q$ is directly finite.

(b) By using the same argument as in (a), we can assume that $R$ is directly finite. By [4, Corollary 4.7], $R$ and all its factor rings are stably finite. Consequently, $P/PI$ is directly finite.

(c) Let $e_0$ be a nonzero idempotent in $I_0$, and note that $e_0Re_0$ is a simple regular ring satisfying 2-comparability. If $e_0Re_0$ is directly infinite, then every nonzero $Z \in \mathrm{FP}(e_0Re_0)$ is directly infinite, by the first part of the proof of Theorem 4.4, which applies to $e_0Re_0$ because it is strictly unperforated (Remark 2.3(c)). It follows that the only directly finite projective $R$-module is 0. If $e_0Re_0$ is artinian, then $X$ must be finitely generated, so $P \oplus X$ is directly finite by (a). So we can assume that $e_0Re_0$ is directly finite and nonartinian. Assume that $P \oplus X$ is directly infinite. Then there is a nonzero cyclic right ideal $Y$ such that $YI_0 = Y$ with $P \oplus X \oplus Y \lesssim P \oplus X$. Write $X = \bigoplus_{i=1}^{\infty} X_i$ for $X_i \in \mathrm{FP}(R)$. Let $D$ be the dimension function on $I_0$ normalized at $e_0R$. Since $X$ is directly finite, we must have $D(X) = \sum D(X_i) < \infty$, so there is $n_0 \geq 1$ such that $D(X_{n_0+1} \oplus \cdots \oplus X_n) < D(Y)$ for all $n > n_0$. By [3, Corollary 4.5], we obtain $X_{n_0+1} \oplus \cdots \oplus X_n \prec Y$ for all $n > n_0$. Since $P \oplus X \oplus Y \lesssim P \oplus X$, there exists $m \geq n_0$ such that $P \oplus Y \oplus X_1 \oplus \cdots \oplus X_{n_0} \lesssim P \oplus X_1 \oplus \cdots \oplus X_m$. Therefore

$$P \oplus X_1 \oplus \cdots \oplus X_m \prec P \oplus X_1 \oplus \cdots \oplus X_{n_0} \oplus Y \lesssim P \oplus X_1 \oplus \cdots \oplus X_m,$$

showing that $P \oplus X_1 \oplus \cdots \oplus X_m$ is directly infinite, in contradiction to (a).
∎

Now we are ready to describe all the directly finite countably generated projective modules over a regular ring with 2-comparability.

THEOREM 4.6. *Let $R$ be a regular ring satisfying 2-comparability.*

(a) *Assume that $R$ has a minimal ideal $I_0$. Then the directly finite countably generated projective modules are the modules of the form $P \oplus Q$, where $P$ is a directly finite finitely generated projective module and $Q$ is a countably generated directly finite projective module such that $Q = QI_0$.*

(b) *Assume that $R$ does not have a minimal ideal. Then the directly finite countably generated projective modules which are not finitely generated are the modules of the form $P = \bigoplus_{i=1}^{\infty} P_i$, where $P_i$ are directly finite finitely generated projective modules and $\operatorname{tr}(P_{i+1}) \subset \operatorname{tr}(P_i)$ for all $i$, and $\bigcap_{i=1}^{\infty} \operatorname{tr}(P_i) = 0$.*

*Proof.* Let $P$ be a directly finite countably generated projective module, and let $P = \bigoplus_{i=1}^{\infty} P_i$, where $P_i \in \mathrm{FP}(R)$. We assume that $P$ is not finitely generated and $P_i \neq 0$ for all $i$. Let $\Gamma_1 = \{i \in \mathbb{N} \mid \operatorname{tr}(P_1) \subset \operatorname{tr}(P_i)\}$. By Theorem 2.6(a), $P_1 \prec P_i$ for all $i \in \Gamma_1$. If $\Gamma_1$ is infinite, then $\aleph_0 P_1 \lesssim^{\oplus} P$, and so $P$ is directly infinite, a contradiction. So $\Gamma_1$ is finite, and collecting in the first position all $P_i$'s with $i \in \Gamma_1$, we can assume that $\operatorname{tr}(P_j) \subseteq \operatorname{tr}(P_1)$ for all $j$. Applying the same argument to $P_2$ and the indexes $\geq 2$, we can assume as well that $\operatorname{tr}(P_1) \supseteq \operatorname{tr}(P_2) \supseteq \operatorname{tr}(P_j)$ for all $j > 2$. Continuing in this way, we see that, without loss of generality, we can assume that the decomposition $P = \bigoplus_{i=1}^{\infty} P_i$ satisfies $\operatorname{tr}(P_i) \supseteq \operatorname{tr}(P_{i+1})$ for all $i$. Assume first that the sequence $\operatorname{tr}(P_1) \supseteq \operatorname{tr}(P_2) \supseteq \cdots$ stabilizes. Then there is $n_0$ such that $\operatorname{tr}(P_{n_0}) = \operatorname{tr}(P_n)$ for all $n \geq n_0$. Write $I_0 = \operatorname{tr}(P_{n_0})$, and note that $I_0$ is a nonzero principal ideal of $R$. If $I_0$ is not a minimal ideal there exists a nonzero $a \in I_0$ such that $RaR \subset I_0$. By Theorem 2.6(a) we then have $aR \lesssim P_i$ for all $i$, and so $P$ is directly infinite. So $I_0$ must be a minimal ideal of $R$. Write $Q = \bigoplus_{n=n_0}^{\infty} P_n$. Then $Q$ is a directly finite countably generated projective module such that $Q = QI_0$, and $P = P' \oplus Q$, where $P' = P_1 \oplus \cdots \oplus P_{n_0 - 1}$ is a directly finite finitely generated projective module. So we showed that $P$ is as in (a) if the chain $\operatorname{tr}(P_1) \supseteq \operatorname{tr}(P_2) \supseteq \cdots$ stabilizes. Assume now that that sequence does not stabilize. By a new arrangement of terms we can then assume that $\operatorname{tr}(P_i) \supset \operatorname{tr}(P_{i+1})$ for all $i$. Write $I = \bigcap_{i=1}^{\infty} \operatorname{tr}(P_i)$. If $I \neq 0$, then we get a contradiction as before. So $I = 0$ and $P$ is as in (b) in this case.

It remains to prove that the modules in (a) and (b) are directly finite. Assume first that $R$ has a minimal ideal $I_0$, and let $P \oplus Q$ be a module as in (a). Then the result follows from Lemma 4.5(c).

Finally assume that $R$ does not have a minimal ideal, and let $P = \bigoplus_{i=1}^{\infty} P_i$, where $P_i$ are directly finite finitely generated projective modules with $\operatorname{tr}(P_i) \supset \operatorname{tr}(P_{i+1})$ for all $i$, and $\bigcap_{i=1}^{\infty} \operatorname{tr}(P_i) = 0$. Assume that $P \oplus X$

$\lesssim^{\oplus} P$ for some nonzero $X \in \mathrm{FP}(R)$. Write $J = \mathrm{tr}(X)$. By comparability of ideals, there is $n_0$ such that $\mathrm{tr}(P_n) \subset J$ for all $n > n_0$ (otherwise, $0 \neq J \subseteq \bigcap_{i=1}^{\infty} \mathrm{tr}(P_i) = 0$). Let $M$ be the unique maximal ideal of the principal ideal $J$. Then we get

$$(*) \qquad P/PM \oplus X/XM \lesssim^{\oplus} P/PM.$$

Now $P/PM = P_1/P_1M \oplus \cdots \oplus P_{n_0}/P_{n_0}M$ is directly finite by Lemma 4.5(a)(b), and $X/XM \neq 0$, so $(*)$ gives a contradiction. Therefore, all the modules in (b) are directly finite, as desired. ∎

COROLLARY 4.7 ([19, Theorem 2.12]). *Let $R$ be a regular ring satisfying 2-comparability. Then $R$ satisfies property* (DF).

*Proof.* Assume first that $R$ has a minimal ideal $I_0$, and let $e_0$ be a nonzero idempotent in $I_0$. On the one hand, it follows from Lemma 4.5(a) that the finite direct sums of directly finite modules in $\mathrm{FP}(R)$ are again directly finite. On the other hand, we have $\mathrm{CP}(I_0) \cong \mathrm{CP}(e_0Re_0)$ by Lemma 2.5, and, since $e_0Re_0$ is a simple regular ring satisfying 2-comparability, property (DF) holds for $e_0Re_0$ by Theorem 4.4. Therefore, the finite direct sums of directly finite modules in $\mathrm{CP}(I_0)$ are again directly finite. It follows from Theorem 4.6(a) and Proposition 4.2 that $R$ satisfies property (DF).

Assume now that $R$ does not have a minimal ideal. Let $P = \bigoplus_{n=1}^{\infty} P_n$ and $Q = \bigoplus_{n=1}^{\infty} Q_n$ be as in Theorem 4.6(b). Then $P \oplus Q = \bigoplus_{n=1}^{\infty}(P_n \oplus Q_n)$, and $P_n \oplus Q_n$ are directly finite by Lemma 4.5(a). It is easily checked that $\mathrm{tr}(P_{n+1} \oplus Q_{n+1}) \subset \mathrm{tr}(P_n \oplus Q_n)$ for all $n$, and that $\bigcap_{n=1}^{\infty} \mathrm{tr}(P_n \oplus Q_n) = 0$. So $R$ satisfies property (DF) by Theorem 4.6(b) and Proposition 4.2. ∎

*Remark* 4.8. (a) Let $R$ be a regular ring with 2-comparability. Since $R$ satisfies property (DF) by Corollary 4.7, the proof of [16, Proposition 4] applies to show that every noncountably generated projective $R$-module is directly infinite (see also [19, Proposition 2.6] for a direct proof of this fact). So Theorem 4.6 describes in fact all the directly finite projective $R$-modules.

(b) Kutami gives in [16] a classification of unit-regular rings satisfying 2-comparability in three classes (A), (B), (C), according with the possible types of directly finite projective modules. As in Theorem 4.6, these types are reflected in the ideal structure of the ring, see [16, Sect. 4]. A similar classification could be established by using Theorem 4.6 for a general regular ring satisfying 2-comparability. So, for example, the regular rings with 2-comparability such that every countably generated directly finite projective is finitely generated are those such that either $\mathrm{Soc}(R_R) \neq 0$, or there are no nonzero directly finite cyclic projectives, or there are neither minimal ideals nor sequences $\{I_n\}_{n=1}^{\infty}$ of nonzero ideals of $R$ such that $\bigcap_{i=1}^{\infty} I_n = 0$.

(c) It is easy to give examples of rings satisfying the hypothesis of Theorem 4.6(a). We refer the reader to [19, Example 3] for examples of rings satisfying the hypothesis of Theorem 4.6(b).

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