Mathematical Models and Methods in Applied Sciences
Vol. 10, No. 2 (2000) 247–260

EXISTENCE OF SOLUTION TO NONLINEAR ELLIPTIC SYSTEMS ARISING IN TURBULENCE MODELLING

MACARENA GÓMEZ MÁRMOL*

Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, C/Tarfia, s/n, 41080 Sevilla, Spain

FRANCISCO ORTEGÓN GALLEGO†

Departamento de Matemáticas, Universidad de Cádiz, CASEM, Campus del Río San Pedro, 11510 Puerto Real, Cádiz, Spain

Communicated by G. P. Galdi
Received 2 November 1998

We study some nonlinear elliptic systems governing the steady-state of a two-equation turbulence model that has been derived from the so-called k–ε model. Two kinds of problems are considered: in the first one, we drop out transport terms and we deduce the existence of a solution for $N \geq 2$; in the second one we take into account all transport terms; in this case, the existence result holds for $N = 2$ or 3. Positivity and L^∞-regularity of the scalar quantities are also shown here.

1. Introduction

Mathematical problems arising in turbulence modelling have led to major challenges from both standpoints, theoretical and numerical. In general, these problems involve systems of nonlinear partial differential equations describing the balance of mean quantities (velocity, pressure, density and internal energy) coupled with mean turbulent quantities (kinetic energy or dissipation, for instance).

One of the turbulence models which scientists have paid much attention is the so-called two equations k–ε model, introduced in 1972 by Launder and Jones.[7,11] In the incompressible case, this model is written in terms of the mean velocity field u, the mean pressure p, the mean kinetic energy k and the mean rate of viscous dissipation ε, as the system

*E-mail: macarena@numerpc1.us.es
†E-mail: francisco.ortegon@uca.es

$$\begin{cases} u_{,t} + (u \cdot \nabla)u + \nabla p - \nabla \cdot \left[\left(\nu_0 + c_1 \dfrac{k^2}{\varepsilon}\right)(\nabla u + \nabla u^{\mathrm{T}})\right] = f\,, \\ \nabla \cdot u = 0\,, \\ k_{,t} + u\nabla k - \nabla \cdot \left[\left(\nu_0 + c_1 \dfrac{k^2}{\varepsilon}\right)\nabla k\right] = c_1 \dfrac{k^2}{\varepsilon}|\nabla u + \nabla u^{\mathrm{T}}|^2 - \varepsilon\,, \\ \varepsilon_{,t} + u\nabla \varepsilon - \nabla \cdot \left[\left(\nu_0 + c_4 \dfrac{k^2}{\varepsilon}\right)\nabla \varepsilon\right] = \dfrac{1}{2}c_2 k|\nabla u + \nabla u^{\mathrm{T}}|^2 - c_3 \dfrac{\varepsilon^2}{k}\,, \end{cases}$$

where ν_0 is the viscosity of the fluid and $c_1, \ldots, c_4$ are positive constant values obtained via experimentation; also, f stands for an external force and $|\cdot|$ is the Euclidean norm.

This model has been used by physicists and engineers leading to some extent to satisfactory results. However, like all turbulence models, it lacks generality and many variants have been introduced ever since. But from the mathematical point of view, this system has not yet been solved and very few may be said about it. Some numerical tests have shown[10,11] that the model became unstable in the sense that k may blow-up whereas ε attained negative values, which is physically unacceptable. In an attempt to stabilize the system, in the early '90s Mohammadi proposed a new approach based on the introduction of two new quantities, namely $\theta = k\varepsilon^{-1}$ and $\varphi = k^{-3}\varepsilon^2$. Then, he derived the respective transport equations for both θ and φ from those of k and ε. Neglecting higher order derivatives terms, the θ–φ model may be written as[9,10]

$$\begin{cases} u_{,t} + (u \cdot \nabla)u + \nabla p - \nabla \cdot [A(\theta, \varphi)(\nabla u + \nabla u^{\mathrm{T}})] = f\,, \\ \nabla \cdot u = 0\,, \\ \theta_{,t} + u\nabla \theta - \nabla \cdot [A(\theta, \varphi)\nabla \theta] = -c_1' \theta^2 |\nabla u + \nabla u^{\mathrm{T}}|^2 + c_2'\,, \\ \varphi_{,t} + u\nabla \varphi - \nabla \cdot [A(\theta, \varphi)\nabla \varphi] = -\varphi\left(c_3'|\nabla u + \nabla u^{\mathrm{T}}|^2 + \dfrac{c_4'}{\theta}\right)\,, \end{cases}$$

where $A(\theta, \varphi) = (\nu_0 + \frac{c_1}{\theta\varphi})I$ and the constants $c_j' > 0$, $j = 1, \ldots, 4$. A reduced version of this system was first studied by Lewandowski and Mohammadi[9] where it was assumed that $u \in L^\infty(0, T; W^{1,\infty}(\Omega))$ is a given data (not necessarily verifying the Reynolds equation) and subject to the constraints

$$\operatorname*{ess\,inf}_{\Omega \times (0,T)} |\nabla u + \nabla u^{\mathrm{T}}| > 0\,, \quad \operatorname*{ess\,sup}_{\Omega \times (0,T)} |\nabla u + \nabla u^{\mathrm{T}}| \leq \frac{1}{a}\,.$$

Recently, the analysis of the transient system has been carried out by the authors[3–5]; it is shown that the regularity of both θ and φ is strongly related to the summability $L^1(\Omega)$–$L^\infty(\Omega)$ of the initial conditions imposed on θ and φ.

The goal of this paper is to show some existence results for two kinds of elliptic systems governing the steady-state of models like the previous one, namely, the Stokes-like problem:

$$\begin{cases} -\nabla \cdot (A(\theta,\varphi)\nabla u) + \nabla p = f\,, \quad \nabla \cdot u = 0\,, & \text{in } \Omega \\ -\nabla \cdot (A(\theta,\varphi)\nabla \theta) = 1 - \theta g_\theta(\theta,\varphi,\nabla u)\,, & \text{in } \Omega \\ -\nabla \cdot (A(\theta,\varphi)\nabla \varphi) = -\varphi g_\varphi(\theta,\varphi,\nabla u)\,, & \text{in } \Omega \\ u = 0\,, \quad \theta = a\,, \quad \varphi = b\,, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

and the Navier–Stokes like problem:

$$\begin{cases} (u\cdot\nabla)u - \nabla \cdot (A(\theta,\varphi)\nabla u) + \nabla p = f\,, \quad \nabla \cdot u = 0\,, & \text{in } \Omega \\ u\nabla\theta - \nabla \cdot (A(\theta,\varphi)\nabla \theta) = 1 - \theta g_\theta(\theta,\varphi,\nabla u)\,, & \text{in } \Omega \\ u\nabla\varphi - \nabla \cdot (A(\theta,\varphi)\nabla \varphi) = -\varphi g_\varphi(\theta,\varphi,\nabla u)\,, & \text{in } \Omega \\ u = 0\,, \quad \theta = a\,, \quad \varphi = b\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

Remarks. 1. Notice that with the choices

$$A(\theta,\varphi) = \left(\nu_0 + \frac{c_1}{\theta\varphi + r}\right) I\,,$$

$$g_\theta(\theta,\varphi,\nabla u) = \theta|\nabla u|^2\,,$$

and

$$g_\varphi(\theta,\varphi,\nabla u) = \theta|\nabla u|^2 + \frac{1}{\theta + r}$$

we recover the steady-state of the θ–φ model, though we have slightly modified the definition of A and g_φ by inserting a small parameter $r > 0$, in order to avoid zero denominator.

Here, we shall consider general expressions for these three nonlinear terms.

2. As we are just considering our analysis from the mathematical point of view, we may simplify the expression $\nabla u + \nabla u^{\mathrm{T}}$ to ∇u, and the existence results (Theorems 1–4) will still hold due to Korn's inequality.[1] Also, all experimental constants have been taken equal to one. In numerical tests, we must consider the original expression of the system.

3. System (1.1) lacks all transport terms $(u\cdot\nabla)u$, $u\nabla\theta$, $u\nabla\varphi$ we may find in (1.2); in this sense, (1.2) is more realistic than (1.1), but the mathematical analysis of system (1.1) is also interesting.

2. Functional Spaces

In the sequel, we will make use of the following standard notations: $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lispchitz boundary $\partial\Omega$, $N \geq 2$ being the space dimension.

$$\mathcal{D}(\Omega) \stackrel{\text{def}}{=} \text{space of } C^\infty \text{ functions with compact support in } \Omega\,.$$

$$H^1(\Omega) \stackrel{\text{def}}{=} \{v \in L^2(\Omega) : \nabla v \in L^2(\Omega)^N\}\,,$$

here $\nabla v = (\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N})^{\mathrm{T}}$ is the gradient of v, all derivatives being taken in the distribution sense in Ω; this is a Hilbert space endowed with the inner product

$$(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)}, \quad (u,v)_{L^2(\Omega)} = \int_{\Omega} u(x)v(x)\,dx$$

(dx indicates Lebesgue's measure on $\mathbb{R}^N$, we will drop out this symbol hereafter)

$$H_0^1(\Omega) \stackrel{\text{def}}{=} \text{closure of } \mathcal{D}(\Omega) \text{ in } H^1(\Omega)\,;$$

$H_0^1(\Omega)$ being a closed linear space in $H^1(\Omega)$, is also a Hilbert space. By Poincaré's inequality, the bilinear form

$$(u,v)_{H_0^1(\Omega)} = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)}, \quad u,v \in H_0^1(\Omega)$$

defines a new inner product on $H_0^1(\Omega)$ equivalent to $(u,v)_{H^1(\Omega)}$ on $H_0^1(\Omega)$. Also, since we assume the boundary $\partial\Omega$ to be Lipschitz, $H_0^1(\Omega)$ may be identified to the space $\{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$.

$$\begin{aligned}
H^{-1}(\Omega) &\stackrel{\text{def}}{=} \text{dual space of } H_0^1(\Omega)\,, \\
\nabla \cdot v &\stackrel{\text{def}}{=} \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_N}{\partial x_N}, \quad \text{divergence of } v = (v_1, \dots, v_N)^{\mathrm{T}}\,, \\
V &\stackrel{\text{def}}{=} \{v \in H_0^1(\Omega)^N : \nabla \cdot v = 0 \text{ in } \Omega\}\,, \\
W &\stackrel{\text{def}}{=} H_0^1(\Omega) \cap L^\infty(\Omega)\,.
\end{aligned}$$

For $u = (u_1, \dots, u_N)^{\mathrm{T}}$ and $v = (v_1, \dots, v_N)^{\mathrm{T}}$, the vector $(u \cdot \nabla)v$ stands for the transport term (the field v is transported by u) and is defined as

$$[(u \cdot \nabla)v]_i \stackrel{\text{def}}{=} \sum_{j=1}^{N} u_j \frac{\partial v_i}{\partial x_j}, \quad i = 1, \dots, N\,,$$

and if z is a scalar quantity, the transport term for z is given by

$$u\nabla z \stackrel{\text{def}}{=} \sum_{j=1}^{N} u_j \frac{\partial z}{\partial x_j}\,.$$

Finally, we will use the abbreviation "a.e." meaning "almost everywhere".

3. Main Results

We will consider the following hypotheses on the data:

$(H1)$ $f \in H^{-1}(\Omega)^N$;
$(H2)$ $a \geq 0$, $b \geq 0$ are real constants;

$(H3)$ $A : \Omega \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^{N\times N}$ is a Caratheodory mapping ($x \mapsto A(x, s_1, s_2)$ is measurable in Ω, $\forall\ s_1, s_2 \in \mathbb{R}$, and $(s_1, s_2) \mapsto A(x, s_1, s_2)$ is continuous in $\mathbb{R} \times \mathbb{R}$, a.e. in Ω) and there exists a constant $\alpha > 0$ such that

$$A(x, s_1, s_2)\xi\xi \geq \alpha|\xi|^2\,, \quad \forall\ s_1, s_2 \in \mathbb{R}, \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega\,;$$

$(H4)$ there exists a continuous function $d : \mathbb{R}^2 \mapsto \mathbb{R}^+$ such that

$$A(x, s_1, s_2)\xi\xi \leq d(s_1, s_2)|\xi|^2\,, \quad \forall\ s_1, s_2 \in \mathbb{R}, \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega\,.$$

$(H5)$ g_θ, g_φ : $\Omega \times \mathbb{R}_\theta \times \mathbb{R}_\varphi \times \mathbb{R}^{N\times N}_{\nabla u} \mapsto \mathbb{R}$ are Caratheodory functions and there exists $c \in L^1(\Omega)$ such that

$$\begin{cases} 0 \leq g(x, s_1, s_2, B) \leq d(s_1, s_2)(c(x) + |B|^2) \\ \forall\ s_1, s_2 \in \mathbb{R}, B \in \mathbb{R}^{N\times N} \text{ and a.e. } x \in \Omega\,; \end{cases}$$

for both $g = g_\theta$ and $g = g_\varphi$.

Remarks. 1. If θ and φ are two measurable functions, then by $A(\theta, \varphi)$ we are denoting the function $x \mapsto A(x, \theta(x), \varphi(x))$. Since A is Caratheodory [hypothesis $(H3)$], $A(\theta, \varphi)$ is also measurable. The same is true for $g_\theta(\theta, \varphi, \nabla u)$ and $g_\varphi(\theta, \varphi, \nabla u)$.
2. In a more general situation, instead of just A, we may think of three different tensor viscosities A_u, A_θ and A_φ for the respective equations of u, θ and φ. If these three matrix functions verify $(H3)$ and $(H4)$, then Theorems 1–4 below still hold. In any case, $A(\theta, \varphi)$ is not assumed to be symmetric.
3. Hypotheses $(H4)$ and $(H5)$ assume general asymptotic behavior of A, g_θ and g_φ with respect to θ and φ. We can afford this since we are going to show the existence of solutions θ and φ lying in L^∞.

Now we introduce the weak formulation problems corresponding to systems (1.1) and (1.2) respectively (remember that the pressure is gone since all test functions are divergence free; it may be retrieved by the standard de Rham's argument[12,15]):

$$\begin{cases} \text{to find } u \in V, \theta, \varphi \in H^1(\Omega) \text{ such that } \theta - a, \varphi - b \in W\,, \\ \displaystyle\int_\Omega A(\theta, \varphi)\nabla u \nabla v = \langle f, v\rangle\,, \quad \forall\ v \in V\,, \\ \displaystyle\int_\Omega A(\theta, \varphi)\nabla\theta\nabla\phi + \int_\Omega \theta g_\theta(\theta, \varphi, \nabla u)\phi = \int_\Omega \phi\,, \quad \forall\ \phi \in W\,, \\ \displaystyle\int_\Omega A(\theta, \varphi)\nabla\varphi\nabla\phi + \int_\Omega \varphi g_\varphi(\theta, \varphi, \nabla u)\phi = 0\,, \quad \forall\ \phi \in W\,; \end{cases} \tag{3.3}$$

$$\begin{cases} \text{to find } u \in V, \theta, \varphi \in H^1(\Omega) \text{ such that } \theta - a, \varphi - b \in W\,, \\ \displaystyle\int_\Omega (u\cdot\nabla)uv + \int_\Omega A(\theta,\varphi)\nabla u\nabla v = \langle f, v\rangle\,, \quad \forall\, v\in V\,, \\ \displaystyle\int_\Omega u\nabla\theta\phi + \int_\Omega A(\theta,\varphi)\nabla\theta\nabla\phi + \int_\Omega \theta g_\theta(\theta,\varphi,\nabla u)\phi = \int_\Omega \phi\,, \quad \forall\,\phi\in W\,, \\ \displaystyle\int_\Omega u\nabla\varphi\phi + \int_\Omega A(\theta,\varphi)\nabla\varphi\nabla\phi + \int_\Omega \varphi g_\varphi(\theta,\varphi,\nabla u)\phi = 0\,, \quad \forall\,\phi\in W\,, \end{cases} \tag{3.4}$$

and the main results now follow (see also Sec. 6).

Theorem 1. *Under hypotheses* $(H1)$*–*$(H5)$*, there exists* (u,θ,φ) *solution to* (3.3)*, such that*

$$0 \le \theta \le a + \frac{C(N,\Omega)}{\alpha}\,, \quad a.e.\ in\ \Omega\,, \tag{3.5}$$

$$0 \le \varphi \le b\,, \quad a.e.\ in\ \Omega\,, \tag{3.6}$$

where $C(N,\Omega)$ *is a constant value.*

Theorem 2. *Let* $N = 2$ *or* 3 *and assume* $(H1)$*–*$(H5)$*. Then there exists* (u,θ,φ) *solution to* (3.4)*, such that* (3.5) *and* (3.6) *still hold.*

The next sections are devoted to describe the proof of these two results.

Proof of Theorem 1. We will make use of the following auxiliary lemma (see appendix for the proof).

Lemma 1. *Let* $A \in L^\infty(\Omega)^{N\times N}$*,* $u \in L^2(\Omega)^N$*,* $\nabla\cdot u = 0$ *in* Ω *and* $u\cdot n = 0$ *on* $\partial\Omega$*,* $f \in L^\infty(\Omega)$ *with* $f \ge 0$ *a.e. in* Ω*,* $h \in L^1(\Omega)$ *with* $h \ge 0$ *a.e. in* Ω *and, finally,* $a \in \mathbb{R}^+$ *be given. We assume that there exists* $\alpha > 0$ *such that*

$$A(x)\xi\xi \ge \alpha|\xi|^2\,, \quad \forall\,\xi\in\mathbb{R}^N,\ a.e.\ in\ \Omega\,.$$

Consider the linear problem

$$\begin{cases} z \in H^1(\Omega)\,, \quad z - a \in W \\ \displaystyle\int_\Omega A\nabla z\nabla\phi + \int_\Omega u\nabla z\phi + \int_\Omega hz\phi = \int_\Omega f\phi\,, \quad \forall\,\phi\in W\,. \end{cases} \tag{3.7}$$

Then, there exists a unique solution z *to* (3.7) *and furthermore*

$$\begin{cases} 0 \le z \le a + \dfrac{C_1}{\alpha}\|f\|_{L^\infty(\Omega)} \quad a.e.\ in\ \Omega\,, \\ \|z\|_{H^1(\Omega)} \le C_2\,, \end{cases} \tag{3.8}$$

where $C_1 = C_1(\Omega,N)$ *and* $C_2 = C_2(a,\alpha,\Omega,N,\|f\|_{L^\infty(\Omega)},\|h\|_{L^1(\Omega)})$*.*

Remarks. 1. Though (3.7) is a linear problem, it contains coefficients with low regularity (u and h). This rather low regularity is compensated with the (strong) hypothesis $f \in L^\infty(\Omega)$ and the positivity conditions on both f and h.

2. Note that the L^∞-estimate in (3.8) depends neither on h, on u nor on $\|A\|_{L^\infty(\Omega)^{N\times N}}$. This is the key which will lead us to the resolution of problems (3.3) and (3.4).
3. Lemma 1 is still valid if $f \in L^q(\Omega)$ for some $q > N/2$ (respectively $f \in W^{-1,q}(\Omega)$ for some $q > N$) instead of $f \in L^\infty(\Omega)$. Notice that under the assumption $f \in W^{-1,q}(\Omega)$, the positivity condition $f \geq 0$ reads in the distribution sense, i.e.

$$\langle f, \phi \rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} \geq 0\,, \quad \forall\, \phi \in W_0^{1,q'}(\Omega)\,, \quad \phi \geq 0 \text{ a.e. in } \Omega\,,$$

where $1/q + 1/q' = 1$.

Now, in order to prove the existence of solution to (3.3), we are going to apply a Schauder fixed point technique.[2] To do so, we introduce the set $B_R = \{(\bar{\theta}, \bar{\varphi}) \in L^2(\Omega) \times L^2(\Omega) \,:\, 0 \leq \bar{\theta} \leq R, 0 \leq \bar{\varphi} \leq b \text{ a.e. in } \Omega\}$ where $R \geq a + C_1/\alpha$ (C_1 is the constant appearing in (3.8)). Then, B_R is a non-empty, closed and convex set of $L^2(\Omega) \times L^2(\Omega)$. Next we define the operator $\Phi : B_R \mapsto L^2(\Omega) \times L^2(\Omega)$, $\Phi(\bar{\theta}, \bar{\varphi}) = (\theta, \varphi)$ where (θ, φ) is given by the following procedure:

(i) $u \in V$ such that

$$\int_\Omega A(\bar{\theta}, \bar{\varphi}) \nabla u \nabla v = \langle f, v \rangle\,, \quad \forall\, v \in V\,.$$

(ii) $\theta \in H^1(\Omega)$ such that $\theta - a \in W$ and

$$\int_\Omega A(\bar{\theta}, \bar{\varphi}) \nabla \theta \nabla \phi + \int_\Omega \theta g_\theta(\bar{\theta}, \bar{\varphi}, \nabla u) \phi = \int_\Omega \phi\,, \quad \forall\, \phi \in W\,.$$

(iii) $\varphi \in H^1(\Omega)$ such that $\varphi - b \in W$ and

$$\int_\Omega A(\bar{\theta}, \bar{\varphi}) \nabla \varphi \nabla \phi + \int_\Omega \varphi g_\varphi(\bar{\theta}, \bar{\varphi}, \nabla u) \phi = 0\,, \quad \forall\, \phi \in W\,.$$

Thanks to hypotheses $(H1)$–$(H5)$, all terms in (i)–(iii) make sense. Step (i) is a linear Stokes system, and therefore it has a unique solution; on the other hand, by applying Lemma 1, problems in steps (ii) and (iii) have a unique solution in θ and φ, respectively.

It is straightforward that Φ is a continuous operator. Also, thanks to the first estimate in (3.8), we have $\Phi(B_R) \subset B_R$. It remains to show that Φ is compact. To do so, we make use of hypothesis $(H5)$ and the estimates (3.8). We have, for $g = g_\theta$ and $g = g_\varphi$

$$0 \leq g(\bar{\theta}, \bar{\varphi}, \nabla u) \leq D(c(x) + |\nabla u|^2)\,,$$

where $D = \max\{d(s_1, s_2) : |s_1| \leq R, |s_2| \leq b\}$. Hence, for all $(\bar{\theta}, \bar{\varphi}) \in B_R$

$$\|g(\bar{\theta}, \bar{\varphi}, \nabla u)\|_{L^1(\Omega)} \leq D(\|c\|_{L^1(\Omega)} + \|u\|^2_{H_0^1(\Omega)^N}) \leq D\left(\|c\|_{L^1(\Omega)} + \frac{\|f\|^2_{H^{-1}(\Omega)^N}}{\alpha^2}\right)$$

and again, using the second estimate in (3.8), we finally deduce that, for $(\theta, \varphi) = \Phi(\bar{\theta}, \bar{\varphi})$,

$$\|\theta\|_{H^1(\Omega)}, \|\varphi\|_{H^1(\Omega)} \leq C(a, b, \alpha, \Omega, N, \|f\|_{H^{-1}(\Omega)^N}, \|c\|_{L^1(\Omega)}) .$$

This means that $\Phi(B_R)$ lies in a bounded set of $H^1(\Omega) \times H^1(\Omega)$, which implies that $\Phi(B_R)$ is relatively compact in $L^2(\Omega) \times L^2(\Omega)$. Consequently, Φ is compact.

By Schauder's fixed point theorem, there exists (θ, φ) such that $(\theta, \varphi) = \Phi(\theta, \varphi)$; writing down (i)–(iii) for this (θ, φ) yields that (u, θ, φ) is a solution of system (3.3). This ends the proof of Theorem 1.

Proof of Theorem 2. Now we show Theorem 2. We proceed as in the previous section. Let us introduce the set $B_R = \{(\bar{u}, \bar{\theta}, \bar{\varphi}) \in L^4(\Omega)^N \times L^2(\Omega) \times L^2(\Omega), 0 \leq \bar{\theta} \leq R, 0 \leq \bar{\varphi} \leq b, \|\bar{u}\|_{L^4(\Omega)^N} \leq \frac{C_0(N,\Omega)}{\alpha}\|f\|_{H^{-1}(\Omega)^N}, \}$, where $C_0(N, \Omega)$ is the Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ (remember here that $N = 2$ or 3); and the operator $\Phi : B_R \mapsto L^4(\Omega)^N \times L^2(\Omega) \times L^2(\Omega)$, $\Phi(\bar{u}, \bar{\theta}, \bar{\varphi}) = (u, \theta, \varphi)$ is defined as follows:

(i) $u \in V$ such that

$$\int_\Omega (\bar{u} \cdot \nabla) uv + \int_\Omega A(\bar{\theta}, \bar{\varphi}) \nabla u \nabla v = \langle f, v \rangle , \quad \forall\, v \in V ;$$

(ii) $\theta \in H^1(\Omega)$ such that $\theta - a \in W$ and

$$\int_\Omega \bar{u} \nabla \theta \phi + \int_\Omega A(\bar{\theta}, \bar{\varphi}) \nabla \theta \nabla \phi + \int_\Omega \theta g_\theta(\bar{\theta}, \bar{\varphi}, \nabla u) \phi = \int_\Omega \phi , \quad \forall\, \phi \in W ;$$

(iii) $\varphi \in H^1(\Omega)$ such that $\varphi - b \in W$ and

$$\int_\Omega \bar{u} \nabla \varphi \phi + \int_\Omega A(\bar{\theta}, \bar{\varphi}) \nabla \varphi \nabla \phi + \int_\Omega \varphi g_\varphi(\bar{\theta}, \bar{\varphi}, \nabla u) \phi = 0 , \quad \forall\, \phi \in W .$$

Step (i) is a transport Stokes system and it has a unique solution by a simple application of the Lax–Milgram theorem. For steps (ii) and (iii), we apply Lemma 1. Consequently, Φ is well defined.

It is straightforward that Φ is continuous and, since $N \leq 3$, it is also compact. Moreover, taking again $R \geq a + C_1/\alpha$, we have $\Phi(B_R) \subset B_R$. Consequently, Φ has a fixed point (u, θ, φ), which in turn is a solution of system (3.4). This ends the proof of Theorem 2.

Remark. In the case $N = 4$, problems (i)–(iii) are well posed and Φ is still well defined. Unfortunately, since the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ is no longer compact, the operator Φ is not compact either, and the procedure described here does not work.

4. Final Comments and Generalizations

The uniqueness of solution for this kind of problems is difficult to obtain, even in the case of the Stokes like system (in general, we cannot expect uniqueness for system

(3.4) since this does not hold for the steady state Navier–Stokes equations, unless $\|f\|_{H^{-1}(\Omega)}$ were small enough). Mainly, this is due to the strong coupling of the unknowns through the nonlinear diffusion matrix $A(\theta,\varphi)$ and the nonlinear terms $\theta g_\theta(\theta,\varphi,\nabla u)$ and $\varphi g_\varphi(\theta,\varphi,\nabla u)$. We may try to relax this strong coupling in order to obtain the uniqueness of positive solutions; for instance, we may take $A=A(\theta)$ verifying the local Lipschitz condition: for every $M>0$, there exists $L_M\ge 0$ such that $\|A(x,s_1)-A(x,s_2)\|\le L_M|s_1-s_2|$, for all s_1, s_2, $0\le s_1$, $s_2\le M$ and a.e. $x\in\Omega$ ($\|\cdot\|$ being a matrix norm); and also $g_\theta=g_\theta(\theta)$ and $g_\varphi=g_\varphi(\theta,\varphi)$, verifying $(s_1g_\theta(x,s_1)-s_2g_\theta(x,s_2))(s_1-s_2)\ge 0$, $(s_1g_\varphi(x,s,s_1)-s_2g_\varphi(x,s,s_2))(s_1-s_2)\ge 0$, for all s, s_1, $s_2\ge 0$, and a.e. $x\in\Omega$. But as one can readily see, these conditions are very restrictive as the system becomes practically uncoupled.

Along the proof of Theorems 1 and 2, we observe that the L^∞-regularity has played a fundamental role. We can extend these results to more general situations in which the nonlinear terms g_θ and g_φ may also depend on u: $g(\theta,\varphi,u,\nabla u)$. In fact, it will be sufficient that these terms lie in $L^1(\Omega)$. To assure this, we may change $(H5)$ to $(H5)'$, namely:

$(H5)'$ g_θ, $g_\varphi:\Omega\times\mathbb{R}_\theta\times\mathbb{R}_\varphi\times\mathbb{R}^N_u\times\mathbb{R}^{N\times N}_{\nabla u}\mapsto\mathbb{R}$ are Caratheodory functions and such that

— if $N>2$: there exists $c\in L^1(\Omega)$ with

$$\begin{cases} 0\le g(x,s_1,s_2,v,B)\le d(s_1,s_2)(c(x)+|v|^{2N/(N-2)}+|B|^2) \\ \forall\ s_1,s_2\in\mathbb{R}, v\in\mathbb{R}^N, B\in\mathbb{R}^{N\times N} \text{ and a.e. } x\in\Omega\,; \end{cases}$$

— if $N=2$: $\forall\ \delta>0$ there exists $c_\delta\in L^1(\Omega)$ with

$$\begin{cases} 0\le g(x,s_1,s_2,v,B)\le d(s_1,s_2)(c_\delta(x)+\exp(\delta|v|^2)+|B|^2) \\ \forall\ s_1,s_2\in\mathbb{R}, v\in\mathbb{R}^N, B\in\mathbb{R}^{N\times N} \text{ and a.e. } x\in\Omega\,; \end{cases}$$

for both $g=g_\theta$ and $g=g_\varphi$.

In effect, in the case $N>2$, we make use of the continuous Sobolev embedding $H^1_0(\Omega)\hookrightarrow L^{2N/(N-2)}(\Omega)$; on the other hand, when $N=2$ we may use the Trudinger–Moser inequality.[6] Now, the two problems are:

$$\begin{cases} \text{to find } u\in V,\theta,\varphi\in H^1(\Omega) \text{ such that } \theta-a,\varphi-b\in W\,, \\ \displaystyle\int_\Omega A(\theta,\varphi)\nabla u\nabla v=\langle f,v\rangle\,,\quad\forall\ v\in V\,, \\ \displaystyle\int_\Omega A(\theta,\varphi)\nabla\theta\nabla\phi+\int_\Omega\theta g_\theta(\theta,\varphi,u,\nabla u)\phi=\int_\Omega f_\theta\phi\,,\quad\forall\ \phi\in W\,, \\ \displaystyle\int_\Omega A(\theta,\varphi)\nabla\varphi\nabla\phi+\int_\Omega\varphi g_\varphi(\theta,\varphi,u,\nabla u)\phi=\int_\Omega f_\varphi\phi\,,\quad\forall\ \phi\in W\,; \end{cases} \tag{4.9}$$

$$\begin{cases} \text{to find } u \in V, \theta, \varphi \in H^1(\Omega) \text{ such that } \theta - a, \varphi - b \in W\,, \\ \displaystyle\int_\Omega (u\cdot\nabla)uv + \int_\Omega A(\theta,\varphi)\nabla u\nabla v = \langle f, v\rangle\,, \quad \forall\, v \in V\,, \\ \displaystyle\int_\Omega u\nabla\theta\phi + \int_\Omega A(\theta,\varphi)\nabla\theta\nabla\phi + \int_\Omega \theta g_\theta(\theta,\varphi,u,\nabla u)\phi = \int_\Omega f_\theta\phi\,, \quad \forall\, \phi \in W\,, \\ \displaystyle\int_\Omega u\nabla\varphi\phi + \int_\Omega A(\theta,\varphi)\nabla\varphi\nabla\phi + \int_\Omega \varphi g_\varphi(\theta,\varphi,u,\nabla u)\phi = \int_\Omega f_\varphi\phi\,, \quad \forall\, \phi \in W\,, \end{cases} \tag{4.10}$$

then, we have the following results:

Theorem 3. *Let* f_θ, $f_\varphi \in L^\infty(\Omega)$ *be given such that*, $f_\theta \geq 0$, $f_\varphi \geq 0$ *a.e. in* Ω. *Assume hypotheses* $(H1)$–$(H4)$ *and* $(H5)'$. *Then there exists* (u,θ,φ) *solution to* (4.9), *such that*

$$0 \leq \theta \leq a + \frac{C(N,\Omega)}{\alpha}\|f_\theta\|_{L^\infty(\Omega)}\,, \quad \textit{a.e. in } \Omega\,, \tag{4.11}$$

$$0 \leq \varphi \leq b + \frac{C(N,\Omega)}{\alpha}\|f_\varphi\|_{L^\infty(\Omega)}\,, \quad \textit{a.e. in } \Omega\,. \tag{4.12}$$

Theorem 4. *Let* $N = 2$ *or* 3. *Under the assumptions of Theorem* 3, *there exists* (u,θ,φ) *solution to* (4.10) *such that* (4.11) *and* (4.12) *still hold.*

Remarks. Theorems 3 and 4 still hold with the hypothesis f_θ, $f_\varphi \in L^\infty(\Omega)$ change to f_θ, $f_\varphi \in L^q(\Omega)$ for some $q > N/2$ (respectively f_θ, $f_\varphi \in W^{-1,q}(\Omega)$ for some $q > N$) and in the estimates (4.11) and (4.12), we have $\|f_\theta\|_{L^q(\Omega)}$ and $\|f_\varphi\|_{L^q(\Omega)}$ (respectively $\|f_\theta\|_{W^{-1,q}(\Omega)}$ and $\|f_\varphi\|_{W^{-1,q}(\Omega)}$) instead of $\|f_\theta\|_{L^\infty(\Omega)}$ and $\|f_\varphi\|_{L^\infty(\Omega)}$.

The function d appearing in $(H4)$, $(H5)$ or $(H5)'$ needs not be continuous. In fact, it is sufficient to assume that $d : \mathbb{R}^2 \mapsto \mathbb{R}^+$ maps bounded sets in $\mathbb{R}^2$ onto bounded sets in $\mathbb{R}^+$ and Theorems 1–4 remain true.

There are an intermediate situation between problems (4.9) and (4.10) in which Theorem 3 applies. In effect, consider the problem in which we drop out the transport term $(u\cdot\nabla)u$ in the equation for the velocity field in (4.10), but we still keep $u\nabla\theta$ and $u\nabla\varphi$ in the respective equations for θ and φ. It is straightforward that in this case Theorem 3 still holds.

We have considered in our analysis only Dirichlet boundary conditions on all the unknowns. We may generalize to other boundary conditions (Fourier, mixed Dirichlet–Neumann, etc.) on the velocity field, provided that the corresponding Stokes problem is well-posed. In those cases, all the results shown here are still valid. But we have not analyzed what happens when the constant Dirichlet boundary conditions on θ and φ are not considered. This seems a very difficult task to take in, even in the "simple" case of non-constant Dirichlet boundary conditions.

To end this work, we notice that the steady-state k–ε turbulence model (see the Introduction) is not a particular case of the problems studied here. This is not only because of the singular coefficients of this model (k^2/ε and ε^2/k which may be approximated by $k^2/\gamma(\varepsilon)$ and $\varepsilon^2/\gamma(k)$, γ being a continuous and nonvanishing function which approximates the identity [this is an idea of Lewandowski[8]]) but also, the nonlinear term $c_1\frac{k^2}{\varepsilon}|\nabla u + \nabla u^{\mathrm{T}}|^2$ does not satisfy hypothesis ($H5$). That is the main difference between the two models k–ε and θ–φ, and that is why we may consider this last model as a stabilization of the first one.

Appendix. Proof of the Auxiliary Lemma 1

Along this Appendix we develop the proof of Lemma 1.

The uniqueness of solution is straightforward. Due to the hypotheses under the field u, there exists a sequence $(u_k) \subset \mathcal{D}(\Omega)$ such that $\nabla \cdot u_k = 0$ and $u_k \to u$ in $L^2(\Omega)$-strongly. We also consider $(h_k) \subset L^\infty(\Omega)$ such that $0 \le h_k \le h$ a.e. in Ω, and $h_k \to h$ in $L^1(\Omega)$-strongly (take for example $h_k = \min(k, h)$, $k = 1, 2, \ldots$).

Then we set up the approximated problems, namely

$$\begin{cases} z_k \in a + H_0^1(\Omega) \\ \displaystyle\int_\Omega A\nabla z_k \nabla v + \int_\Omega u_k \nabla z_k v + \int_\Omega h_k z_k v = \int_\Omega f v\,, \quad \forall\, v \in H_0^1(\Omega)\,. \end{cases} \tag{A.1}$$

For every $k \ge 1$, by the Lax–Milgram theorem, there exists a unique solution to (A.1). Then, Lemma 1 will be shown if we prove that the estimates (3.8) hold uniformly for all z_k, $k \ge 1$.

To begin with, we show the existence of a constant, not dependent on k, such that $0 \le z_k \le C$, a.e. in Ω.

Taking the test function $v = z_k^- = \max\{0, -z_k\} \in H_0^1(\Omega)$ (since $a \ge 0$), we deduce $z_k^- = 0$, which means that $z_k \ge 0$, a.e. in Ω.

Now, in order to state the uniform bound of (z_k) we are going to apply Stampacchia's technique based on truncations.[13,14] To do so, we first consider the following problems

$$\begin{cases} y_k \in H_0^1(\Omega) \\ \displaystyle\int_\Omega A\nabla y_k \nabla v + \int_\Omega u_k \nabla y_k v = \int_\Omega v\,, \quad \forall\, v \in H_0^1(\Omega) \end{cases} \tag{A.2}$$

and let's see that there exists a constant $C = C(N, \Omega, \alpha) > 0$ such that $y_k \le C$, $\forall\, k \ge 1$ and a.e. in Ω.

For every $M > 0$, take $v = (y_k - M)^+ \in H_0^1(\Omega)$ as a test function in (A.2); it yields

$$\alpha \int_{y_k \ge M} |\nabla y_k|^2 \le \int_{y_k \ge M} (y_k - M)^+\,.$$

On the other hand, by the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ [$2^* = 2N/(N-2)$ if $N > 2$, $2^* = q$, any finite q] we have

$$\left(\int_\Omega |(y_k - M)^+|^{2^*}\right)^{2/2^*} \le C^*(\Omega, N) \int_\Omega |\nabla (y_k - M)^+|^2$$

hence

$$\frac{\alpha}{C^*(\Omega,N)}\left(\int_\Omega |(y_k-M)^+|^{2^*}\right)^{2/2^*} \le \int_\Omega |(y_k-M)^+|\,.$$

On the other hand,

$$\begin{aligned}\left(\int_{y_k\ge M}(y_k-M)^+\right)^2 &\le \left(\int_{y_k\ge M}|(y_k-M)^+|^{2^*}\right)^{2/2^*}|\{y_k\ge M\}|^{2-2/2^*}\\ &\le \frac{C^*(\Omega,N)}{\alpha}\left(\int_{y_k\ge M}(y_k-M)^+\right)|\{y_k\ge M\}|^{2-2/2^*}\end{aligned}$$

and therefore

$$\int_{y_k\ge M}(y_k-M)^+ \le \frac{C^*(\Omega,N)}{\alpha}|\{y_k\ge M\}|^{2-2/2^*}\,.$$

Let $H>M$; we have

$$\int_{y_k\ge M}(y_k-M)^+ \ge \int_{y_k\ge H}(y_k-M)^+ \ge (H-M)|\{y_k\ge H\}|^{2-2/2^*}\,.$$

We introduce the function $\psi:\mathbb{R}^+\mapsto\mathbb{R}^+$ given by $\psi(H)=|\{y_k\ge H\}|$, so that

$$\psi(H)\le \frac{C^*(\Omega,N)}{\alpha(H-M)}\psi(M)^{2-2/2^*}\,,\quad \forall\, H>M\,.$$

At this point, we make use of the following result due to Stampacchia.[13,14]

Lemma A.1. *Let $k_0\in\mathbb{R}$ and $\psi:[k_0,+\infty)\mapsto\mathbb{R}^+$ be a decreasing function such that*

$$\psi(h)\le\frac{C}{(h-k)^\alpha}(\psi(k))^\beta\,,\quad \forall\, h>k>k_0$$

for some non-negative constants C, α and β. Then,

- *if $\beta>1$, $\psi(k_0+d)=0$, with $d=C(\psi(k_0))^{\beta-1}2^{\alpha\beta/(\beta-1)}$.*
- *if $\beta=1$, $\psi(h)\le e\exp(-\xi(h-k_0))\psi(k_0)$ where $\xi=(eC)^{-1/\alpha}$.*
- *if $\beta<1$ and $k_0>0$, $\psi(h)\le 2^{\mu/(1-\beta)}(C^{1/(1-\beta)}+(2k_0)^\mu\psi(k_0))h^{-\mu}$ where $\mu=\alpha/(1-\beta)$.*

Since $2-\frac{2}{2^*}=\frac{N+2}{N}=\beta>1$, we apply Lemma A.1 with $k_0=0$ and deduce that $\psi(d)=0$, for $d=\frac{C^*(\Omega,N)}{\alpha}|\Omega|^{(N+2)/N-1}2^\beta$.

Consequently, there exists a constant value $C(\Omega,N)$ such that

$$y_k\le\frac{C(\Omega,N)}{\alpha}\,,\quad \text{a.e. in }\Omega\,.$$

Consider the function $\tilde z_k=z_k-a-\|f\|_{L^\infty(\Omega)}y_k$, which is a solution to the problem

$$\begin{cases}\tilde z_k\in H_0^1(\Omega)\\ \displaystyle\int_\Omega A\nabla\tilde z_k\nabla v+\int_\Omega u_k\nabla\tilde z_k v=\int_\Omega(f-\|f\|_{L^\infty(\Omega)}-h_kz_k)v\,,\quad \forall\, v\in H_0^1(\Omega)\,.\end{cases}$$

By the maximum principle we have $\tilde{z}_k \leq 0$, i.e. $z_k \leq a + \|f\|_{L^\infty(\Omega)} y_k$. So, in conclusion we have proved that

$$0 \leq z_k \leq a + \frac{C(\Omega, N)}{\alpha} \|f\|_{\infty(\Omega)}, \quad \forall\, k \geq 1, \quad \text{a.e. in } \Omega.$$

It remains to show that the sequence (z_k) is bounded in $H^1(\Omega)$. To do so, it will be sufficient to show that the sequence $(\bar{z}_k)$, $\bar{z}_k = z_k - a$ is bounded in $H^1_0(\Omega)$. We have

$$\begin{cases} \displaystyle\int_\Omega A\nabla \bar{z}_k \nabla v + \int_\Omega u_k \nabla \bar{z}_k v + \int_\Omega h_k \bar{z}_k v \\ \qquad \displaystyle = \int_\Omega f v - \int_\Omega a h_k v, \quad \forall\, v \in H^1_0(\Omega) \end{cases}$$

taking $v = \bar{z}_k$ as a test function in this formulation yields

$$\int_\Omega A\nabla \bar{z}_k \nabla \bar{z}_k + \int_\Omega h_k |\bar{z}_k|^2 = \int_\Omega f \bar{z}_k - \int_\Omega a h_k \bar{z}_k,$$

hence

$$\|\bar{z}_k\|^2_{H^1_0(\Omega)} \leq \frac{1}{\alpha^2} \max(a\alpha, C(\Omega, N)\|f\|_{L^\infty(\Omega)})(|\Omega| \|f\|_{L^\infty(\Omega)} + a\|h\|_{L^1(\Omega)}).$$

Acknowledgments

The authors wish to thank Dr. J. Casado Díaz for fruitful discussions and useful comments and suggestions.

This research was partially supported by DGICYT under grant PB93–1196.

References

1. G. Duvaut and J. L. Lions, **Les Inéquations en Mécanique et en Physique** (Dunod, 1972).
2. I. Fonseca and W. Gangbo, **Degree Theory in Analysis and Applications** (Oxford Univ. Press, 1995).
3. M. Gómez Mármol, **Estudio Matemático de Algunos Problemas no Lineales de la Mecánica de Fluidos Incompresibles**. Tesis doctoral, Universidad de Sevilla (1998).
4. M. Gómez Mármol and F. Ortegón Gallego, *Las ecuaciones de Stokes y de Navier–Stokes acopladas a dos ecuaciones parabólicas no lineales*, Actas del XV CEDYA/V Congreso de Matemática Aplicada, Vigo (1997).
5. M. Gómez Mármol and F. Ortegón Gallego, *Coupling the Stokes and Navier–Stokes equations with two scalar nonlinear parabolic equations*, *Modèles Math. Anal. Numér.* **33** (1999) 157–167.
6. O. Kavian, **Introduction à la Théorie des Points Critiques et Application aux Problèmes Elliptiques**, *Mathématiques & Applications 13* (Springer-Verlag, 1993).
7. R. Launder, **Mathematical Models of Turbulence** (Academic Press, 1972).
8. R. Lewandowski, **Analyse Mathématique et Océanographie** (Masson, 1997).

9. R. Lewandowski and B. Mohammadi, *Existence and positivity results for the ϕ–θ model and a modified k–ε turbulence model*, *Math. Model Methods Appl. Sci.* **3** (1993) 195–215.
10. B. Mohammadi, **Etude du Modèle k–ε de la Turbulence Pour les Écoulments Compressibles**. Thèse de doctorat, Paris VI (1991).
11. B. Mohammadi and O. Pironneau, **Analysis of the k–ε Turbulence Model**, *Research in Applied Mathematics* (Wiley-Masson, 1994).
12. J. Simon, *Démonstration constructive d'un théorème de G. de Rham*, *C. R. Acad. Sci. Paris* t. 316, Série I (1993) 1167–1172.
13. G. Stampacchia, *Some limit cases of L^p-estimates for solutions of second order elliptic equations*, in *Inter. Symp. on Lin. Spaces* (1960), pp. 399–408.
14. G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, *Ann. Inst. Fourier* (1965) 189–258.
15. L. Tartar, **Topics in Nonlinear Analysis** (Publ. Math. d'Orsay, 1978).