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A Sufficient Condition for Generalized Lorenz Order

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In this paper, a sufficient condition for non-negative random variables to be ordered in the Generalized Lorenz sense is presented. This condition does not involve inverse distribution functions. Applications of this result to several income distribution models are given. *Journal of Economic Literature* Classification Numbers: D36, D69.

1. INTRODUCTION

In economics, the relationship between income-welfare-inequality and its principal field of application, namely the comparison and ranking of income distribution of different social states, has been the subject of numerous studies (see Atkinson [3], Rothschild and Stiglitz [14], Dasgupta *et al.* [6] or Kakwani [9]).We may suppose that there exists a social welfare function

$$\omega = \omega(\mathbf{x}) = \omega(x_1, x_2, ..., x_n)$$

where x_i is the income of individual *i*. We may reasonably assume that $\omega(\cdot)$ is a Schur-concave non-decreasing function of all incomes (the assumption that $\omega(\cdot)$ is Schur-concave is equivalent to the presumption that society favours a more equitable distribution [see Dasgupta *et al.*, [6]). Let Ω denote the set of non-decreasing Schur-concave welfare functions and write

$$\mathbf{x} \succcurlyeq \mathbf{x}' \Leftrightarrow \omega(\mathbf{x}) \geqslant \omega(\mathbf{x}') \qquad \text{for all} \quad \omega(\cdot) \in \Omega.$$

A sufficient condition for this to hold is that $\mathbf{x}$ has both a higher mean and a higher Lorenz curve than $\mathbf{x}'$ (see Dasgupta *et al.* [6]). This sufficient condition tends to obscure many important situations in which distributions can be ranked. The Lorenz curve of any income distribution is the graph

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of the fraction of the total income owned by the lowest u-th fraction ($0 \leqslant u \leqslant 1$) of the population, as a function of u. If a non-negative random variable X represents the income of a society or community, with distribution function $F(x)$ and finite expectation μ_X, then the Lorenz curve $L_X(u)$ of X is given by (Gastwirth, [8])

$$L_X(u) = \frac{1}{\mu_X} \int_0^u F^{-1}(t)\, dt \qquad 0 \leqslant u \leqslant 1$$

where F^{-1} denotes the inverse of F:

$$F^{-1}(a) = \inf\{x : F(x) \geqslant a\}, \qquad a \in [0, 1].$$

There is extensive discussion of the Lorenz curve in Gail and Gastwirth [7] and a concise account of its properties in Dagum [5]. Shorrocks [16] introduces the notion of a Generalized Lorenz curve, $GL_X(u)$, constructed by scaling up the ordinary Lorenz curve by the mean of the distribution, i.e.

$$GL_X(u) = \mu_X \cdot L_X(u)$$

and shows that

$$\omega(\mathbf{x}) \geqslant \omega(\mathbf{x}') \quad \text{for all} \quad \omega(\cdot) \in \Omega \quad \text{iff} \quad GL_{\mathbf{x}}(u) \geqslant GL_{\mathbf{x}'}(u) \quad \text{for all } u.$$

Thus, scaling up the Lorenz curves to form the Generalized Lorenz curves will often reveal a dominance relationship that is not apparent from an examination of means and Lorenz curves on their own.

Let X and Y be two random variables with distribution functions F and G respectively. The Generalized Lorenz curve can be used to define a partial ordering on the class of distribution functions as follows:

$$F \leqslant_{gl} G \Leftrightarrow GL_X(u) \geqslant GL_Y(u) \qquad \text{for every} \quad 0 \leqslant u \leqslant 1.$$

In this case, we say that Y is at least as unequal as X in the Shorrocks or Generalized Lorenz sense.

In many distribution families, $GL_X(u)$ and the inverse of F do not have simple closed forms. In this paper, a sufficient condition for non-negative random variables to be ordered in the Generalized Lorenz sense is presented. This condition does not involve inverse distribution functions. In Section 3 we apply our results to three income distribution models: the Lognormal, the Pareto and the Gamma distributions. We conclude in Section 4 with a brief discussion about other related orders.

2. RESULTS

In what follows, we consider non-negative random variables X and Y with finite means, having distribution functions F and G, respectively, with supports $\operatorname{supp}(G) \subseteq \operatorname{supp}(F) \equiv [a, b]$, where $0 \leqslant a < b \leqslant \infty$. Assume that F and G are continuous and strictly increasing on their supports. The following theorem shows that a "single-crossing property" on F and G implies the Generalized Lorenz order.

THEOREM 2.1. *Suppose $E[X] \leqslant E[Y]$ and there is some k in $[a, b]$ such that $F(x) \geqslant G(x)$ for x in $[a, k]$ and $F(x) \leqslant G(x)$ for x in $[k, b]$. Then $G \leqslant_{gl} F$.*

Proof. By the assumptions on F and G, there exists $t_0 \in [0, 1]$ such that $F^{-1}(t) \leqslant G^{-1}(t)$ for t in $[0, t_0]$. Therefore,

$$\int_0^u F^{-1}(t)\,dt \leqslant \int_0^u G^{-1}(t)\,dt \tag{1}$$

for all u in $[0, t_0]$. If

$$\int_0^c F^{-1}(t)\,dt > \int_0^c G^{-1}(t)\,dt$$

for some c in $(t_0, 1]$, we must have $\int_0^1 F^{-1}(t)\,dt > \int_0^1 G^{-1}(t)\,dt$, because $F^{-1}(t) \geqslant G^{-1}(t)$ for all t in $[c, 1]$. But this means $E[X] > E[Y]$, a contradiction. Hence, (1) holds for all u in $[0, 1]$ and, consequently, $G \leqslant_{gl} F$.

Remark 2.1. If $k = b$ in Theorem 2.1, then G is said to be stochastically larger than F and the relation is denoted $F \leqslant_{st} G$.

Suppose now that F and G are absolutely continuous with density functions f and g, respectively (note that the sufficient condition in Theorem 2.1 does not involve the existence of f and g). By relating the unimodality of the ratio g/f (where we understand unimodality of the function $g(t)/f(t)$ to be for t restricted to $\operatorname{supp}(f)$) to single-crossing property we obtain the next result, which provides a convenient sufficient condition for the Generalized Lorenz comparison of two random variables.

THEOREM 2.2. *If $E[X] \leqslant E[Y]$ and $g(t)/f(t)$ is unimodal, where the mode is a supremum, then $G \leqslant_{gl} F$.*

Proof. Let $S(h)$ be the number of sign changes of the function $h(t)$. Since the function $g(t)/f(t)$ is unimodal, with the mode yielding a supremum, we have that

$$S(g/f-1)=S(g-f)\leqslant 2 \tag{2}$$

with sign sequence $-$, $+$, $-$ in case of equality.

By Lemma 2.1 of Shaked [15], condition (2) implies that there exists a k in $[a, b]$ such that $F(t)\geqslant G(t)$ for t in $[a, k]$ and $F(t)\leqslant G(t)$ for t in $[k, b]$. Hence, by Theorem 2.1 it follows that $G\leqslant_{gl} F$.

Corollary 2.1. *If $E[X]\leqslant E[Y]$ and $g(t)/f(t)$ is log-concave, then $G\leqslant_{gl} F$.*

Proof. The proof is immediate considering that a sufficient condition for f/g to be unimodal is for f/g to be log-concave (Keilson and Gerber [11]).

3. APPLICATIONS

In this section we will apply the results of Section 2 to three models of income distributions: the Lognormal, the Pareto and the Gamma distributions.

3.1. *The Lognormal Distribution*

Let X be a lognormal random variable with parameters μ and σ. Its probability density function is

$$f(x)=\frac{1}{x\sqrt{2\pi\sigma}}\exp\left\{-\frac{1}{2}\left[\frac{\ln x-\mu}{\sigma}\right]^2\right\}, \qquad x>0, \qquad \mu\in\mathbb{R}, \qquad \sigma>0. \tag{3}$$

Fix $\sigma>0$ and for μ_i $(i=1, 2)$, we denote by F_{μ_i} the corresponding distribution function. It is easy to see that $\mu_1<\mu_2$ implies $F_{\mu_1}\leqslant_{st} F_{\mu_2}$. Applying Theorem 2.1 it is concluded that $F_{\mu_2}\leqslant_{gl} F_{\mu_1}$ whenever $\mu_1<\mu_2$.

Now, assume that X_1 and X_2 be lognormal random variables with parameters (μ_1, σ_1) and (μ_2, σ_2) respectively and let f_1 and f_2 be the corresponding densities. Let $\sigma_1<\sigma_2$. Since $E[X_i]=\exp(\mu_i+\frac{1}{2}\sigma_i^2)$, $(i=1, 2)$, it follows that $E[X_1]\geqslant E[X_2]$ if and only if $\sigma_2^2-\sigma_1^2\leqslant 2(\mu_1-\mu_2)$. Besides, elementary manipulation shows that the ratio $f_1(x)/f_2(x)$ is unimodal for $x>0$ and, by Theorem 2.2, $F_1\leqslant_{gl} F_2$ holds.

3.2. *The Pareto Distribution*

Let X be the Pareto random variable with parameters α and ε. Its probability density function is:

$$f(x)=\frac{\alpha}{x}\left(\frac{x}{\varepsilon}\right)^{-\alpha}, \qquad x\geqslant\varepsilon, \qquad \alpha>0\,, \quad \varepsilon>0.$$

Its corresponding distribution function is

$$F(x)=1-\left(\frac{x}{\varepsilon}\right)^{-\alpha}, \qquad x\geqslant\varepsilon. \tag{4}$$

From (4) it is easy to verify that $F_{\varepsilon_1}\leqslant_{st} F_{\varepsilon_2}$ for $\varepsilon_1<\varepsilon_2$ (α fixed) and that $F_{\alpha_1}\leqslant_{st} F_{\alpha_2}$ whenever $\alpha_1<\alpha_2$ (ε fixed). It follows from Theorem 2.1 that Pareto distributions are ordered in the Generalized Lorenz sense either by their parameter α or by their parameter ε.

Now consider the general case in which X_1 and X_2 have respective parameters $(\alpha_1, \varepsilon_1)$ and $(\alpha_2, \varepsilon_2)$, densities f_1 and f_2 and distribution functions F_1 and F_2 and suppose $\varepsilon_1\leqslant\varepsilon_2$, that is, $\operatorname{supp}(f_2)\subseteq\operatorname{supp}(f_1)$. Since $E[X_i]=\alpha_i\varepsilon_i/(\alpha_i-1)$ for $\alpha_i>1$, it follows that $E[X_1]\leqslant E[X_2]$ whenever $\alpha_1\geqslant\alpha_2>1$. In these conditions, $f_2(x)/f_1(x)$ is unimodal on $\operatorname{supp}(f_1)$. Hence, from Theorem 2.2 it follows that

$$\varepsilon_1\leqslant\varepsilon_2 \qquad \text{and} \qquad \alpha_1\geqslant\alpha_2>1 \qquad \text{implies} \quad F_2\leqslant_{gl} F_1.$$

3.3. *The Gamma Distribution*

A random variable X follows the Gamma distribution with parameters α, β and γ if its density has the form

$$f(x)=\frac{(x-\gamma)^{\alpha-1}\exp[-(x-\gamma)/\beta]\,\beta^{-\alpha}}{\Gamma(\alpha)}, \qquad x>\gamma, \quad \alpha>0, \quad \beta>0, \quad \gamma>0 \tag{5}$$

where $\Gamma(\cdot)$ denotes the complete Gamma function.

It is known that Gamma distributions are stochastically ordered by their parameters. Applying Theorem 2.1 it is concluded that these distributions are ordered in the Generalized Lorenz sense according to their parameters. Thus, we have the following results:

$$\alpha_1<\alpha_2\Rightarrow F_{\alpha_2}\leqslant_{gl} F_{\alpha_1} \qquad (\beta \text{ and } \gamma \text{ fixed})$$

$$\beta_1<\beta_2\Rightarrow F_{\beta_2}\leqslant_{gl} F_{\beta_1}, \qquad (\alpha \text{ and } \gamma \text{ fixed}),$$

$$\gamma_1<\gamma_2\Rightarrow F_{\gamma_2}\leqslant_{gl} F_{\gamma_1}, \qquad (\alpha \text{ and } \beta \text{ fixed}).$$

Now, fix γ (assume without loss of generality $\gamma=0$). Let X_1 and X_2 be Gamma random variables with parameters α_1, β_1 and α_2, β_2 respectively and assume $\alpha_1 \leqslant \alpha_2$. Then, clearly $\text{supp}(f_1)= \text{supp}(f_2)=\{x : x>0\}$. It follows from (5) that $f_2(x)/f_1(x)$ is log-concave whenever $\alpha_1 \leqslant \alpha_2$. On the other hand, the mathematical expectation of the Gamma distribution with parameters α and β is $\alpha\beta$ so that $E[X_1] \leqslant E[X_2]$ if and only if $\alpha_1\beta_1 \leqslant \alpha_2\beta_2$. From Corollary 2.1 it follows that if $\alpha_1 \leqslant \alpha_2$ and $\alpha_1\beta_1 \leqslant \alpha_2\beta_2$ then $F_2 \leqslant_{gl} F_1$.

4. RELATED TOPICS

The Generalized Lorenz order is closely related to the usual Lorenz order (see Arnold [1]). We say that Y is at least as unequal as X in the Lorenz sense if the Lorenz curve of X is nowhere below that of Y. Obviously, if both random variables have the same mean, then Lorenz and Generalized Lorenz orders are equivalent. The Lorenz order within the Lognormal, Pareto or Gamma parametric families has been studied in Arnold *et al.* [2]. They show that any two members of each family which differ in their form parameters have nested Lorenz curves.

The relationship between the Lorenz curve and some notions of interest in reliability theory has been studied by several authors (see, for example, Klefsjo [12]). In this context, it can be seen that the sufficient condition given in Theorem 2.1 can be also derived, by using Theorem 1.6 (i) of Chong [4], from the inequality

$$\int_a^x \bar{F}(t)\,dt \leqslant \int_a^x \bar{G}(t)\,dt \qquad \text{for all} \quad x \text{ in } (a, b) \tag{6}$$

where $\bar{F}=1-F$ and $\bar{G}=1-G$ are the survival functions of X and Y, respectively. If $E[X]=E[Y]$, then (6) is equivalent to

$$\int_x^b \bar{G}(t)\,dt \leqslant \int_x^b \bar{F}(t)\,dt \qquad \text{for all} \quad x \text{ in } (a, b). \tag{7}$$

Stoyan [17] says that Y is “smaller in mean residual life” than X if (7) holds and calls this ordering the “convex order.” If X and Y are non-negatives random variables and $E[X]=E[Y]$, then it follows from Theorem 3.2 of Arnold [1] and Theorem 1.3.1 of Stoyan [17] that the convex and the Lorenz orders are equivalent. The convex order within the Lognormal and Gamma distributions has also been studied (Stoyan [17]).

On the other hand, the origins of the “single-crossing property” may be found in Karlin and Novikoff [10]. This property has often been used to compare distribution functions (see, for example, Arnold [1], Marshall and Proschan [13] and Stoyan [17]). Theorem 6.4 in Arnold [1] provides a result for the Lorenz order similar to Theorem 2.1. When $E[X] = E[Y]$, both results are equivalent. In Theorem 6.5 he also provides a sufficient condition for the Lorenz order in terms of two sign changes of the densities corresponding to absolutely continuous distribution functions. Stoyan [17] (see Sections 1.3–1.6) discusses a very similar property (called the "cut criterion") which provides a suitable condition for the convex order.

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