Singular sector of the Kadomtsev–Petviashvili hierarchy, $\bar{\partial}$ operators of nonzero index, and associated integrable systems

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Integrable hierarchies associated with the singular sector of the Kadomtsev–Petviashvili (KP) hierarchy, or equivalently, with $\bar{\partial}$ operators of nonzero index are studied. They arise as the restriction of the standard KP hierarchy to submanifolds of finite codimension in the space of independent variables. For higher $\bar{\partial}$ index these hierarchies represent themselves as families of multidimensional equations with multidimensional constraints. The $\bar{\partial}$-dressing method is used to construct these hierarchies. Hidden Korteweg–de Vries, Boussinesq, and hidden Gelfand–Dikii hierarchies are considered, too. © *2000 American Institute of Physics.* [S0022-2488(99)02812-1]

I. INTRODUCTION

It is well established now that the Kadomtsev–Petviashvili (KP) hierarchy is the key ingredient in a number of remarkable nonlinear problems, both in physics and mathematics (see, e.g., Refs. 1–4). In physics, its applications range from the shallow water waves (see Refs. 1–4) to the modern string theory (see, e.g., Refs. 5–8). Resolution of the famous Schottky problem is one of the most impressive manifestations of the KP hierarchy in pure mathematics.[9] Several methods have been developed to describe and analyze the KP hierarchy and other integrable hierarchies, for instance, the inverse scattering transform method,[1–4] Grassmannian approach,[10–13] or $\bar{\partial}$-dressing method.[14–16,4,17] These methods have arisen to study generic properties of the KP hierarchy and other integrable equations. In particular, the construction of everywhere regular solutions of integrable systems (solitons, lumps, dromions, etc.), which may have applications in physics, was a main interest.

Much less attention was paid to singular solutions of integrable equations. Pole-type solutions of the Korteweg–de Vries (KdV) equation have been known for a long time. However, interest in this class of solutions increased only when it was shown that the motion of poles for the KdV equation is governed by the Calogero–Moser model.[18,19] Similar results for rational singular solutions of the KP equation have been obtained in Ref. 20. The general study of generic singularity manifolds began with the formulation of the Painlevé analysis method for partial differential equations in Refs. 21 and 22. The structure of generic singularities of integrable equations turns out to be connected with all their remarkable properties (Lax pairs, Backlund-transformations, etc.) (see, e.g., Refs. 23 and 3). Characteristic singular manifolds (i.e., singular manifolds with additional constraints) have been discussed in Refs. 24–26.

A new method to analyze singular sectors of integrable equations has been proposed in Ref.

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27. It uses the Birkhoff decomposition of the Grassmannian, its relation with zero sets of the τ function and its derivatives, and properties of Backlünd transformations. This method provides a regular way to construct desingularized wave functions near the blow-up locus (Birkhoff strata). Note that the connection between the Painlevé analysis and cell decomposition for the Toda lattice has been discussed in Ref. 28 (see also Ref. 29). Note also that the characteristic singular manifolds considered in Refs. 24 and 26 correspond to the second Birkhoff stratum ($\tau=0$, $\tau_x=0$).

A problem closely related to the study of these singular sectors is the following: A standard KP (and KdV) hierarchy flows in the so-called big cell of the Grassmannian (dense open subset of the Grassmannian). The Birkhoff strata are subsets with finite codimension. Are there any integrable systems associated with the Birkhoff strata? A positive answer to this question has been given recently in Ref. 30. It occurs that in the KdV case the corresponding integrable hierarchies are connected to the Schrödinger equation with energy-dependent potential.

In the present paper we study integrable hierarchies associated with the singular sector of the KP hierarchy. This sector consists of different Birkhoff strata or equivalently of different Schubert cells. The Schubert cells have finite dimension and are connected with the family of the Calogero–Moser-type models which describe motion of poles. Here we will concentrate on integrable systems associated with Birkhoff strata. We show that they can be constructed by restricting the standard KP hierarchy to submanifolds of finite codimension in the space of independent variables. To build these hierarchies we will mainly use the $\bar{\partial}$-dressing method. Integrable systems associated with Birkhoff strata are rather complicated as well as the corresponding linear problems. They are of high order, though there are effectively $2+1$ dimensional hierarchies. For higher Birkhoff strata these integrable equations clearly demonstrate a sort of quasi-multidimensionality. We also discuss hidden Gelfand–Dikii hierarchies. Besides illustrating by simpler formulas some of the results of this work, we can in this case provide useful methods to construct solutions.

An important property of the hidden KP hierarchies is that they are associated with $\bar{\partial}$ operators of nonzero index. This result is due to the interpretation of the Grassmannian as the space of boundary conditions for the $\bar{\partial}$ operator acting on the Hilbert space of square integrable functions. We prove that the codimension of Birkhoff strata coincides with the index of corresponding $\bar{\partial}$ operator up to sign.

We finish this introduction by describing the plan of the work. In Sec. II we recall some basic facts about the KP hierarchy. In Sec. II A we briefly present the $\bar{\partial}$-dressing method. In Sec. II B we review the Grassmannian and its stratification. The relation between singular sectors of the KP hierarchy and $\bar{\partial}$ operators of nonzero index is considered in Sec. II C. Section III is devoted to the construction of the hidden KP hierarchies. In Sec. III A the case index $\bar{\partial}=-1$ is carefully analyzed. The case index $\bar{\partial}=-2$ is studied in Sec. III B and the cases of index $\bar{\partial}=-3$ and higher indices are discussed in Sec. III C. Finally, hidden Gelfand–Dikii hierarchies are analyzed in Sec. IV. We prove that under certain conditions the only hidden Gelfand–Dikii hierarchies are the KdV hidden hierarchy and the Boussinesq hidden hierarchy. The former is studied in Sec. IV A, the latter in Sec. IV B. A method of constructing solutions is developed in Sec. IV C.

II. THE STANDARD KP HIERARCHY AND SOME GENERAL METHODS

We start by recalling some basic facts about the KP hierarchy and some of the methods developed to its study. The KP hierarchy can be described in various ways (see, e.g., Refs. 1–8). The most compact form of it is given by the Lax equation

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n=1,2,3,..., \tag{2.1}$$

where

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + u_3\partial^{-3} + \cdots \tag{2.2}$$

is the formal pseudodifferential operator, $\partial = \partial/\partial t_1$ and $(L^n)_+$ denotes the differential part of L^n. Equation (2.1) is an infinite set of equations for scalar functions $u_1, u_2, u_3, \ldots$. These equations allow one to express $u_2, u_3, \ldots$ via u_1 and its derivatives with respects to t_1 and t_2. As a result one gets the usual form of the KP hierarchy given by

$$\frac{\partial u}{\partial t_n} = f_n \left(u, \frac{\partial u}{\partial t_1}, \frac{\partial u}{\partial t_2}, \ldots, \frac{\partial^{-1} u}{\partial t_1}, \ldots \right), \quad n = 3,4,5,\ldots, \tag{2.3}$$

where $u = u_1$ and f_n are certain functions on $u, \partial u/\partial t_1, \partial u/\partial t_2, \ldots, \partial^{-1} u/\partial t_1, \ldots$. The simplest of these equations is

$$\frac{\partial u}{\partial t_3} = \frac{\partial^3 u}{\partial t_1^3} + 6u \frac{\partial u}{\partial t_1} + 3 \left(\frac{\partial}{\partial t_1} \right)^{-1} \frac{\partial^2 u}{\partial t_2^2}. \tag{2.4}$$

Equation (2.1) arises as the compatibility condition of the linear equations

$$L\psi = \lambda\psi \tag{2.5}$$

and

$$\frac{\partial \psi}{\partial t_n} = (L^n)_+ \psi, \quad n = 1,2,3,\ldots, \tag{2.6}$$

where $\psi = \psi(\mathbf{t}, \lambda)$ is the wave function of the KP hierarchy, λ is a complex parameter (spectral parameter), and $\mathbf{t} = (t_1, t_2, \ldots, t_n, \ldots) \in \mathbb{C}^\infty$. This wave function has the form

$$\psi = \exp \left(\sum_{n=1}^\infty \lambda^n t_n \right) \chi(\mathbf{t}, \lambda), \tag{2.7}$$

where

$$\chi(\mathbf{t}, \lambda) = 1 + \frac{\chi_1(\mathbf{t})}{\lambda} + \frac{\chi_2(\mathbf{t})}{\lambda^2} + \cdots \quad \text{for large } \lambda. \tag{2.8}$$

The functions $\psi(\mathbf{t}, \lambda)$ and the adjoin wave function $\psi^*(\mathbf{t}, \lambda)$ [solution of equations formally adjoined to Eqs. (2.5) and (2.6)] obey the famous Hirota bilinear equation

$$\int_{S_\infty} d\lambda \, \psi(\mathbf{t}, \lambda) \psi^*(\mathbf{t}', \lambda) = 0 \quad (\text{all } \mathbf{t} \text{ and } \mathbf{t}'), \tag{2.9}$$

where S_∞ is a small circle around $\lambda = \infty$.

Finally, the wave function is connected with the τ function via

$$\chi(\mathbf{t}, \lambda) = \frac{\tau(\mathbf{t} - [\lambda^{-1}])}{\tau(\mathbf{t})}, \tag{2.10}$$

where $[a] := (a, \frac{1}{2}a^2, \frac{1}{3}a^3, \ldots)$.

A. The $\bar{\partial}$-dressing method

Now, we present a sketch of the $\bar{\partial}$-dressing method (see, e.g., Refs. 14–16). It is based on the nonlocal $\bar{\partial}$ problem:

$$\frac{\partial \chi(\mathbf{t},\lambda,\bar{\lambda})}{\partial \bar{\lambda}} = \int\int_G d\mu \Lambda d\bar{\mu} \chi(\mathbf{t},\mu,\bar{\mu}) g(\mathbf{t},\mu) R_0(\mu,\bar{\mu},\lambda,\bar{\lambda}) g^{-1}(\mathbf{t},\lambda), \tag{2.11}$$

where χ is a scalar function, $R_0(\mu,\bar{\mu},\lambda,\bar{\lambda})$ is an arbitrary function, the bar means complex conjugation, G is a domain in $\mathbb{C}$, and $g(\mathbf{t},\mu)$ is a certain function of $\mathbf{t}$ and the spectral parameter. It is assumed that χ is properly normalized $[\chi \xrightarrow{\lambda\to\lambda_0} \eta(\lambda_0)]$ and Eq. (2.11) is uniquely solvable. By virtue of the generalized Cauchy formula, the $\bar{\partial}$ problem (2.11) is equivalent to a linear integral equation. The form of this linear equation [and corresponding nonlinear equations, associated with (2.11)] is encoded in the dependence of the function g on $\mathbf{t}$. To extract these equations, we introduce long derivatives

$$\nabla_n = \frac{\partial}{\partial t_n} + g^{-1}(\mathbf{t},\lambda) g_{t_n}(\mathbf{t},\lambda), \tag{2.12}$$

where $g_{t_n} \equiv \partial_g / \partial t_n$. Then, we consider the Manakov ring of differential operators of the form

$$L = \sum_{n_1,n_2,\ldots} u_{n_1 n_2 n_3 \ldots}(\mathbf{t}) \nabla_1^{n_1} \nabla_2^{n_2} \nabla_3^{n_3} \cdots, \tag{2.13}$$

where $u_{n_1 n_2 n_3 \ldots}(\mathbf{t})$ are scalar functions. In this ring we select those L which obey the conditions

$$\left[L, \frac{\partial}{\partial \bar{\lambda}} \right] \chi = 0 \tag{2.14}$$

and $L\chi \to 0$ as $\lambda \to \infty$. Condition (2.14) means that $L\chi$ has no singularities in G. The unique solvability of (2.11) implies that for such L one has

$$L_i \chi = 0. \tag{2.15}$$

The set of equations (2.15) is known as the system of linear problems. Note that taking into account that $(\partial / \partial t_n)\psi = g \nabla_n \chi$, Eq. (2.15) can be equivalently written as

$$L_i \psi = 0, \tag{2.16}$$

where in operators L_i one has to substitute ∇_n by $\partial / \partial t_n$. The compatibility conditions of (2.15) [or (2.16)] are equivalent to nonlinear equations for $u_{n_1 n_2 n_3 \ldots}(\mathbf{t})$, which are solvable by the $\bar{\partial}$-dressing method. One has to select a basis among an infinite set of linear equations (2.15) [or (2.16)]. If one considers an infinite family of times $t_n (n=1,2,3,\ldots)$ one has an infinite basis of operators L_i and, consequently an infinite hierarchy of nonlinear integrable equation associated with (2.11).

To get the standard KP hierarchy one can choose the canonical normalization of χ (i.e., $\chi \to 1 + \chi_1 / \lambda + \chi_2 / \lambda^2 + \cdots$ as $\lambda \to \infty$) and put

$$g = \exp\left(\sum_{n=1}^{\infty} \lambda^n t_n \right).$$

The long derivatives ∇_n are $\nabla_n = \partial / \partial t_n + \lambda^n$ $(n=1,2,3,\ldots)$ and the corresponding linear problems take the form

$$L_n \chi = \left(\nabla_n - \sum_{k=0}^{n} u_k(\mathbf{t}) \nabla_1^k \right) \chi = 0, \tag{2.17}$$

or equivalently

$$\frac{\partial \psi}{\partial t_n} = \sum_{k=0}^{n} u_k(\mathbf{t}) \frac{\partial^k \psi}{\partial t_1^k}, \quad n=1,2,3,\ldots, \tag{2.18}$$

where $u_k(\mathbf{t})$ are scalar functions. Equation (2.18) is just equation (2.6) and their compatibility conditions are equivalent to the KP hierarchy in the usual form. The $\bar{\partial}$-dressing method provides a wide class of exact explicit solutions of the KP hierarchy which correspond to degenerate kernels $R_0(\mu,\bar{\mu},\lambda,\bar{\lambda})$ of the $\bar{\partial}$-problem (2.11) (see Refs. 14–16, and 4). It is worth realizing that the $\bar{\partial}$ problem for ψ and the adjoin $\bar{\partial}$ problem for $\psi^*(\mathbf{t},\lambda)$ imply the Hirota bilinear identity (2.9).

Note that in the KP case, the domain G is $D_0 = \mathbb{C} - D_\infty$, where D_∞ is a small disk around $\lambda = \infty (\partial D_\infty = S_\infty)$. In a similar manner, one can formulate the KP hierarchy if one chooses G such that $\partial G = S$ (S being the unit circle).

B. Grassmannian and stratification

Next, we comment on some basic facts about the Grassmannian approach in relation to the standard KP hierarchy. Following Refs. 8 and 27, we consider the Grassmannian Gr as the set of linear subspaces W of formal Laurent series on the circle S_∞. That means that W possesses an algebraic basis

$$W = \{w_0(\lambda), w_1(\lambda), w_2(\lambda), \ldots\} \tag{2.19}$$

with the basis elements

$$w_n(\lambda) = \sum_{i=-\infty}^{s_n} a_i \lambda^i \tag{2.20}$$

of finite order. Here $s_0 < s_1 < s_2 < \cdots$ and $s_n = n$ for large n. It can be proved that Gr is a connected Banach manifold which exhibits a stratified structure.[13,27] To describe this structure one introduces the set $\mathcal{S}_0$ of increasing sequences of integers

$$S = \{s_0, s_1, s_2, \ldots\}, \tag{2.21}$$

such that $s_n = n$ for large n. One can associate to each $W \in$ Gr the set of integers

$$S_W = \{n \in \mathbb{Z} : \exists w \in W \text{ of order } n\} \in \mathcal{S}_0.$$

On the other hand, given $S \in \mathcal{S}_0$ one may define the subset of Gr,

$$\Sigma_s = \{W \in \text{Gr} : S_W = S\}, \tag{2.22}$$

which is called the Birkhoff stratum associated with S. The stratum Σ_s is a submanifold of Gr of finite codimension $l(S) = \Sigma_{n \geqslant 0}(n - s_n)$. In particular, if $S = \{0,1,2,3,\ldots\}$ the corresponding stratum has codimension zero and it is a dense open subset of Gr which is called the principal stratum or the big cell. Lower Birkhoff strata correspond to $S = \{s_0, s_1, s_2, \ldots\}$ different from $\{0,1,2,3,\ldots\}$.

The KP hierarchy wave function $\psi(\mathbf{t},\lambda)$ (2.7), (2.8) leads naturally to a family $W(\mathbf{t})$ in Gr.[27] In order to see it we start by introducing ($\lambda \in S_\infty$):

$$W = \text{span}\{\psi(\mathbf{t},\lambda), \text{ all } \mathbf{t}\}. \tag{2.23}$$

Using (2.6) and Taylor expansions, one gets

$$W = \text{span}\{\psi, \partial_1 \psi, \partial_1^2 \psi, \ldots\}, \tag{2.24}$$

where $\partial \equiv \partial/\partial t_1$. Now, the flow defined as

$$W(\mathbf{t}) := \exp\left(-\Sigma_{k \geq 1}\lambda^n t_n\right)W,$$

can be characterized as

$$W(\mathbf{t}) = \operatorname{span}\{\chi(\mathbf{t},\lambda), \nabla_1\chi(\mathbf{t},\lambda), \nabla_1^2\chi(\mathbf{t},\lambda),...\}, \tag{2.25}$$

where $\nabla_1 = \partial/\partial t_1 + \lambda$. Since $\nabla_1^n\chi = \lambda^n + O(\lambda^{n-1})$, one has

$$S_{W(\mathbf{t})} = \{0,1,2,...\}. \tag{2.26}$$

So the flows $W(\mathbf{t})$ generated by the standard KP hierarchy belong to the principal Birkhoff stratum.[27,31] Then, it seems natural to wonder if there exist integrable structures associated with other Birkhoff strata. In this sense, only recently some progress has been made. In Ref. 30 it was shown that for the KdV hierarchy, (reduction of the KP hierarchy) evolutions associated with the Birkhoff strata are given by integrable hierarchies arising from the Schrödinger equations with energy dependent potentials. One of the main goals of the present paper is the study of integrable structures associated with the full KP hierarchy outside the principal stratum.

C. $\bar{\partial}$ operators of nonzero index and singular sector of KP

Finally, we propose a wider approach which reveals the connection of stratification of the Grassmannian with the analytic properties of the $\bar{\partial}$ operators. This approach is based on the observation that the Grassmannian can be viewed as the space of boundary conditions for the $\bar{\partial}$ operator.[5] Let us consider the Hilbert space H of square integrable functions $w = w(\lambda,\bar{\lambda})$ on $\Omega := \mathbb{C} - D_\infty$ (where D_∞ is a small disk around the point $\lambda = \infty$), with respect to the bilinear form:

$$\langle u,v \rangle = \int\int_\Omega u(\lambda,\bar{\lambda})v(\lambda,\bar{\lambda})\frac{d\lambda \wedge d\bar{\lambda}}{2\pi i\lambda}. \tag{2.27}$$

Then, given $W \in \mathrm{Gr}$ (described above) there is an associated domain $\mathcal{D}_W$ on H for $\bar{\partial}$, given by those functions w for which $\bar{\partial}w \in H$ and such that their boundary values on S_∞ are in W. Thus, we have an elliptic boundary value problem. To formulate it correctly, that is to have a skew-symmetric $\bar{\partial}$ operator:

$$\langle v,\bar{\partial}u \rangle = -\langle \bar{\partial}v,u \rangle \quad \forall u \in \mathcal{D}_W, \quad \forall v \in \mathcal{D}_{\widetilde{W}} \tag{2.28}$$

one has to define $\widetilde{W}$, the dual of an element W in the Grassmannian, as the space of formal Laurent series $v(\lambda)$ of $\lambda \in S_\infty(S_\infty = \partial D_\infty)$ which obey the condition

$$\int_{S_\infty}\frac{d\lambda}{2\pi i\lambda}v(\lambda)u(\lambda) = 0, \quad \forall u \in W. \tag{2.29}$$

Properties of W and $\widetilde{W}$ are convenient to evaluate the index of the $\bar{\partial}$ operator. Let S_W and $S_{\widetilde{W}}$ be subsets of integers determined by the orders of elements in W and $\widetilde{W}$. Then, we have

$$S_{\widetilde{W}} = \{-n | n \notin S_W\}. \tag{2.30}$$

Let $\bar{\partial}_W$ denote the operator $\bar{\partial}$ acting on the domain $\mathcal{D}_W$. The index of this operator is defined as

$$\text{index } \bar{\partial}_W := \dim(\ker \bar{\partial}_W) - \dim(\operatorname{coker} \bar{\partial}_W).$$

It can be determined as

$$\text{index } \bar{\partial}_W = \text{card}(S_W - \mathbb{N}) - \text{card}(S_{\tilde{W}} - \mathbb{N}), \tag{2.31}$$

where $\mathbb{N} \equiv \{0,1,2,...\}$. Note that the index of the $\bar{\partial}_W$ operator is closely connected to the notion of virtual dimension of W used in Ref. 13,

$$\text{v.d.}(W) = \text{card}(S_W - \mathbb{N}) - \text{card}(\mathbb{N} - S_{\tilde{W}}).$$

Indeed,

$$0 \in S_W \Rightarrow \text{index } \bar{\partial}_W = \text{v.d.}(W). \tag{2.32}$$

Let us consider now subspaces $W(\mathbf{t})$ generated by the standard KP hierarchy. Since $S_{W(\mathbf{t})} = \{0,1,2,...\}$ [see (2.26)] and $S_{\tilde{W}(\mathbf{t})} = \{1,2,...\}$, one has that index $\bar{\partial}_{W(\mathbf{t})} = 0$. Thus, all the equations of the standard KP hierarchy are associated with a sector of Gr with zero index of $\bar{\partial}$.

Now, what about the case of nonzero index? How to characterize these sectors in Gr? Are there integrable systems associated with them? Addressing these questions is the main subject of our paper. The answer can be formulated as follows: Given the wave function $\psi(\mathbf{t},\lambda)$ of the standard KP hierarchy, we consider submanifolds $\mathcal{M}$ of finite codimension in the space $\mathbb{C}^\infty$ which are defined by m constraints

$$f_i(\mathbf{t}) = 0, \quad i = 1,2,...,m \tag{2.33}$$

(f_i are some analytic functions) imposed on the independent variables $\mathbf{t} = (t_1, t_2,...)$. The point is that under appropriate conditions, the restriction ψ_{res} of the wave function ψ on $\mathcal{M}$ determines families $W_{\text{res}}(\mathbf{s})$ in Gr which correspond to $\bar{\partial}$ operators of nonzero index. We will show that these sectors of Gr are associated with integrable hierarchies.

The nonzero $\bar{\partial}$-index sector of Gr is closely connected to its singular sector. Indeed, suppose that the restriction of ψ on the manifold (2.33) defines the corresponding family $W_{\text{res}}(\mathbf{s})$ such that

$$S_{W_{\text{res}}(\mathbf{s})} = \mathbb{N} - \{r_1, r_2,...,r_l\}. \tag{2.34}$$

Then it is clear that

$$\lambda^{-l} W_{\text{res}}(\mathbf{s}) \tag{2.35}$$

are elements of Gr with zero virtual dimension and, consequently, the same holds for $\lambda^{-l} W_{\text{res}}$. Therefore, there is a τ function, $\tau(\mathbf{t})$, associated with $\lambda^{-l} W_{\text{res}}$ such that

$$\tilde{\chi}(\mathbf{t},\lambda) = \frac{\tau(\mathbf{t} - [\lambda^{-1}])}{\tau(\mathbf{t})}.$$

On the other hand, the elements of minimal order in (2.35) are $\lambda^{-l}\chi_{\text{res}}(\mathbf{s},\lambda)$. This means that the function $\tilde{\chi}$ is singular on the submanifold $\mathcal{M}$ or, equivalently

$$\tau|_{\mathcal{M}} \equiv 0. \tag{2.36}$$

Then, the submanifolds $\mathcal{M}$ leading to the nonzero index sector of Gr are zero manifolds of the τ-function of the KP hierarchy. This property is rather obvious if one observes that (2.35) determines a domain for the $\bar{\partial}$ operator with a nontrivial kernel. Consequently, the determinant of the $\bar{\partial}$ operator, which is proportional to $\tau(\mathbf{t}(\mathbf{s}))$,[13] vanishes. Therefore, the constrained wave function $\psi_{\text{res}}(\mathbf{s},\lambda)$ is a regularization of the wave function $\tilde{\psi}(\mathbf{t},\lambda)$ on the blow-up submanifold $\mathcal{M}$.

The above coincidence between a singular and nonzero $\bar{\partial}$ index sectors is an important feature of the KP hierarchy. It demonstrates close relation between singular and nonzero $\bar{\partial}$ index aspects of the integrable hierarchies.

Note that integrable hierarchies with constrained independent variables have been discussed in different context also in Refs. 32 and 33.

III. HIDDEN KP HIERARCHIES

Now, we proceed to the construction of nonlinear systems associated with manifolds of finite codimensions given by constraints (2.33). First, we realize that for "good" functions f_i, the theorem of implicit function implies that one can solve Eq. (2.33) with respect to any m variables, i.e., one can express m variables $t_{n_1}, t_{n_2}, \ldots, t_{n_m}$ as functions of the others $s = (\ldots, t_n, \ldots)$ $\times (n \notin \{n_1, \ldots, n_m\})$ in the form

$$t_{n_i} = b_i(s), \quad i = 1, \ldots, m. \tag{3.1}$$

Formula (3.1) gives us the parametrization of the manifold given by (2.33) by the independent variables s. Since any set of m times can be chosen as t_{n_i} in (3.1), one has an infinite number of different parametrizations of the same manifold (2.33).

The KP wave function $\psi(\mathbf{t}, \lambda)$ restricted to the manifold (3.1), i.e., the function $\psi_{\text{res}}(s, \lambda)$ is regularizable.[24] Since in what follows, we have to consider $\psi_{\text{res}}(s, \lambda)$ and $\chi_{\text{res}}(s, \lambda)$ instead of ψ and χ we will omit the label res in both cases. In order to construct restricted KP hierachies we start with the $\bar{\partial}$-problem (2.11) for the regularized restricted function $\chi(s, \lambda)$. This function has canonical normalization, the corresponding $\bar{\partial}$ problem is uniquely solvable and

$$g(s, \lambda) = \exp\left(\sum_{n=1}^{\infty} \lambda^n t_n(s)\right). \tag{3.2}$$

Then, one can use all the machinery of the $\bar{\partial}$-dressing method.

A. The case index $(\bar{\partial}) = -1$

We start our study of the hidden KP hierarchies by considering the simplest case $m = 1$. Suppose first that we solve the constraint $f(\mathbf{t}) = 0$ with respect to t_1, then we have

$$t_1 = b_1(s_2, s_3, \ldots), \tag{3.3}$$

$$g(s, \lambda) = \exp\left(\lambda b_1(s) + \sum_{n=2}^{\infty} \lambda^n s_n\right) \tag{3.4}$$

and the long derivatives ∇_n are

$$\nabla_n = \frac{\partial}{\partial s_n} + \lambda \frac{\partial b_1(s)}{\partial s_n} + \lambda^n, \quad n = 2, 3, \ldots. \tag{3.5}$$

Since in this case the operator of the first order in λ is missing in the basis (2.25) one has

$$S_{W_{\text{res}}(s)} = \mathbb{N} - \{1\},$$

consequently, due to (2.30) and (2.31),

$$\text{index}(\bar{\partial}_{W_{\text{res}}(s)}) = -1. \tag{3.6}$$

In order to get the linear problems (2.16) associated to (3.4), one has to construct the operators L of the form (2.13) which obey conditions (2.14) and $L\chi \to 0$ as $\lambda \to \infty$. This is a tedious but straightforward calculation after which one gets the infinite set of linear problems

$$\partial_x^3 \psi - \partial_y^2 \psi = u_0 \psi + u_1 \partial_x \psi + u_2 \partial_y \psi + u_3 \partial_x^2 \psi + u_4 \partial_x \partial_y \psi, \tag{3.7}$$

and

$$\partial_{s_4} \psi - \partial_x^2 \psi = v_0 \psi + v_1 \partial_x \psi + v_2 \partial_y \psi,$$

$$\partial_{s_5} \psi = p_0 \psi + p_1 \partial_x \psi + p_2 \partial_y \psi + p_3 \partial_x^2 \psi + \partial_x \partial_y \psi, \dots \tag{3.8}$$

$$\partial_{s_n} \psi = \partial_x^{n_2} \partial_y^{n_3} \psi + \cdots,$$

where $n = 4,5,6,\dots$; $n = 2n_2 + 3n_3$ with $n_2 = 2,3,\dots$ and $n_3 = 0,1$, and we have made $x := s_2$, $y := s_3$. By denoting $b := b_1$, we have that the coefficients in (3.7) and (3.8) are given by

$$u_0 = b_{xxx}\chi_1 + 3b_{xx}\chi_{1x} + 3\chi_{2xx} + 3b_x\chi_{1xx} + 3b_x^2\chi_{2x} + 3b_x\chi_{3x} + 3\chi_{4x} - b_{yy}\chi_1 - 2b_y\chi_{1y} - 3b_x b_{xy}\chi_1$$
$$+ b_x b_y \chi_{1x} - 3b_x^2 \chi_{1y} - 3b_x \chi_{2y} + 4b_y \chi_{2x} - b_y b_{xx}\chi_1 - 3b_y \chi_1 \chi_{1x} - 3\chi_3 \chi_{1x} - 3b_x^2 b_{xx}\chi_1$$
$$- 3b_x \chi_1 \chi_{2x} + 2b_x \chi_1 \chi_{1y} - 3b_x \chi_2 \chi_{1x} - 3\chi_2 \chi_{2x} + 3\chi_1 \chi_2 \chi_{1x} + 2\chi_2 \chi_{1y} - 2\chi_{3y}, \tag{3.9}$$

$$u_1 = 3b_x b_{xx} + 3b_x \chi_{1x} + 3\chi_{2x} - 3\chi_{1x}\chi_1 - 2\chi_{1y} - b_y^2 - b_x^2 b_y,$$

$$u_2 = 3b_{xx} + b_x^3 + 3\chi_{1x} + b_x b_y, \qquad u_3 = -2b_y, \qquad u_4 = 3b_x,$$

$$v_0 = 2\chi_1 \chi_{1x} - 2\chi_{2x} - 2b_x \chi_{1x}, \qquad v_1 = -b_x^2, \qquad v_2 = -2b_x.$$

$$p_3 = -b_x, \qquad p_2 = 2b_x^2 - b_y, \qquad p_1 = b_x^3 - b_x b_y - \chi_{1x},$$

$$p_0 = \chi_1 \chi_{2x} + \chi_1 \chi_{1y} - 2b_x \chi_1 \chi_{1x} - \chi_1^2 \chi_{1x} + \chi_{1x}\chi_2 - \chi_{3x} - \chi_{2y} - b_y \chi_{1x} - b_x \chi_{1y} + 2b_x \chi_{2x} + 2b_x^2 \chi_{1x}.$$

The absence of an operator ∇ of the pure first-order λ imposes the constraints:

$$b_{xxx} + 3\chi_{1xx} + 3\chi_{3x} - b_{yy} - 2\chi_{2y} - 3b_x b_{xy} - b_x \chi_{1y} + b_y \chi_{1x} - b_{xx} b_y$$
$$- 3\chi_{1x}\chi_2 - 3b_x^2 b_{xx} - 3\chi_{2x}\chi_1 + 3\chi_{1x}\chi_1^2 + 2\chi_{1y}\chi_1 = 0,$$

$$b_{s_4} - b_{xx} - 2\chi_{1x} + 2b_x b_y + b_x^3 = 0, \tag{3.10}$$

$$b_{s_5} - b_{xy} + b_x b_{xx} + b_y^2 - b_x^2 b_y - b_x^4 - \chi_{2x} - \chi_{1y} + 3b_x \chi_{1x} + \chi_1 \chi_{1x} = 0$$

and so on. Higher linear problems and the expressions of the coefficients are complicated and we omit them. The linear problems (3.7) and (3.8) represent themselves as an infinite hierarchy of the linear problems for the restricted (hidden) KP hierarchy with index $\bar{\partial} = -1$. All these problems are compatible by construction and the compatibility conditions for them give rise to an infinite hierarchy of nonlinear evolution equations, a restricted (hidden) KP hierarchy. The simplest system of this hidden KP hierarchy is the one associated with (3.7) and the first equation in (3.8) and has the form

$(t:=s_4)$:

$$u_{4t} = -u_{4xx} - \tfrac{8}{9}u_4 u_{4y} - \tfrac{5}{9}u_4^2 u_{4x} + \tfrac{4}{3}u_3 u_{4x} + 2u_{2x},$$

$$u_{3t} = -\tfrac{2}{3}u_4 u_{4xx} - \tfrac{2}{3}u_{4x}^2 + 3v_{0x} + u_{3xx} + 2u_{1x} - \tfrac{1}{9}u_4^2 u_{3x} - \tfrac{4}{3}u_3 u_{4y} - \tfrac{2}{3}u_4 u_{3y} + \tfrac{2}{9}u_4^2 u_{4y} - \tfrac{2}{9}u_3 u_4 u_{4y},$$

$$(3.11)$$

$$u_{2t} = -\tfrac{2}{3}u_{4xxx} + u_{2xx} - \tfrac{1}{9}u_4^2 u_{2x} - \tfrac{2}{3}u_2 u_{4y} + \tfrac{2}{3}u_{4yy} - \tfrac{2}{3}u_4 u_{2y} - 2v_{0y} + \tfrac{2}{3}u_4 u_{4xy} - \tfrac{2}{3}u_2 u_4 u_{4x} - u_4 v_{0x}$$

$$+ \tfrac{2}{3}u_3 u_{4xx} + \tfrac{2}{3}u_1 u_{4x},$$

$$u_{1t} = -\tfrac{2}{9}u_4 u_{4xxx} - \tfrac{8}{9}u_{4x}u_{4xx} + 3v_{0xx} + u_{1xx} + 2v_{0x} - \tfrac{1}{9}u_4^2 u_{1x} - \tfrac{4}{3}u_1 u_{4y} + \tfrac{2}{9}u_4 u_{4yy} + \tfrac{2}{9}u_{4y}^2 - \tfrac{2}{3}u_4 u_{1y}$$

$$- u_4 v_{0y} + \tfrac{2}{9}u_4^2 u_{4xy} + \tfrac{2}{9}u_4 u_{4x}u_{4y} - \tfrac{4}{9}u_1 u_4 u_{4x} - \tfrac{2}{9}u_3 u_4 u_{4xx} - \tfrac{2}{9}u_3 u_{4x}^2 - 2u_3 v_{0x} + \tfrac{2}{9}u_2 u_4 u_{4y},$$

$$u_{0t} = v_{0xxx} + u_{0xx} - \tfrac{1}{9}u_4^2 u_{0x} - \tfrac{4}{3}u_0 u_{4y} - v_{0yy} - \tfrac{2}{3}u_4 u_{0y} - u_4 v_{0xy} - \tfrac{2}{3}u_0 u_4 u_{4x} - u_3 v_{0xx} - u_2 v_{0y} - u_1 v_{0x}$$

where

$$v_0 = \tfrac{1}{27}u_4^2 u_3 - \tfrac{1}{6}u_3^2 + \tfrac{2}{9}u_4 u_{4x} - \tfrac{2}{3}u_1 + \tfrac{4}{9}u_{4y} - \tfrac{4}{9}\partial_y \partial_x^{-1}(u_2 - \tfrac{1}{27}u_4^3 + \tfrac{1}{6}u_3 u_4).$$

Note that since due to (3.9) $u_4 = 3b_x$ and $u_3 = 2b_y$, one can rewrite the system (3.11) as a system of four equations for the variables b, u_2, u_1, and u_0. Equation (3.11) and higher equations are solvable by the $\bar{\partial}$-dressing method, though all solutions would be expressed in implicit form. Then, we have an infinite hierarchy of integrable $2+1$-dimensional equations. This hierarchy is associated with a restricted element in the Grassmannaian which satisfies

$$W_{\text{res}}(s) = \text{span}\{\nabla_x^n \nabla_y^m \chi(s,\lambda), n \geq 0, m = 0,1\}. \tag{3.12}$$

Note that the basis of the space $W_{\text{res}}(s)$ is formed by two-dimensional jets of special form in contrast to the one-dimensional jets (2.25) for the standard KP hierarchy with index $\bar{\partial} = 0$.

Note also that if one tries to construct the linear problem starting with g of the form (3.4) with $b_1 = \text{const}$, the procedure collapses. In this case (3.10) gives too strong constraints on the function χ ($\chi_{1x} = 0, \chi_{2x} + \chi_{1y} = 0, \dots$).

As has been mentioned before, the formula (3.3) gives only one possible parametrization of the manifold defined by the equation $f(\mathbf{t}) = 0$. Let us take the one given by

$$t_2 = b_2(s_1, s_3, s_4, \dots), \tag{3.13}$$

so

$$g(\mathbf{s}, \lambda) = \exp\left(\lambda s_1 + \lambda^2 b_2(\mathbf{s}) + \sum_{i=3}^{\infty} \lambda^i s_i \right) \tag{3.14}$$

and long derivatives are

$$\nabla_n = \frac{\partial}{\partial s_n} + \lambda^2 \frac{\partial b_2}{\partial s_n} + \lambda^n, \quad n = 1,3,4,\dots. \tag{3.15}$$

Similar to the previous case, one gets an infinite family of linear problems with the corresponding family of constraints. The first linear problem is of the form ($x:=s_1$, $y:=s_3$, $t:=s_4$)

$$\partial_x^3 \psi = u_5 \partial_y^2 \psi + u_4 \partial_x \partial_y \psi + u_3 \partial_x^2 \psi + u_2 \partial_y \psi + u_0 \psi,$$

$$\partial_t \psi = v_3 \partial_x^2 \psi + v_2 \partial_y \psi + v_1 \partial_x \psi + v_0 \psi.$$

The compatibility condition for this system leads to the first integrable system of this hidden KP hierarchy. The expressions of the potentials, the constraints and the system are a bit more complicated than in the above-studied case and we omit them here. Note that in this case we have

$$W_{res}(\mathbf{s}) = \text{span}\{\nabla_x^n \nabla_y^m \chi(\mathbf{s}, \lambda), n \geq 0, m = 0, 1\} \tag{3.16}$$

and consequently

$$\text{index}(\bar{\partial}_{W_{res}(\mathbf{s})}\text{res}) = -1. \tag{3.17}$$

Let us consider now the next choice,

$$t_3 = b_3(s_1, s_2, s_4, \ldots). \tag{3.18}$$

One has

$$g = \exp\left(\lambda s_1 + \lambda^2 s_2 + \lambda^3 b_3(\mathbf{s}) + \sum_{i=4}^{\infty} \lambda^i s_i \right) \tag{3.19}$$

and the long derivatives have the form

$$\nabla_n = \frac{\partial}{\partial s_n} + \lambda^3 \frac{\partial b_3}{\partial s_n} + \lambda^n, \quad n = 1, 2, 4, 5, \ldots . \tag{3.20}$$

Using these long derivatives we can find the corresponding hierarchy of linear problems and nonlinear integrable equations. On first sight, one could think that as $\nabla_x := \nabla_1$ and $\nabla_y := \nabla_2$ are third-order operators, the operators of the first and the second order are missing and consequently index $\bar{\partial} = -2$. However, by taking the linear combination of long derivatives ($b := b_3$ here)

$$b_y \nabla_x - b_x \nabla_y = b_y \partial_x + \lambda b_y - b_x \partial_y - \lambda^2 b_x,$$

we see that only the first order in λ is missing and then, we have index $\bar{\partial} = -1$.

A similar situation takes place in the general case

$$t_n = b_n(s_1, \ldots, s_{n-1}, s_{n+1}, \ldots) \tag{3.21}$$

for $n \geq 3$. One has

$$\nabla_m = \frac{\partial}{\partial s_m} + \lambda^n \frac{\partial b_n}{\partial s_m} + \lambda^m, \quad m \neq n. \tag{3.22}$$

In a similar way to the previous case, one gets the operators of the second, third, ..., $(n-1)$th order by taking linear combinations of the operators $\nabla_1, \nabla_2, \ldots, \nabla_{n-1}$. Then, we have that in the general case index $\bar{\partial} = -1$.

We finally point out that all the hierarchies of linear problems and integrable systems considered in this section are closely connected. In fact, since they are associated with different parametrizations of the same manifold $f(\mathbf{t}) = 0$, they are related to each other by change of independent and dependent variables.

B. The case index $(\bar{\partial}) = -2$

Suppose now that we take $m = 2$ in (2.33), i.e., the manifold $\mathcal{M}$ is defined as

$$f_1(\mathbf{t}) = 0, \quad f_2(\mathbf{t}) = 0.$$

Then, one possible parametrization is

$$t_1 = b_1(s_3, s_4, \dots), \quad t_2 = b_2(s_3, s_4, \dots),$$

and the function g and the long derivatives are given by

$$g(\mathbf{s}, \lambda) = \exp\left(\lambda b_1(s) + \lambda^2 b_2(s) + \sum_{i=3}^{\infty} \lambda^i s_i \right),$$

$$\nabla_k = \frac{\partial}{\partial s_k} + \lambda \frac{\partial b_1}{\partial s_k} + \lambda^2 \frac{\partial b_2}{\partial s_k} + \lambda^k, \quad k = 3, 4, 5, \dots .$$

In order to construct linear problems we look for the operators of the form

$$L = \sum u_{n_3 n_4 n_5 \dots} \nabla_3^{n_3} \nabla_4^{n_4} \nabla_5^{n_5} \dots$$

satisfying $L\chi = 0$. It is easy to see that for $n \neq 1, 2$ we have an order n operator of the form $\nabla_3^{n_3} \nabla_4^{n_4} \nabla_5^{n_5}$ with $n_3 = 0, 1, 2, \dots$ and n_4, $n_5 = 0, 1$. On the other hand, as each long derivative is of a order ≥ 3, there are not operators of order 1 and 2. Then,

$$W_{\text{res}}(\mathbf{s}) = \text{span}\{ \nabla_3^{n_3} \nabla_4^{n_4} \nabla_5^{n_5} \chi(\mathbf{s}, \lambda), n_3 = 0, 1, 2, \dots, n_4, n_5 = 0, 1 \},$$

$$S_{W_{\text{res}}(\mathbf{s})} = \mathbb{N} - \{1, 2\} \tag{3.23}$$

and consequently

$$\text{index}(\bar{\partial}_{W_{\text{res}}(\mathbf{s})}) = -2.$$

In order to get the linear problems involving the minimum number of independent variables we use, instead of (3.23), the more convenient system of generators of $W_{\text{res}}(\mathbf{s})$ given by

$$W_{\text{res}}(\mathbf{s}) = \text{span}\{ \nabla_3^{n_3} \nabla_4^{n_4} \chi(\mathbf{s}, \lambda), n_3, n_4 = 0, 1, 2, \dots, \nabla_5 \chi(\mathbf{s}, \lambda) \}. \tag{3.24}$$

The linear problems corresponding to the lowest orders are then

$$\partial_4^3 \psi - \partial_3^4 \psi + u_{11} \partial_3 \partial_4^2 \psi + u_{10} \partial_3^2 \partial_4 \psi + u_9 \partial_3^3 \psi + u_8 \partial_4^2 \psi + u_7 \partial_3 \partial_4 \psi + u_6 \partial_3^2 \psi$$

$$+ u_5 \partial_5 \psi + u_4 \partial_4 \psi + u_3 \partial_3 \psi + u_0 \psi = 0, \tag{3.25}$$

$$\partial_3 \partial_5 \psi - \partial_4^2 \psi + v_7 \partial_3 \partial_4 \psi + v_6 \partial_3^2 \psi + v_5 \partial_5 \psi + v_4 \partial_4 \psi + v_3 \partial_3 \psi + v_0 \psi = 0, \tag{3.26}$$

$$\partial_5 \partial_4 \psi - \partial_3^3 \psi + p_8 \partial_3 \partial_5 \psi + p_7 \partial_3 \partial_4 \psi + p_6 \partial_3^2 \psi + p_5 \partial_5 \psi + p_4 \partial_4 \psi + p_3 \partial_3 \psi + p_0 \psi = 0, \tag{3.27}$$

where u_i, $i = 0, 3, 4, \dots, 11$, v_k $k = 0, 3, \dots, 7$, and p_l, $l = 0, 3, \dots, 8$ can be expressed in terms of b_1, b_2, and χ_n ($n \geq 1$). Due to the absence of operators of the first and second order in λ we have two constraints on b_1, b_2, and $\chi_n (n \geq 1)$ associated to each equation (3.25)–(3.27).

Note that only two linear equations among (3.25)–(3.27) are independent (form the basis of the Manakov ring). For instance, the problems (3.26) and (3.27) are equivalent modulo the problem (3.25). Indeed, acting by operator ∂_4 on (3.26) and by the operator ∂_3 on (3.27), subtracting the equations obtained and using (3.25) one gets an identity. Then, choosing, for instance, (3.25) and (3.26) we have a system of two three-dimensional linear equations (variables s_3, s_4, s_5). However, it implies that ψ satisfies also a two-dimensional linear equation. Indeed, using (3.25), one can express $\partial_5 \psi$ via the derivatives of ψ with respect to s_3 and s_4 (since in general $u_5 \not\equiv 0$).

By substituting this expression into (3.26), one gets

$$\partial_3\partial_4^3\psi - \partial_3^5\psi + r_{14}\partial_3^2\partial_4^2\psi + r_{13}\partial_3^3\partial_4\psi + r_{12}\partial_4^3\psi + \bar{r}_{12}\partial_3^4\psi + r_{11}\partial_3\partial_4^2\psi + r_{10}\partial_3^2\partial_4\psi$$
$$+ r_9\partial_3^3\psi + r_8\partial_4^2\psi + r_7\partial_3\partial_4\psi + r_6\partial_3^2\psi + r_4\partial_4\psi + r_3\partial_3\psi + r_0\psi = 0, \tag{3.28}$$

where r_i ($i=0,3,4,6,7,...,14$) and $\bar{r}_{12}$ can be expressed in terms of u_i ($i=0,3,4,...,11$) and v_j ($j=0,3,4,...,7$). Thus, one has the two-dimensional equation (3.28) with variables s_3 and s_4 and the three-dimensional equation (3.25). The compatibility condition of the linear problem constituted by these two equations [or equivalently (3.25) and (3.26)] gives rise to a three-dimensional non-linear integrable system with independent variables s_3, s_4, s_5.

In the same way, taking into account (3.24), the equations in linear problems which involve higher times s_n ($n=6,7,...$) can be written in the form

$$\frac{\partial\psi}{\partial s_k} = \partial_3^{n_3}\partial_4^{n_4}\psi + \cdots + p_{k5}\partial_5\psi + \cdots + p_{k0}\psi, \quad k=6,7,..., \tag{3.29}$$

where $k=3n_3+4n_4$ and $p_{k\alpha}$ are certain functions and we have two constraints associated with each equation (3.29). In order to eliminate $\partial_5\psi$ in (3.29), we use (3.25) so that

$$\frac{\partial\psi}{\partial s_k} = \partial_3^{n_3}\partial_4^{n_4}\psi + \sum_{m_3,m_4} \bar{p}_{m_3m_4}(s)\partial_3^{m_3}\partial_4^{m_4}\psi, \quad k=6,7,.... .$$

By construction, these last equations are compatible with Eq. (3.28), thus, we have an infinite hierarchy of commuting $2+1$-dimensional integrable systems for the coefficients.

Finally, from (3.25) we can express $\nabla_5\chi$ as a linear combination of elements of the form $\nabla_3^{n_3}\nabla_4^{n_4}\chi$, we can eliminate $\nabla_5\chi$ for the system of generators of $W_{\text{res}}(s)$ [see (3.23), (3.24)]. In fact, by using (3.25)–(3.27) in (3.23) we get

$$W_{\text{res}}(s) = span\{\nabla_3^{n_3}\nabla_4^{n_4}\chi(s,\lambda), \quad n_3=0,1,2,...,n_4=0,1,2\}.$$

C. Higher Indices of the $\bar{\partial}$ operator

We finish Sec. III by discussing the basic properties of the case index $\bar{\partial} \leqslant -3$ [or equivalently $m \geqslant 3$ in (2.33)]. We will show that we have a hierarchy of integrable systems associated with each manifold defined by (2.33) with $m \geqslant 3$, but in this case the hierarchy consists of $3+1$-dimensional nonlinear systems, instead of the $2+1$ dimensional systems found in the cases index $\bar{\partial} = -1$ and index $\bar{\partial} = -2$. As all the basic properties are exhibited for $m=3$, we start by considering this particular case. Suppose then, that we take $m=3$ in (2.33) and solve the constraints with respect to t_1, t_2, and t_3. We have

$$t_1 = b_1(s) \quad t_2 = b_2(s), \quad t_3 = b_3(s),$$

consequently

$$g(s,\lambda) = \exp\left(\lambda b_1(s) + \lambda^2 b_2(s) + \lambda^3 b_3(s) + \sum_{i=4}^{\infty} \lambda^i s_i \right),$$

and the long derivatives are

$$\nabla_k = \frac{\partial}{\partial s_k} + \lambda\frac{\partial b_1}{\partial s_k} + \lambda^2\frac{\partial b_2}{\partial s_k} + \lambda^3\frac{\partial b_3}{\partial s_k} + \lambda^k, \quad k=4,5,...$$

From the expressions of the long derivatives, it is easy to see that, for $n \geq 4$, we have an operator of order n in λ of the form $\nabla_4^{n_4} \nabla_5^{n_5} \nabla_6^{n_6} \nabla_7^{n_7}$ with $n_4 \geq 0$, n_5, n_6, $n_7 = 0,1$ (at most one of them equal to 1). On the other hand, as every long derivative is of order ≥ 4, there are not operators of order 1, 2, and 3. Then

$$W_{\mathrm{res}}(s) = span\{\nabla_4^{n_4} \nabla_5^{n_5} \nabla_6^{n_6} \nabla_7^{n_7} \chi(s,\lambda), n_4$$

$$\geq 0, n_5, n_6, n_7$$

$$= 0,1 \text{ at most one of } n_5, n_6, n_7 \text{ equal to } 1\}, \tag{3.30}$$

so that

$$S_{W_{\mathrm{res}}(s)} = \mathbb{N} - \{1,2,3\}$$

and consequently

$$index(\bar{\partial}_{W_{\mathrm{res}}(s)}) = -3.$$

In order to get the linear problems involving the minimum number of independent variables we use, instead of (3.30), the more convenient description of $W_{\mathrm{res}}(s)$ as

$$W_{\mathrm{res}}(s) = span\{\nabla_4^{n_4} \nabla_5^{n_5} \chi, n_4 \geq 0, n_5 = 0,1,2,3, \nabla_6 \chi, \nabla_7 \chi, \nabla_5 \nabla_6 \chi\}. \tag{3.31}$$

The lowest order linear equations constructed using the system of generators in (3.31) are

$$\partial_5^4 \psi - \partial_4^5 \psi + u_{19} \partial_5^3 \partial_4 \psi + u_{18} \partial_5^2 \partial_4^2 \psi + u_{17} \partial_4^3 \partial_5 \psi + u_{16} \partial_4^4 \psi + u_{15} \partial_5^3 \psi + u_{14} \partial_5^2 \partial_4 \psi + u_{13} \partial_5 \partial_4^2 \psi + u_{12} \partial_4^3 \psi$$
$$+ u_{11} \partial_5 \partial_6 \psi + u_{10} \partial_5^2 \psi + u_9 \partial_4 \partial_5 \psi + u_8 \partial_4^2 \psi + u_7 \partial_7 \psi + u_6 \partial_6 \psi + u_5 \partial_5 \psi + u_4 \partial_4 \psi + u_0 \psi = 0, \tag{3.32}$$

$$\partial_4 \partial_6 \psi - \partial_5^2 \psi + v_9 \partial_4 \partial_5 \psi + v_8 \partial_4^2 \psi + v_7 \partial_7 \psi + v_6 \partial_6 \psi + v_5 \partial_5 \psi + v_4 \partial_4 \psi + v_0 \psi = 0, \tag{3.33}$$

$$\partial_4 \partial_7 \psi - \partial_5 \partial_6 \psi + p_{10} \partial_5^2 \psi + p_9 \partial_4 \partial_5 \psi + p_8 \partial_4^2 \psi + p_7 \partial_7 \psi + p_6 \partial_6 \psi + p_5 \partial_5 \psi + p_4 \partial_4 \psi + p_0 \psi = 0, \tag{3.34}$$

$$\partial_5 \partial_7 \psi - \partial_4^3 \psi + q_{11} \partial_5 \partial_6 \psi + q_{10} \partial_5^2 \psi + q_9 \partial_4 \partial_5 \psi + q_8 \partial_4^2 \psi + q_7 \partial_7 \psi + q_6 \partial_6 \psi + q_5 \partial_5 \psi + q_4 \partial_4 \psi + q_0 \psi$$
$$= 0, \tag{3.35}$$

$$\partial_6^2 \psi - \partial_4^3 \psi + w_{11} \partial_5 \partial_6 \psi + w_{10} \partial_5^2 \psi + w_9 \partial_4 \partial_5 \psi + w_8 \partial_4^2 \psi + w_7 \partial_7 \psi + w_6 \partial_6 \psi + w_5 \partial_5 \psi + w_4 \partial_4 \psi + w_0 \psi$$
$$= 0, \tag{3.36}$$

where u_i, $i = 0,4,5,\ldots,19$; v_j, $j = 0,4,5,\ldots,9$; p_k, $k = 0,4,5,\ldots,10$; q_l, $l = 0,4,5,\ldots,11$; and w_r, $r = 0,4,5,\ldots,11$ are functions of b_1, b_2, b_3, χ_n $(n \geq 1)$ and their derivatives. Besides, we have three constraints on the coefficients of the wave functions for each equation (3.32)–(3.36).

Again, among these five equations, there are only three independent ones modulo (3.32). For instance, acting on (3.33) by ∂_5^2, on (3.35) by ∂_4^2 and using (3.32), (3.33), and (3.34) one gets an identity. So (3.35) and similarly (3.36) are satisfied due to Eqs. (3.32)–(3.34). These three equations provides a four-dimensional system (being the independent variables s_4, s_5, s_6, s_7). However, using Eq. (3.33) one can get $\partial_7 \psi$ in terms of derivatives of ψ with respect to the three others independent variables, i.e.,

$$\partial_7 \psi = -\frac{1}{v_7}(\partial_4 \partial_6 \psi - \partial_5^2 \psi + v_9 \partial_4 \partial_5 \psi + v_8 \partial_4^2 \psi + v_6 \partial_6 \psi + v_5 \partial_5 \psi + v_4 \partial_4 \psi + v_0 \psi), \tag{3.37}$$

and then this term can be eliminated from (3.32) and (3.34). As a result one gets two linear equations for ψ which contain only ∂_4, ∂_5, and ∂_6. The linear problem determined by this couple of equations is compatible by construction, and the compatibility condition gives rise to a system of nonlinear equations in three dimensions (s_4, s_5, and s_6). A new system in the hierarchy can be obtained as the compatibility condition of one of the equations considered before (where ∂_7 has been eliminated) and (3.37). In this case we have a system of $3+1$-dimensional nonlinear equations (spatial variables s_4, s_5, s_6, and time s_7). Linear problems containing higher times $s_8, s_9, \ldots$ have the form

$$\partial_k \psi = \sum_{n_4 n_5 n_6} v_{n_4 n_5 n_6} \partial_4^{n_4} \partial_5^{n_5} \partial_6^{n_6} \psi, \quad k = 8, 9, \ldots, \tag{3.38}$$

where n_4, $n_5 \geq 0$, $n_6 = 0,1$ and $v_{n_4 n_5 n_6}$ are certain functions. Note that in (3.38) we have already eliminated the term $\partial_7 \psi$ by using (3.33).

The compatibility conditions of (3.38) with the equation obtained by eliminating $\partial_7 \psi$ in (3.32) defines an infinite hierarchy of $3+1$ dimensional nonlinear systems with four independent variables: s_4, s_5, s_6, and time s_k. Now, one could eliminate also $\partial_6 \psi$ from Eqs. (3.32), (3.33), and (3.34), in order to get a single linear equation containing derivatives with respect to only two independent variables, s_4 and s_5. But in order to do it, one has to invert an involved differential operator. Consequently, the corresponding $2+1$ dimensional equation is a complicated integro-differential one.

Finally, from the above discussion, it is clear that we can eliminate $\nabla_7 \chi$ from (3.31), then we get a basis of $W_{\mathrm{res}}(s)$ in the form:

$$W_{\mathrm{res}}(\mathbf{s}) = span\{\nabla_4^{n_4} \nabla_5^{n_5} \chi, n_4 \geq 0, n_5 = 0,1,2,3, \nabla_6 \chi, \nabla_5 \nabla_6 \chi\}.$$

We finish the study of the hidden KP hierarchies by summarizing the results for the general case $m \geq 3$. Solving Eq. (2.33) with respect to the first m times one has

$$t_k = b_k(s_{m+1}, s_{m+2}, \ldots), \quad k = 1, 2, \ldots, m,$$

then

$$g = \exp\left\{ \sum_{k=1}^{m} \lambda^k b_k(\mathbf{s}) + \sum_{k=m+1}^{\infty} \lambda^k s_k \right\}$$

and

$$\nabla_k = \frac{\partial}{\partial s_k} + \sum_{l=1}^{m} \lambda^l \frac{\partial b_l}{\partial s_k} + \lambda^k, \quad k = m+1, m+2, \ldots.$$

In this case

$$S_{W_{\mathrm{res}}(s)} = \mathbb{N} - \{1, 2, \ldots, m\},$$

and consequently

$$index(\bar{\partial}_{W_{\mathrm{res}}(s)}) = -m.$$

Now, by constructing operators of the form

$$L = \sum_{n_{m+1} n_{m+2}, \ldots} u_{n_{m+1} n_{m+2}, \ldots} \nabla_{m+1}^{n_{m+1}} \nabla_{m+2}^{n_{m+2}} \cdots,$$

which satisfy the condition (2.14), one gets an infinite hierarchy of linear equations. One can see that there are m of them which form a basis of Manakov ring of operators of lowest order with minimal number of independent variables $(s_{m+1}, s_{m+2},..., s_{2m+1})$. As above, one can in general eliminate $\partial_{2m+1}\psi, \partial_{2m}\psi,..., \partial_{m+4}\psi$ from these subsystem, and get a system of three-dimensional linear problems. In this way, the whole hierarchy consists in $3+1$ dimensional nonlinear systems with constraints.

We finally point out that $3+1$ hierarchies of integrable systems with constraints have been discussed in a different situations in Refs. 34–36.

IV. HIDDEN GELFAND–DIKII HIERARCHIES ON THE GRASSMANNIAN

As a particular case of hidden KP hierarchies associated with sectors of Gr with nonzero $\bar{\partial}$ index, we discuss here the hidden Gelfand–Dikii (GD) hierarchies, $1+1$-dimensional integrable hierarchies associated with energy-dependent spectral problems. We prove that under certain assumptions the only hidden GD hierarchies are those associated with Schrödinger equations with energy-dependent potentials (hidden KdV hierarchies)

$$\partial_x^2 \psi = \left(k^{2m+1} + \sum_{n=0}^{2m} u_n(\mathbf{s}) k^n \right) \psi, \quad k := \lambda^2,$$

and that associated to the third-order equation (hidden Boussinesq hierarchy)

$$\partial_x^3 \psi = (k^2 + u_1(\mathbf{s})k + u_0(\mathbf{s})) \psi + (v_1(\mathbf{s})k + v_0(\mathbf{s})) \partial_x \psi, \quad k := \lambda^3.$$

The hidden KdV hierarchies were already introduced and studied from the point of view of the Hamiltonian formalism in Ref. 37 and they were further generalized and analyzed in Refs. 38 and 39. As for the hidden Boussinesq hierarchy, it is one of the four cases studied in Ref. 40 in connection with the theory of energy-dependent third-order Lax operators. In both cases we manage to formulate a general solution method.

The start point here is an l-GD wave function. It is a particular KP wave function $\psi(\mathbf{t}, \lambda)$ verifying the reduction conditions:

$$\partial_{ml}\chi = 0, \quad m = 1,2,.... \tag{4.1}$$

As a consequence, its corresponding flow can be characterized as

$$W(\mathbf{t}) = span\{\lambda^{ml}\nabla_l^n \chi(\mathbf{t}, \lambda), m \geq 0, 0 \leq n \leq l-1\}.$$

Note that $W(\mathbf{t})$ does not depend on the parameters $t_{ml} m = 1,2,...,$ so from now on they will be supposed to be set equal zero, or equivalently

$$\mathbf{t} = (t_1,...,t_n,...), \quad n \notin (l) := \{l,2l,3l,...\}.$$

In the $\bar{\partial}$-dressing method, reductions (4.1) correspond to kernels of the form

$$R_0(\mu, \bar{\mu}, \lambda, \bar{\lambda}) = \delta(\mu^l - \lambda^l) \bar{R}_0(\mu),$$

then, we have the $\bar{\partial}$ problem

$$\frac{\partial \chi(\mathbf{t}, \lambda)}{\partial \bar{\lambda}} = \sum_{\alpha=1}^{l} \chi(\mathbf{t}, q^\alpha \lambda) \bar{R}_\alpha(\mathbf{t}, \lambda), \tag{4.2}$$

where

$$q = \exp\left(\frac{2\pi i}{l}\right), \quad \tilde{R}_\alpha(\mathbf{t},\lambda) = g_0(\mathbf{t},\lambda)\tilde{R}_\alpha(\lambda)g_0^{-1}(\mathbf{t},q^\alpha\lambda)$$

with $g_0(\mathbf{t},\lambda) = \exp(\Sigma_{i\notin(l)}\, t_i\lambda^i)$ and $\tilde{R}_\alpha(\lambda)\,\alpha = 1,2,...,l$ are l arbitrary functions. Note that the $\bar{\partial}$ problem (4.2) is invariant under multiplication by λ^l.

We now focus our attention on the study of the hidden 1-GD hierarchy. In order to do that, given an integer number $r>0, r\notin(l)$, we consider restriction under $d_r = r-1-[(r-1)/l]$ constraints (2.33), (here $[\cdot]$ denotes integer part), solved with respect to the first consecutive d_r parameters. It means

$$t_i = b_i(\mathbf{s}_r) \quad 1\leqslant i < r, \quad i\notin(l)$$

with $\mathbf{s}_r = (s_r,...,s_n,...)n > r, n\notin(l)$. We have then that

$$W_{\text{res}}(\mathbf{s}_r) = \text{span}\{\lambda^{ml}\nabla_{i_1}^{n_1}\nabla_{i_2}^{n_2}...\chi(\mathbf{s}_r,\lambda),m,n_1,n_2,...\geqslant 0,i_1,i_2,...\geqslant r,i_1,i_2,...\notin(l)\} \quad (4.3)$$

and long derivatives are here defined as

$$\nabla_n = \frac{\partial}{\partial s_n} + \sum_{i<r}\lambda^i\frac{\partial b_i}{\partial s_n} + \lambda^n, \quad n\geqslant r.$$

Clearly, if we look for $(1+1)$-dimensional hierarchies associated with these restrictions we need

$$W_{\text{res}}(\mathbf{s}_r) = span\{\lambda^{ml}\nabla_x^n\chi(\mathbf{s}_r,\lambda),m\geqslant 0,0\leqslant n\leqslant l-1\} \quad (4.4)$$

where x stands now for s_r. Consequently we are interested in those submanifolds $\mathcal{M}$ for which the corresponding $W_{\text{res}}(\mathbf{s}_r)$ verifies (4.4). In this sense we have:

Proposition 1: The family $W_{\text{res}}(\mathbf{s}_r)$ satisfies (4.4) if and only if the function $\psi(\mathbf{s}_r,\lambda)$ obeys an infinite system of linear equations of the form

$$\partial_x^l\psi = \Sigma_{m=0}^{l-1}u_m(\mathbf{s}_r,k)\partial_x^m\psi,$$

$$\partial_n\psi = \Sigma_{m=0}^{l-1}\alpha_{nm}(\mathbf{s}_r,k)\partial_x^m\psi, \quad n>r, \quad n\notin(l), \quad (4.5)$$

where u_m and α_{nm} are polynomials in $k := \lambda^l$.

Proof: The function $\chi(\mathbf{s}_r,\lambda)$ as well as its long derivatives of all orders with respect to the variables s_n belong to $W_{\text{res}}(\mathbf{s}_r)$. Then, if (4.4) holds all these functions can be decomposed in terms of the basis $k^m\nabla_x^n\chi$. Therefore (4.5) follows.

Reciprocally, if χ satisfies a system of the form (4.5) then from (4.3) and by using Taylor expansion we deduce (4.4) at once. $\square$

The next statement describes the cases in which (4.4) may arise.

Proposition 2: Only two classes of parametrized submanifolds $\mathcal{M}$ satisfying (4.4) are allowed:

(i) *Submanifolds $\mathcal{M}_m^{(2)}$ of the form*

$$t_{2i-1} = b_i(\mathbf{s}_{2m+1}), \quad i=1,...,m, \quad m\geqslant 1,$$

for the second GD hierarchy.

(ii) *Submanifolds $\mathcal{M}^{(3)}$ of the form*

$$t_1 = b(\mathbf{s}_2),$$

for the third GD hierarchy.

Proof: Let us assume that (4.4) holds then $W_{\text{res}}(\mathbf{s}_r)$ has no elements with order n such that $0 < n < \min(r,l)$. Moreover

$$\text{order}(\lambda^{lm}\nabla_x^n\chi)=lm+nr.$$

Let us first consider the case $r>l$. It implies that $l+r<2r$, and therefore if there are $i \notin (l)$ such that $r<i<l+r$ then the functions $\nabla_i\chi$ are elements of order i which cannot be decomposed in terms of the basis $\lambda^{lm}\nabla_x^n\chi$. The only way to avoid these functions is to take $l=2$, and consequently the allowed r are the odd integers $r=2m+1$ $(m\geqslant 1)$.

Consider now the case $r<l$. We have

$$\text{order}(\nabla_x\chi)<\text{order}(\nabla_x^2\chi)<\text{order}(\lambda^l\nabla_x\chi).$$

Thus, given $i\notin(l)$ such that $r<i<2r$ then the functions $\nabla_i\chi$ are elements of order i which cannot be decomposed in terms of the basis $\lambda^{lm}\nabla_x^n\chi$. But it is obvious that these functions will arise unless we take $r=2$ and $l=3$. □

As it will be proved below one can construct explicit examples of submanifolds $\mathcal{M}_m^{(2)}$ and $\mathcal{M}^{(3)}$ satisfying (4.4). Observe that in these cases the corresponding families of subspaces in the Grassmannian lead to the following values of the index of $\bar{\partial}$:

(1) For $\mathcal{M}_m^{(2)}$:

$$S_{W_{\text{res}}(s_{2m+1})}=\mathbb{N}-\{1,3,...,2m-1\},$$

so that

$$index(\bar{\partial}_{W_{\text{res}}(s_{2m+1})})=-m.$$

(2) For $\mathcal{M}^{(3)}$:

$$S_{W_{\text{res}}(s_2)}=\mathbb{N}-\{1\},$$

and as a consequence

$$index(\bar{\partial}_{W_{\text{res}}(s_2)})=-1.$$

In both cases the families of subspaces in the Grassmannian lie outside the zero index sector of the $\bar{\partial}$ operator. Next, we are going to show that for both classes of submanifolds described above there exist hierarchies of integrable systems.

Before analyzing these two cases, it is worth noticing that we have only looked for submanifolds associated to $(1+1)$-dimensional hierarchies of integrable systems, obtained by solving constraints (2.33) with respect to the first variables. Nevertheless, by using the methods of Sec. III and solving the constraints with respect to any set of variables, we can get in general multidimensional hidden l-Gelfand–Dikii hierarchies for arbitrary l and $r(r\notin(l))$. It is also clear that all these hierarchies belong to sectors in the Grassmannian with nonzero index.

A. Hidden KdV hierarchies

Consider first submanifolds $\mathcal{M}_m^2$ verifying (4.4). From Proposition 1 the constrained wave function $\psi(s_{2m+1},\lambda)$ satisfies an infinite linear system of the form

$$\partial_x^2\psi=u(s_{2m+1},k)\psi,$$

$$\partial_{2n+1}\psi=\alpha_n(s_{2m+1},k)\psi+\beta_n(s_{2m+1},k)\partial_x\psi, \quad n>m, \tag{4.6}$$

where $k:=\lambda^2$, $u=u(s_{2m+1},k):=k^{2m+1}+\sum_{n=0}^{2m}k^n u_n(s_{2m+1})$, and α_n, β_n are polynomials in k. By introducing the bilinear form

$$B(\psi,\varphi):=-\frac{1}{2\lambda^{2m+1}}\begin{vmatrix} \psi(\lambda) & \varphi(\lambda) \\ \psi(-\lambda) & \varphi(-\lambda) \end{vmatrix},$$

we may write the coefficients α_n and β_n in (4.6) as

$$\alpha_n=\frac{B(\partial_{2n+1}\psi,\partial_x\psi)}{B(\psi,\partial_x\psi)},\quad \beta_n=\frac{B(\psi,\partial_{2n+1}\psi)}{B(\psi,\partial_x\psi)}. \tag{4.7}$$

Then, we have $\alpha_n=-\frac{1}{2}\beta_{nx}$. Furthermore, the compatibility conditions for (4.6) imply

$$\partial_{2n+1}u=J\beta_n,$$

where

$$J:=-\tfrac{1}{2}\partial_x^3+2u\partial_x+u_x. \tag{4.8}$$

On the other hand, the function β_n is related to the trace of the resolvent of the Schrödinger operator

$$R(s_{2m+1},k):=\frac{\psi(s_{2m+1},\lambda)\,\psi(s_{2m+1},-\lambda)}{B(\psi,\partial_x\psi)}.$$

Thus, from (4.7) and the polynomial character of β_n as a function of k it follows that

$$\beta_n=(k^{n-m}R)_+,$$

where $(k^{n-m}R)_+$ stands for the polynomial part of $k^{n-m}R$ with R being substituted by its expansion as $k\to\infty$,

$$R=1+\sum_{n\geqslant 1}\frac{R_n(s_{2m+1})}{k^n}.$$

Therefore

$$\partial_{2n+1}u=J(k^{n-m}R)_+. \tag{4.9}$$

It turns out[37] that the coefficients R_n are differential polynomials in the potential functions $(u_0,u_1,\ldots,u_{2m})$. They can be determined by identifying coefficients of powers of k in the equation

$$JR=0.$$

In this way the set of equations (4.9) constitutes a hierarchy of integrable systems associated with the Schrödinger operator in (4.6). We will refer to this hierarchy as KdV$_{2m+1}$. Solutions of the members of the hierarchies can be derived from the functions b_i and χ_n.

For example, for $m=1$, using our standard techniques and the method considered above to construct the hierarchy, we have that the potential function is given by

$$u_2=2b_x,$$

$$u_1=b_x^2+2\chi_{1x}, \tag{4.10}$$

$$u_0=2\chi_{3x}-2\chi_2\chi_{1x}+\chi_1 b_{zz}+2\chi_{1x}b_x$$

where $b:=b_1$. The first integrable system in the hierarchy corresponds to $t:=s_5$ and takes the form

$$\partial_t u_0 = \tfrac{1}{4} u_{2xxx} - u_0 u_{2x} - \tfrac{1}{2} u_2 u_{0x},$$

$$\partial_t u_1 = -\tfrac{1}{2} u_2 u_{1x} - u_1 u_{2x} + u_{0x}, \qquad (4.11)$$

$$\partial_t u_2 = -\tfrac{3}{2} u_2 u_{2x} + u_{1x}.$$

The second equation in (4.6) is in this case

$$\partial_t \psi = \tfrac{1}{4} u_{2x} \psi + (k - \tfrac{1}{2} u_2) \partial_x \psi,$$

and the absence of the pure first order λ in $W_{res}(s_2)$ means [Eq. (4.6)] the constraints

$$\chi_{1x} = b_x^2 + b_t, \qquad \chi_{2x} = \chi \chi_{1x} - \tfrac{1}{2} b_{xx}. \qquad (4.12)$$

Note also, that using (4.10) and the first equation in (4.12), system (4.11) is equivalent to a system of two equations for b and u_0,

$$u_{0t} = \tfrac{1}{2} b_{xxxx} - 2 b_{xx} u_0 - b_x u_{0x},$$

$$(2b_t + 3b_x^2)_t + 2b_{xx}(2b_t + 3b_x^2) - u_{0x} + b_x(2b_t + 3b_x^2)_x = 0.$$

Analogously, for $m = 2$, we have that the potential function is given by

$$u_4 = 2b_{2x},$$

$$u_3 = 2b_{1x} + b_{2x}^2,$$

$$u_2 = 2b_{1x}b_{2x} + 2\chi_{1x},$$

$$u_1 = 2\chi_{3x} - 2\chi_2\chi_{1x} + \chi_1 b_{2xx} + + 2\chi_{1x}b_{2x} + b_{1x}^2,$$

$$u_0 = 2\chi_{5x} - 2\chi_4\chi_{1x} + \chi_3 b_{2xx} + 2\chi_{3x}b_{2x} + \chi_1 b_{1xx} + 2b_{1x}\chi_{1x} - (u_1 - b_{1x}^2)\chi_{2x}.$$

The second equation in (4.6) is

$$\partial_t \psi = \tfrac{1}{4} u_{4x} \psi + (k - \tfrac{1}{2} u_4) \partial_x \psi,$$

and the absence of orders λ and λ^3 means the constraints

$$b_{2x}^2 + b_{2t} - b_{1x} = 0, \quad \chi_{1x} - b_{2x}b_{1x} - b_{1t} = 0, \quad 2\chi_{2x} + b_{2xx} - 2\chi_{1x}\chi_1 = 0,$$

$$2\chi_{4x} + b_{1xx} - 2(\chi_1\chi_3)_x + b_{2xx}(\chi_2 - \chi_1^2 - b_{2x}) + 2\chi_2\chi_1\chi_{1x} = 0.$$

Finally, we point out that although we have only considered constraints solved with respect to the first parameters, submanifolds of the form $\mathcal{M}_m^{(2)}$ can be parametrized in other ways and we also get hierarchies of integrable systems. Unfortunately, in this case, there are not available direct methods to construct the hierarchies, as the one discussed above to construct the KdV_{2m+1} hierarchy, but we can always use standard techniques to get integrable systems. For example, taking the case $m = 1$ with the parametrization

$$t_3 = b(\mathbf{s}), \quad \mathbf{s} := (s_1, s_5, \ldots, s_{2m+1}, \ldots)$$

and taking $x := s_1$, $t := s_5$ we have the linear problem

$$\partial_x^2 \psi = (u_3 k^3 + u_2 k^2 + u_1 k + u_0) \psi + v_0 \partial_x \psi,$$

$$\partial_t \psi = \alpha_0 \psi + (\beta_1 k + \beta_0) \partial_x \psi, \tag{4.13}$$

where

$$u_3 = b_x^2, \quad u_2 = 2 b_x, \quad u_1 = 1 + 2 b_x \chi_{1x},$$

$$u_0 = -\frac{b_{xx}}{b_x} \chi_1 - 2 b_x \chi_{1x} \chi_2 + 2 \chi_{1x} + 2 b_x \chi_{3x}, \tag{4.14}$$

$$v_0 = \frac{b_{xx}}{b_x}, \quad \alpha_0 = -2 \frac{u_{2x}}{u_2^3}, \quad \beta_1 = \frac{2}{u_2}, \quad \beta_0 = 2 \frac{1 - u_1}{u_2^2},$$

and the absence of order λ in Eq. (4.13) means the constraints

$$\chi_{1x} = \frac{1}{b_x} - b_t, \quad \chi_{2x} = \chi \chi_{1x} + \frac{b_{xx}}{2 b_x^2}. \tag{4.15}$$

By imposing the compatibility condition in the linear problem (4.13) we have the integrable system

$$u_{3t} = 2 \beta_{1x} u_2 + 2 \beta_{0x} u_3 + \beta_1 u_{2x} + \beta_0 u_{3x},$$

$$u_{2t} = 2 \beta_{1x} u_1 + 2 \beta_{0x} u_2 + \beta_1 u_{1x} + \beta_0 u_{2x},$$

$$u_{1t} = 2 \beta_{1x} u_0 + 2 \beta_{0x} u_1 + \beta_1 u_{0x} + \beta_0 u_{1x},$$

$$u_{1t} = \alpha_{0xx} + 2 \beta_{0x} u_0 + \beta_0 u_{0x} - \alpha_{0x} v_0,$$

$$u_{0t} = (2 \alpha_0 + \beta_{0x} + v_0 \beta_0)_x,$$

that using (4.14) and (4.15) can be reduced to a system of two equations for b and u_0:

$$(b_x b_t)_t - \frac{b_{xx}}{b_x^2} u_0 + \left(\frac{b_t}{b_x} - \frac{1}{b_x^2} \right)_x (3 - 2 b_x b_t) + \frac{1}{2 b_x} u_{0x} - \left(\frac{b_t}{b_x} - \frac{1}{b_x^2} \right) b_x b_t = 0,$$

$$u_{0t} + \frac{1}{2} \left(\frac{b_{xx}}{b_x^3} \right)_{xx} - 2 \left(\frac{b_t}{b_x} - \frac{1}{b_x^2} \right)_x u_0 - \left(\frac{b_t}{b_x} - \frac{1}{b_x^2} \right) u_{0x} - \frac{b_{xx}}{2 b_x} \left(\frac{b_{xx}}{b_x^3} \right)_x = 0.$$

B. Hidden Boussinesq hierarchy

Our next task is to show that the submanifolds $\mathcal{M}^{(3)}$ satisfying (4.4) are also associated with a hierarchy of integrable systems. From Proposition 1, now, we have

$$\partial_x^3 \psi = u(s_2, k) \psi + v(s_2, k) \partial_x \psi,$$

$$\partial_n \psi = \alpha_n(s_2, k) \psi + \beta_n(s_2, k) \partial_x \psi + \gamma_n(s_2, k) \partial_x^2 \psi, \tag{4.16}$$

where $k := \lambda^3$, $u := u_0(s_2) + k u_1(s_2) + k^2$, $v := v_0(s_2) + k v_1(s_2)$ and α_n, β_n, and γ_n are polynomials in k. By introducing the trilinear form

$$T(\psi,\varphi,\eta) := \frac{1}{3i\sqrt{3}\lambda^6} \begin{vmatrix} \psi(\lambda) & \varphi(\lambda) & \eta(\lambda) \\ \psi(\epsilon\lambda) & \varphi(\epsilon\lambda) & \eta(\epsilon\lambda) \\ \psi(\epsilon^2\lambda) & \varphi(\epsilon^2\lambda) & \eta(\epsilon^2\lambda) \end{vmatrix},$$

where $\epsilon = (-1+i\sqrt{3})/2$, we may write the coefficients in (4.16) as

$$\alpha_n = \frac{T(\partial_n\psi,\partial_x\psi,\partial_x^2\psi)}{T(\psi,\partial_x\psi,\partial_x^2\psi)}, \quad \beta_n = \frac{T(\psi,\partial_n\psi,\partial_x^2\psi)}{T(\psi,\partial_x\psi,\partial_x^2\psi)},$$

$$\gamma_n = \frac{T(\psi,\partial_x\psi,\partial_n\psi)}{T(\psi,\partial_x\psi,\partial_x^2\psi)}.$$

By now using (4.16) it immediately follows that

$$\alpha_n = -\tfrac{1}{3}[2v\gamma_n + \gamma_{nxx} + 3\beta_{nx}].$$

Furthermore, the compatibility conditions for (4.16) imply

$$\begin{pmatrix} \partial_n v \\ \partial_n u \end{pmatrix} = J\begin{pmatrix} \gamma_{nx} + \beta_n \\ \gamma_n \end{pmatrix},$$

where J is the matrix operator given by

$$J_{11} := -2\partial_x^3 + 2v\partial_x + v_x, \quad J_{12} := = \partial_x^4 - \partial_x^2 \cdot v + 3\partial_x \cdot u - u_x,$$

$$J_{21} := -\partial_x^4 + v\partial_x^2 + 3u\partial_x + u_x,$$

$$J_{22} := \tfrac{1}{3}[2\partial_x^5 - 2(v\partial_x^3 + \partial_x^3 \cdot v) + (v^2 + 3u_x)\partial_x + \partial_x \cdot (v^2 + 3u_x)].$$

The standard technique shows that these equations are related to the resolvent trace functions

$$(Rs_2,k) := \begin{pmatrix} \dfrac{T(\psi,\partial_x\psi,\lambda\partial_x\psi)}{T(\psi,\partial_x\psi,\partial_x^2\psi)} \\ \dfrac{T(\psi,\partial_x\psi,\lambda\psi)}{T(\psi,\partial_x\psi,\partial_x^2\psi)} \end{pmatrix}, \quad S(s_2,k) := \begin{pmatrix} \dfrac{T(\psi,\partial_x\psi,\lambda^2\partial_x\psi)}{T(\psi,\partial_x\psi,\partial_x^2\psi)} \\ \dfrac{T(\psi,\partial_x\psi,\lambda^2\psi)}{T(\psi,\partial_x\psi,\partial_x^2\psi)} \end{pmatrix},$$

in the form

$$\begin{pmatrix} \partial_{3n+1}v \\ \partial_{3n+1}u \end{pmatrix} = J(k^n R)_+, \quad \begin{pmatrix} \partial_{3n-1}v \\ \partial_{3n-1}u \end{pmatrix} = J(k^{n-1}S)_+. \tag{4.17}$$

In these expressions R and S are substituted by their expansion as $k \to \infty$,

$$R = \sum_{n\geqslant 1} \frac{R_n(s_2)}{k^n}, \quad S = \sum_{n\geqslant 0} \frac{S_n(s_2)}{k^n}.$$

The coefficients R_n are S_n are differential polynomials in the potential functions $u_i, v_i (i=0,1)$. They can be determined by identifying coefficients of powers of k in

$$JR = 0, \quad JS = 0.$$

It is easy to prove that Eq. (4.17) constitutes an evolution equation for u and v. The set of these equations is a hierarchy of integrable systems associated with the third-order operator in (4.16). We will refer to this hierarchy as the hidden Boussinesq hierarchy. Solutions of the members of the hierarchy can be derived from the functions b and χ_n:

$$u_0 = b_{xxx}\chi_1 - 3\chi_{1x}\chi_3 + 3\chi_{4x} + 3b_x\chi_{3x} + 3b_x^2\chi_{2x} + 3b_{xx}\chi_{1x} + 3\chi_{2xx} + 3b_x\chi_{1xx} + 3\chi_{1x}\chi_1\chi_2$$

$$- 3\chi_{2x}\chi_2 - 3b_x\chi_{1x}\chi_2 - 3b_{xx}b_x^2\chi_1 + 3\chi_{1x}\chi_1^2 b_x - 3\chi_{2x}\chi_1 b_x - 3b_x^2\chi_{1x}\chi_1,$$

$$u_1 = b_x^3 + 3b_{xx} + 3\chi_{1x},$$

$$v_0 = 3b_{xx}b_x - 3\chi_{1x}\chi_1 + 3\chi_{2x} + 3b_x\chi_{1x},$$

$$v_1 = 3b_x.$$

         (4.18)

The first equation of the hidden Boussinesq hierarchy corresponds to the time parameter $t := s_4$ and takes the form

$$\partial_t u_0 = \tfrac{2}{9}v_1 v_{1xxxx} + \tfrac{8}{9}v_{1x}v_{1xxx} + \tfrac{2}{3}v_{1xx}^2 - \tfrac{2}{9}v_0 v_1 v_{1xx} - \tfrac{2}{9}v_0 v_{1x}^2 - \tfrac{2}{3}u_0 v_1 v_{1x}$$

$$- \tfrac{1}{9}u_{0x}v_1^2 - \tfrac{2}{3}v_{0xxx} + \tfrac{2}{3}v_0 v_{0x} + u_{0xx},$$

$$\partial_t u_1 = -\tfrac{2}{9}v_1^2 v_{1xx} - \tfrac{2}{9}v_1 v_{1x}^2 - \tfrac{2}{3}u_1 v_1 v_{1x} - \tfrac{1}{9}u_{1x}v_1^2 - \tfrac{2}{3}v_{1xxx} + \tfrac{2}{3}(v_0 v_1)_x + u_{1xx},$$

$$\partial_t v_0 = \tfrac{4}{9}v_1 v_{1xxx} + \tfrac{4}{3}v_{1x}v_{1xx} - \tfrac{4}{9}v_0 v_1 v_{1x} - \tfrac{1}{9}v_1^2 v_{0x} - v_{0xx} + 2u_{0x},$$

$$\partial_t v_1 = -\tfrac{5}{9}v_1^2 v_{1x} - v_{1xx} + 2u_{1x},$$

         (4.19)

The second equation in the linear problem (4.16) is

$$\partial_t \psi = (\alpha_0 + \alpha_1 k)\psi + \beta_0 \partial_x \psi + \partial_x^2 \psi$$

with

$$\alpha_0 = \tfrac{2}{9}v_1 v_{1x} - \tfrac{2}{3}v_0, \quad \alpha_1 = -\tfrac{2}{3}v_1, \quad \beta_0 = -\tfrac{1}{9}v_1^2.$$

Finally, the constraints imposed by the absence of the first order in $W_{\mathrm{res}}(s_2)$ are given by

$$\tfrac{1}{3}b_{xxx} - \chi_{1x}\chi_2 + \chi_{3x} + \chi_{1xx} + \chi_{1x}\chi_1^2 - \chi_{2x}\chi_1 - b_{xx}b_x^2 = 0,$$

$$b_x^3 + b_t - b_{xx} - 2\chi_{1x} = 0.$$

         (4.20)

C. Methods of solution of hidden Gelfand–Dikii Hierarchies

A solution method for the hierarchies studied above was discussed for the hidden KdV hierarchies in Ref. 41 and some solutions were exhibited there. The main idea is that taking a particular l-GD wave function ($l=2$ for $\mathcal{M}_m^{(2)}$ and $l=3$ for $\mathcal{M}^{(3)}$) the constraints imposed by the absence of some orders in $W_{\mathrm{res}}(s_r)$ determine differential equations for the functions b_i associated with the submanifolds $\mathcal{M}_m^{(2)}$ and $\mathcal{M}^{(3)}$ satisfying (4.4). If these equations can be solved, they lead to solutions of the corresponding hierarchy. In general, these differential equations are too complicated to be solved. Nevertheless, we may provide appropriate methods of solution directly based on the Grassmannian. To this end it is required an element W of Gr associated to a wave function ψ for the l-Gelfand–Dikii hierarchy, such that the functions of W admit meromorphic expansions in the disk $D_0 = \mathbb{C} - D_\infty$ with fixed poles $\lambda_i, i=1,...,n$ of maximal orders r_i, $i=1,...,n$. Under these conditions any linear functional on W of the form

$$l(\psi) = \sum_{j=1}^{s} c_j \frac{d^{n_j}w}{d\lambda^{n_j}}(q_j), \quad |q_j| < 1, \quad q_j \notin \{\lambda_i : i = 1, \dots, n\},$$

admits a representation

$$l(w) = \oint_{S_\infty} \tilde{w}(\lambda) w(\lambda) \frac{d\lambda}{2\pi i \lambda}, \quad \forall w \in W, \tag{4.21}$$

with a finite order function $\tilde{w}$. For example in the case $r_i = 1, \ \forall i = 1, \dots, n$ we have

$$\frac{1}{n!} \frac{d^n w}{d\lambda^n}(q) = \oint_{S_\infty} \left[\frac{\lambda}{(\lambda - q)^{n+1}} - \sum_j \frac{\lambda}{(\lambda_j - q)^{n+1}} \prod_{i \neq j} \frac{\lambda - \lambda_i}{\lambda_j - \lambda_i} \right] w(k) \frac{d\lambda}{2\pi i \lambda}.$$

Proposition 3: Given $W \in$ Gr associated to a wave function ψ for the KdV hierarchy and a linearly independent set of m functionals $\{l_i\}_{i=1}^{m}$ such that for certain numbers c_{ij}

$$W \subset \mathrm{Ker}\left(\lambda^2 l_i - \sum_j c_{ij} l_j \right), \quad i = 1, \dots, m, \tag{4.22}$$

where $(\lambda^2 l_i)(w) \equiv l_i(\lambda^2 w)$. Then, a submanifold $\mathcal{M}_m^{(2)}$ satisfies (4.4) if

$$l_i(\psi(\mathbf{t}, \lambda)) = 0, \quad i = 1, \dots, m, \tag{4.23}$$

for all $\mathbf{t} \in \mathcal{M}_m^{(2)}$.

Proof: Let $\{\tilde{w}_i : i = 1, \dots, m\}$ be the functions representing the functionals l_i. From (4.22) it follows that

$$\lambda^2 \tilde{w}_i = \sum_j c_{ij} \tilde{w}_j + \tilde{u}_i, \quad i = 1, \dots, m,$$

where $\tilde{u}_i$ are elements of $\tilde{W}$. Moreover, from (4.21) we have that the equations (4.23) are equivalent to

$$\oint_{S_\infty} \tilde{w}_i(\lambda) \psi(\mathbf{t}, \lambda) \frac{d\lambda}{2\pi i \lambda} = 0, \quad i = 1, \dots, m.$$

Therefore, if $t_{2i-1} = b_i(s_{2m+1}), \ i = 1, \dots, m$ are the functions characterizing the parametrized submanifold $\mathcal{M}_m^{(2)}$, then, the restricted wave function $\psi(s_{2m+1}, \lambda)$ generates a subspace W_{res} such that

$$\widetilde{W_{\mathrm{res}}} = \tilde{W} \oplus \mathrm{span}\{\tilde{w}_i, i = 1, \dots, m\}.$$

As a consequence

$$\mathrm{v.d.}(W_{\mathrm{res}}) = \mathrm{v.d.}(W) - m = -m.$$

Moreover, it is known[42] that the virtual dimension of a subspace W does not change under the action of an invertible multiplication operator. Then, by taking (2.32) into account we have

$$\mathrm{Index} \bar{\partial}_{W_{\mathrm{res}}(s_{2m+1})} = \mathrm{v.d.}(W_{\mathrm{res}}(s_{2m+1})) = \mathrm{v.d.}(W_{\mathrm{res}}) = -m.$$

Hence, it is easy to deduce that

$$S_{W_{\mathrm{res}}(s_{2m+1})} = \mathbb{N} - \{1, 3, \dots, 2m-1\},$$

so that the statement follows at once.

In the same way one proves:

Proposition 4: Given W associated to a wave function ψ for the Boussinesq hierarchy and a nontrivial functional l on W verifying

$$W \subset \mathrm{Ker}(k^3 l - cl),$$

for a certain number c. Then a submanifold $\mathcal{M}^{(3)}$ satisfies (4.4) if

$$l(\psi(\mathbf{t}, \lambda)) = 0, \tag{4.24}$$

for all $\mathbf{t} \in \mathcal{M}^{(3)}$.

These results allow us to determine the functions $b_i = b_i(\mathbf{s}_r)$ from constraints of the types (4.23) and (4.24).

We devote the rest of the section to illustrating this method by constructing some solutions. We concentrate first on the hidden KdV hierarchies. Our first example is based on the subspace $W \in \mathrm{Gr}$ of boundary values of functions $w = w(\lambda)$ analytic on the unit disk $|\lambda| < 1$, with the possible exception of a single real pole $-1 < q < 1$ and such that

$$\lambda^2 W \subset W, \quad \mathrm{Res}(w, q) = c w(-q),$$

for a given $c > 0$. This subspace determines a KdV wave function

$$\psi(\mathbf{t}, \lambda) = \exp\left(\sum_{n \geq 1} \lambda^{2n-1} t_{2n-1} \right)\left(1 + \frac{a(\mathbf{t})}{\lambda - q} \right), \quad a(\mathbf{t}) = \frac{2qc(t)}{2q + c(\mathbf{t})},$$

where

$$c(t) := c \exp\left(-2 \sum_{n \geq 1} q^{2n-1} t_{2n-1} \right).$$

We may construct solutions of the KdV$_3$ hierarchy from W by means of the functional

$$l(w) = \frac{dw}{d\lambda}(0),$$

which obviously satisfies $\lambda^2 l = 0$. The implicit equation $l(\psi(\mathbf{t}, \lambda)) = 0$ reads

$$t_1\left(1 - \frac{a(\mathbf{t})}{q} \right) = \frac{a(\mathbf{t})}{q^2}.$$

By introducing the new variables

$$y := 2q t_1, \quad x := s_3,$$

$$z := 2q^3 x - \log c + 2 \sum_{n \geq 2} q^{2n+1} s_{2n+1},$$

the equation reduces to

$$y + z = \log\left[\frac{1}{2q} + \frac{2}{qy} \right]. \tag{4.25}$$

For $q > 0$ it defines two branches $y^{(i)} = 2q b^{(i)}(z)$ ($i = 1, 2$), while for $q < 0$ it leads to only one branch $y^{(3)} = 2q b^{(3)}(z)$. Moreover, from (4.25) we have

$$\frac{dy}{dz} = -1 + \frac{4}{(y+2)^2}.$$

This relation together with (4.10) leads to the following expressions for the corresponding solutions of the KdV$_3$ hierarchy

$$u_0 = 8q^6 \frac{y(y+4)(y^2+4y-4)}{(y+2)^6},$$

$$u_1 = q^4 \frac{y(y+4)(y^2+4y-8)}{(y+2)^4}, \tag{4.26}$$

$$u_2 = -2q^2 \frac{y(y+4)}{(y+2)^2}.$$

They represent coherent structures which propagate freely without deformation. In the case of $y^{(3)}$ it determines a singular solution.

More general solutions of this type can be defined by increasing the number of poles. Thus, one may take the subspace $W \in \text{Gr}$ of boundary values of analytic functions $w = w(\lambda)$ on the disk $|\lambda| < 1$, with the possible exception of n single poles at given real numbers $(0 < |q_i| < 1$ $i = 1,...,n)$ and such that

$$\lambda^2 W \subset W, \quad \text{Res}(w, q_i) = c_i w(-q_i),$$

with $c_i > 0$. The corresponding KdV wave function reads

$$\psi(\mathbf{t}, \lambda) = \exp\left(\sum_{n \geq 1} \lambda^{2n-1} t_{2n-1}\right)\left(1 + \sum_{i=1}^{n} \frac{a_i(\mathbf{t})}{\lambda - q_i}\right),$$

where the coefficients a_i satisfy the system

$$a_i(\mathbf{t}) + c_i(\mathbf{t}) \sum_{j=1}^{n} \frac{a_j(\mathbf{t})}{q_i + q_j} = c_i(\mathbf{t})$$

with

$$c_i(\mathbf{t}) := c_i \exp\left(-2 \sum_{n \geq 1} q_i^{2n-1} t_{2n-1}\right).$$

By using again the functional $l(w) = w'(0)$, the implicit equation $l(\psi) = 0$ is now

$$t_1\left(1 - \sum_{j=1}^{n} \frac{a_j(\mathbf{t})}{q_j}\right) = \sum_{j=1}^{n} \frac{a_j(\mathbf{t})}{q_j^2}.$$

It can be shown that the corresponding solutions of the KdV$_3$ hierarchy represent composite structures which decompose asymptotically into solutions of the form (4.26).

Another kind of solution is obtained by considering the subspace $W \in \text{Gr}$ of boundary values of analytic functions $w = w(\lambda)$ on the disk $|\lambda| < 1$, with the possible exception of a single pole at $\lambda = 0$ and such that

$$\lambda^2 W \subset W, \quad (w, q) = cw(-q),$$

for given $c > 0$ and $q \in \mathbb{R}$ such that $-1 < q < 1$. The wave function is

$$\psi(\mathbf{t},\lambda)=\exp\!\left(\sum_{n\geqslant 1}\lambda^{2n-1}t_{2n-1}\right)\!\left(1+\frac{a(\mathbf{t})}{\lambda}\right),\quad a(\mathbf{t})=q\,\frac{c(\mathbf{t})-1}{c(\mathbf{t})+1},$$

where

$$c(\mathbf{t}):=c\,\exp\!\left(-2\sum_{n\geqslant 1}q^{2n-1}t_{2n-1}\right).$$

Solutions of the KdV_3 hierarchy can be derived by taking the functional

$$l(w)=\frac{dw}{d\lambda}(q)+c\,\frac{dw}{d\lambda}(-q),$$

which verifies

$$W\subset\mathrm{Ker}(\lambda^2 l-q^2 l).$$

The implicit equation $l(\psi(\mathbf{t},\lambda))=0$ leads to

$$\sum_{n\geqslant 0}(2n+1)q^{2n}t_{2n+1}=\frac{1}{4q}\left(c(\mathbf{t})-\frac{1}{c(\mathbf{t})}\right).$$

By introducing the new variables

$$y:=2\Sigma_{n\geqslant 0}q^{2n+1}t_{2n+1}-\log c,\quad x:==s_3,$$

$$z:=4q^3x+\log c+4\Sigma_{n\geqslant 2}nq^{2n+1}s_{2n+1},$$

the equation reduces to

$$y+z=-\sinh y.$$

It defines one implicit branch $y=y(z)$ which satisfies

$$\frac{dy}{dz}=-\frac{1}{1+\cosh y}.$$

From this relation and (4.10) we get the following expressions for the associated solution of the KdV_3 hierarchy:

$$u_0=-\frac{4q^6}{\cosh^6\dfrac{y}{2}},$$

$$u_1=q^4\!\left(1+\frac{2}{\cosh^2(y/2)}+\frac{3}{\cosh^4(y/2)}\right),$$

$$u_2=-2q^2\!\left(1+\frac{1}{\cosh^2(y/2)}\right).$$

They again represent coherent structures propagating freely and without deformation.

The same strategy can be applied for characterizing solutions of the hidden Boussinesq hierarchy. Let us first take the subspace $W\in\mathrm{Gr}$ of boundary values of functions $w=w(\lambda)$ analytic on the unit disk $|\lambda|<1$, with the possible exception of a single pole q and such that

$$\lambda^3 W \subset W, \quad \frac{dw}{d\lambda}(0) = 0.$$

This subspace determines a wave function for the Boussinesq hierarchy given by

$$\psi(\mathbf{t}, \lambda) = g(\mathbf{t}, \lambda) \left(1 + \frac{q^2 t_1}{(1 + q t_1)(k - q)} \right).$$

Consider now the functional

$$l(w) = \frac{d^2 w}{d\lambda^2}(0),$$

which obviously satisfies $\lambda^3 \cdot l = 0$. The equation $l(\psi(\mathbf{t}, \lambda)) = 0$ implies

$$q t_1^2 + 2 t_1 - 2 q t_2 = 0,$$

so that one finds the following explicit solution of the hidden Boussinesq hierarchy:

$$u_0 = -\frac{12 q^6}{(1 + 2 q^2 x)^3}, \quad u_1 = \frac{q^3}{(1 + 2 q^2 x)^{3/2}},$$

$$v_0 = \frac{3 q^4}{(1 + 2 q^2 x)^2}, \quad v_1 = \frac{3 q}{(1 + 2 q^2 x)^{1/2}},$$

Other solutions can be generated by starting with the same subspace $W \in \mathrm{Gr}$ and by taking the functional

$$l(w) = \frac{d^4 w}{d\lambda^4}(0).$$

In this case

$$\lambda^3 l(w) = 24 \frac{dw}{d\lambda}(0),$$

so that $W \subset \mathrm{Ker}(\lambda^3 l)$. The constraint $l(\psi(\mathbf{t}, \lambda)) = 0$ takes the form

$$q^3 t_1^4 + 4 q^2 t_1^3 + 4 q(2 + q^2 t_2) t_1^2 + 8(1 + q^2 t_2) t_1 - 4 q^3 (2 t_4 + t_2^2) = 0.$$

A particular solution $t_1 = b_1(s_2)$ of this equation is

$$b_1 = -\frac{1}{q} + \frac{1}{q} \sqrt{-1 - 2 q^2 s_2 + 2 \sqrt{1 + 2 q^4 s_4 + 2 q^2 s_2 + 2 q^4 s_2^2}}.$$

It can be seen that the corresponding solution, as a function of x, is globally defined on $\mathbb{R}$ only for $s_4 > -1/4 q^4$, otherwise its domain is $\mathbb{R} - [-q^2(1 + \sqrt{-1 - 4 q^4 s_4}), -q^2(1 - \sqrt{-1 - 4 q^4 s_4})]$.

ACKNOWLEDGMENT

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