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Nonclassical symmetries for a family of Cahn–Hilliard equations

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Abstract

In this Letter we make a full analysis of the symmetry reductions of the family of Cahn–Hilliard equations by using the classical Lie method of infinitesimals, the functional forms of the diffusion coefficients for which the Cahn–Hilliard equations can be fully reduced to ordinary differential equations by classical Lie symmetries are derived.

We prove that by using the nonclassical method, we obtain several solutions which are not invariant under any Lie group admitted by the equation and consequently which are not obtainable through the Lie classical method.

For this Cahn–Hilliard equation, we obtain nonclassical symmetries that reduce the original equation to ordinary differential equations with the Painlevé property. We remark that these symmetries have not been derived elsewhere by the singular manifold method.

1. Introduction

The Cahn–Hilliard equation was introduced to study phase separation in binary alloys glasses and polymers [4] and is a good approach to spinodal decomposition. Based in numerical version of the Fourier transformation approach to the nonlinear Cahn–Hilliard diffusion equation, computer simulations of the spinodal decomposition for a model alloy were carried out by Liu and Haasen [14]. The Cahn–Hilliard diffusion equation is also an equation that serves as a model for many problems in physical chemistry, developmental biology [2], as well as population movement [8]. The existence of a weak solution for the Cahn–Hilliard equation with degenerate mobility was proved in [9].

The Cahn–Hilliard flux equation describing diffusion for decomposition of a one-dimensional binary solution can be written as

$$u_t + \left(k u_{xxx} - f(u) u_x\right)_x = 0, \qquad (1)$$

which is appropriate to cases where the motion is isotropic. Here u is the solute concentration at point x, t is the time, $f'(u)$ is the interdiffusion coefficient of solute, which is concentration dependent and $k/2$ is the gradient energy coefficient describing the contribution of the diffuse interface to the decomposition. In [8] bifurcations of the equilibrium to nonuniform states have been discussed for $f(u) = D_0 + D_2 u^2$ and in [15] several nonlinear results were derived.

Although the direct method of Clarkson and Kruskal is found to more to be more powerful than the classical method [5]), similarity reductions for the Cahn–Hilliard equation with $f(u) = u$ and $f(u) = u^2$,

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have been obtained in [17,18], by using classical Lie symmetries as well as the direct method. In [17,18], the direct method did not yield any reduction that could not be obtained by Lie classical symmetries.

In this Letter we solve a complete group classification problem for Eq. (1), by studying those diffusion coefficients $f(u)$ which admit the classical symmetry group. Both the symmetry group and the diffusion coefficients will be found through consistent application of the Lie-group formalism.

Motivated by the fact that symmetry reductions for many partial differential equations (PDE's) are known that are not obtained by using the classical Lie group method, there have been several generalizations of the classical Lie group method for symmetry reductions. The notion of nonclassical symmetries was firstly introduced by Bluman and Cole [3] to study the symmetry reductions of the heat equation. Since then, a great number of papers have been devoted to the study of nonclassical symmetries of nonlinear partial differential equations in both one and several dimensions (see, i.e. [6,12,13]). Clarkson and Mansfield [7] presented an algorithm for calculating the determining equations associated with the nonclassical method.

Recently Zhdanov and Lahno [20] have applied the nonclassical (or conditional) method to the one-dimensional porous medium equation

$$u_t - (uu_x)_x = 0. \tag{2}$$

According to these authors the nonclassical method for (2) as well as for the parabolic type PDE's is inefficient. Once obtained new nonclassical symmetries performing the symmetry reductions gives rise to invariant solutions corresponding to the Lie symmetries of (2).

The aim of this Letter is to prove that the nonclassical method applied to the Cahn–Hilliard equation gives rise to new solutions of (1) which are not group-invariant and consequently cannot be obtained by Lie classical symmetries. Some of these solutions (the characteristic solutions) are solutions of (2) which are not invariant under any Lie group admitted by (2) and consequently cannot been obtained by Lie classical symmetries this result is a counterexample of the statement done in [20] because according to them *all the solutions* of (2) derived by the nonclassical method are group-invariant.

Recently, the family of Cahn–Hilliard equations has arisen a great interest because of an apparent contradiction between the scope of the singular manifold method (SMM) and the nonclassical symmetry reductions [19,11].

In [10] Estévez and Gordoa developed a method for identifying the nonclassical symmetries of PDE's using the SMM based on the Painlevé property (PP) as a tool. They propose the following conjecture: ''The singular manifold method allows us to identify the nonclassical symmetries that reduce the original equation to an ODE with the Painlevé property''.

The combination of this statement with the Ablowitz, Ramani and Segur conjecture [1] means that for equations with the PP, the SMM should identify all the nonclassical symmetries. Nevertheless, for equations with the conditional PP, the SMM is only able to identify the symmetries for which the associated reduced ODE's are of the Painlevé type.

In [11] the authors claim that, for (1) with $f(u)=u$ an= d $f(u)=u^2$, besides a trivial symmetry, the SMM allows them to determine two different symmetries and that these symmetries are *the only ones* in which the associated similarity reduction leads to an ODE of Painlevé type. Nevertheless, for (1) with $f(u)=u$ and $f(u)=u^2$, besides the symmetries derived in [11], we have derived three new nonclassical symmetries for which the corresponding associated similarity reductions leads to three different ODE's of Painlevé type.

Consequently, for (1) with $f(u)=u$ and $f(u)=u^2$, the nonclassical method is more general than the SMM and that this latter method does not allow us to identify all the nonclassical symmetries that reduce Eq. (1), with $f(u)=u$ and $f(u)=u^2$, to ODE's with the PP.

2. Lie classical classification.

To apply the classical method to (1) we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\phi(x,t,u) + O(\epsilon^2), \end{aligned} \tag{3}$$

where ϵ is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of (1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$V=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\phi(x,t,u)\frac{\partial}{\partial u}. \tag{4}$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi\equiv\xi\frac{\partial u}{\partial x}+\tau\frac{\partial u}{\partial t}-\phi=0. \tag{5}$$

We consider the classical Lie group symmetry analysis of Eq. (1). Invariance of Eq. (1) under a Lie group of point transformations with infinitesimal generator (4) leads to a set of forty determining equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. Solving this system we obtain

$$\tau=\tau(t),\quad \xi=\frac{x}{4}\frac{d\tau}{dt}+\xi_1(t),$$

$$\phi=\phi_1(t)u+\phi_2(x,t),$$

where τ, ξ_1, ϕ_1 and ϕ_2 are related by the following conditions:

$$\begin{aligned}
-\phi_1 u\frac{df}{du}-\phi_2\frac{df}{du}-\frac{1}{2}f\frac{d\tau}{dt}&=0,\\
-(\phi_1 u+\phi_2)\frac{d^2f}{du^2}-\phi_1\frac{df}{du}-\frac{1}{2}\frac{df}{du}\frac{d\tau}{d,t}&=0,\\
-\frac{x}{4}\frac{d^2\tau}{dt^2}-2\frac{df}{du}\frac{\partial\phi_2}{\partial x}-\frac{d\xi_1}{dt}&=0,\\
\frac{d\phi_1}{dt}u+k\frac{\partial^4\phi_2}{\partial x^4}-f\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial\phi_2}{\partial t}&=0.
\end{aligned}$$

The solutions of this system depends of $f(u)$. For $f(u)$ arbitrary, the only symmetries admitted by (1) are the group of space and time translations, which are defined by the infinitesimal generators

$$V_1=\frac{\partial}{\partial x},\quad V_2=\frac{\partial}{\partial t}.$$

In this case, we obtain travelling wave reductions

$$z=x-\lambda t,\quad u=h(z),$$

where $h(z)$, after integrating once with respect to z, satisfies

$$kh'''-f(h)h'-\lambda h=k_1. \tag{6}$$

Eq. (6) is invariant under translations, this allow us to reduce the order by one. The only functional forms of $f(u)$, with $f(u)\neq$ const. for which Eq. (1) have extra symmetries are $f(u)=(au+b)^n$ and $f(u)=de^{au}$, and these symmetries are, respectively defined, by the following infinitesimal generators:

$$V_3^1=x\frac{\partial}{\partial x}+4t\frac{\partial}{\partial t}-\frac{2}{an}(au+b)\frac{\partial}{\partial u},$$

$$V_3^2=x\frac{\partial}{\partial x}+4t\frac{\partial}{\partial t}-\frac{2}{a}\frac{\partial}{\partial u}.$$

Without loss of generality we can consider $b=0$ and $a=d=1$. For the sake of completeness, we provide next, the generators of the nontrivial one-dimensional optimal system which are:

- For $f(u)=u^n$, the set

$$\{<V_1>,<\lambda V_1+V_2>,<V_3^1>\}.$$

- For $f(u)=e^u$, the set

$$\{<V_1>,<\lambda V_1+V_2>,<V_3^2>\}.$$

In both sets, $\lambda\in\mathbb{R}$ is arbitrary.

Since Eq. (1), with $f(u)=u^n$ and $f(u)=e^u$ has additional symmetries and the reductions that correspond to V_1 and V_2 have already been derived, we must only determine the similarity variables and similarity solutions corresponding to V_3^1 and V_3^2 which are:

- For V_3^1:

$$z=xt^{-1/4},\quad u=t^{-1/2n}h(z),$$

where $h(z)$ satisfies the ODE

$$kh''''-h^nh''-\frac{z}{4}h'-nh^{n-1}(h')^2-\frac{1}{2n}h=0. \tag{7}$$

Eq. (7) does not admit Lie symmetries. Nevertheless, for $n=2$ this equation can be easily integrated once respect to z yielding

$$kh''' - h^2 h' - \frac{z}{4} h = k_1 .$$

- For V_3^2:

$$z = xt^{-1/4}, \quad u = -\ln\left(t^{1/2} h(z)\right),$$

where $h(z)$ satisfies the ODE

$$4kh^3 h'''' - 4\left[h^2 h'' - 2h(h')^2\right] - 4kh^2\left[4h'h''' + 3(h'')^2\right] - zh^3 h' + 48kh(h')^2 h'' - 24k(h')^4 + 2h^4 = 0. \qquad (8)$$

Eq. (8) does not admit Lie symmetries.

Due to the fact that (1) is only invariant under time and space translations and under a scaling group we have considered nonclassical symmetries.

3. Nonclassical symmetries

The basic idea of the method is that the PDE (1) is augmented with the invariance surface condition (5) which is associated with the vector field (4). By requiring that both (1) and (5) are invariant under the transformation with infinitesimal generator (4) one obtains an overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is in general, larger than for the classical method as in this method one requires only the subset of solutions of (1) and (5) to be invariant under the infinitesimal generator (4).

To obtain nonclassical symmetries of (1) we apply the algorithm described in [7] for calculating the determining equations. We can distinguish two different cases:

- In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,u)=1$. The nonclassical method applied to (1) gives only rise to the classical symmetries.
- In the case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and the determining equation for the infinitesimal ϕ is

$$\begin{aligned} &k\phi_{xxxx} + 4k\phi\phi_{uu}\phi_{xx} + 4k\phi_{ux}\phi_{xx} - f\phi_{xx} \\ &\quad + 3k\phi_{uu}(\phi_x)^2 + 6k\phi^2\phi_{uuu}\phi_x \\ &\quad + 12k\phi\phi_{uux}\phi_x + 10k\phi\phi_u\phi_{uu}\phi_x \\ &\quad + 6k\phi_{uxx}\phi_x + 4k\phi_u\phi_{ux}\phi_x \\ &\quad - 3f'\phi\phi_x + k\phi^4\phi_{uuuu} + 4k\phi^3\phi_{uuux} \\ &\quad + 6k\phi^3\phi_u\phi_{uuu} + 6k\phi^2\phi_{uuxx} \\ &\quad + 12k\phi^2\phi_u\phi_{uux} + 4k\phi^3(\phi_{uu})^2 \\ &\quad + 12k\phi^2\phi_{ux}\phi_{uu} + 7k\phi^2(\phi_u)^2\phi_{uu} \\ &\quad - f\phi^2\phi_{uu} + 4k\phi\phi_{uxxx} + 6k\phi\phi_u\phi_{uxx} \\ &\quad + 8k\phi(\phi_{ux})^2 + 4k\phi(\phi_u)^2\phi_{ux} - 2f\phi\phi_{ux} \\ &\quad - 2f'\phi^2\phi_u + \phi_t - f''\phi^3 = 0 \end{aligned} \qquad (9)$$

The complexity of this equation is the reason why we cannot solve (9) in general. Thus we proceed, by making ansatz on the form of $\phi(x,t,u)$, to solve (9). In this way we get the following functional forms of $f(u)$ and the following similarity reductions which are unobtainable by Lie classical symmetries. Due to the invariance under temporal and spatial translations, we take $t_0 = 0$ and $x_0 = 0$ without loss of generality.

1. Choosing $\phi = \phi(x,t)$, we find that for any function $f(u)$ the infinitesimal generators take the form

$$\xi = 1, \quad \tau = 0, \quad \phi = \phi(x,t),$$

where $\phi(x,t)$ satisfies the following equation

$$k\phi_{xxxx} - \phi^3 f''(u) - 3\phi\phi_x f'(u) + \phi_t = 0. \qquad (10)$$

Setting $f(u) = u^n$ and solving (10) we obtain that $n=1$ is the only value for which we obtain solutions which are not group-invariant. For $f(u)=u$ we get the infinitesimal generators

$$\xi = 1, \quad \tau = 0, \quad \phi(x,t) = -\frac{x}{3t}. \qquad (11)$$

It is easy to check that these generators do not satisfy Lie classical determining equations. Therefore we obtain the nonclassical symmetry reduction

$$z = t, \quad u = -\frac{x^2}{6t} + w(t), \qquad (12)$$

where $w(t)$ satisfies the linear ODE

$$3tw' + w = 0. \tag{13}$$

Consequently, an exact solution of (1) is

$$u = -\frac{x^2}{6t} + \frac{k_1}{t^{1/3}}. \tag{14}$$

We remark that solution (14) *is not* a travelling wave reduction and *it is not invariant* under the scaling group. Solution (14) is a characteristic solution of (1) consequently (14) is a solution of (2). This is in contradiction with the statement done in [20]:''the conditional symmetries of (2) with $\tau = 0$ yield solutions which are nothing else than group-invariant solutions''.

We point out that in (21), the infinitesimals for the independent variables are autonomous with respect to the dependent variable, generating a group of '*fibre-preserving transformations*'. Consequently, the solution (14) should also be obtained by the direct method [16], this solution is missing in [17,18].

2. Choosing $\phi = \frac{cu}{(x+k_1)}$ besides the classical reductions we obtain:

2.1. For

$$f(u) = k_1 u + \frac{k_2}{\sqrt{u}}, \tag{15}$$

we get the infinitesimal generators

$$\xi = 1, \quad \tau = 0, \quad \phi(x,u) = \frac{2u}{x} \tag{16}$$

and we obtain the symmetry reduction

$$z = t, \quad u = x^2 w(t), \tag{17}$$

where $w(t)$ satisfies

$$w' - 6k_1 w^2 = 0. \tag{18}$$

We can choose $k_1 = 1$ without loss of generality. Consequently an exact solution is

$$u = -\frac{x^2}{6t}. \tag{19}$$

We must point out that, when $f(u)$ adopts the the functional form (15), Eq. (1) does not admit any classical symmetry but translations, consequently this solution cannot be obtained by Lie classical symmetries. The scaling reduction can be used to reduce the single PDE (1) to a system of ODE's which has the common solution (19).

2.2. For

$$f(u) = k_1 u^{2/3} + k_2 u^{-2/3} \tag{20}$$

solving (9) we get the infinitesimal generators

$$\xi = 1, \quad \tau = 0, \quad \phi(x,u) = \frac{3u}{x}. \tag{21}$$

The corresponding symmetry reduction is

$$z = t, \quad u = x^3 w(t), \tag{22}$$

where $w(t)$ satisfies

$$w' - 12k_1 w^{5/3} = 0. \tag{23}$$

Consequently, an exact solution is

$$u = \frac{x^3}{16\sqrt{2}\,(-k_1 t)^{3/2}}. \tag{24}$$

We remark that when $f(u)$ adopts the form (20) Eq. (1) does not admit, besides translations, any classical symmetry. Therefore, this solution cannot be obtained by classical reduction. The scaling reduction can be used to reduce the single PDE (1) to a system of ODE's which has the common solution (24).

3. Choosing $\phi = \eta(t)u$, solving (9) we obtain

$$f(u) = -k_1(\log u - 1) + \frac{k_2}{u} + k_3, \tag{25}$$

and we get the infinitesimal generators

$$\xi = 1, \quad \tau = 0, \quad \phi(t,u) = -\frac{u}{\sqrt{2k_1 t}}. \tag{26}$$

The corresponding symmetry reduction is

$$z = t, \quad u = w(t)\exp\left(-\frac{x}{\sqrt{2k_1 t}}\right), \tag{27}$$

where $w(t)$ satisfies

$$4k_1^2 t^2 w' + 2k_1 tw(k_1 \log w - k_3) + kw = 0.$$

We can choose $k_1 = 1$ and the integrating constant $k_4 = 0$ without loss of generality. Consequently, an exact solution is

$$u = \exp\left(-\frac{x}{\sqrt{2t}} + \frac{k}{2t} + k_3\right). \tag{28}$$

When $f(u)$ adopts the functional form (25), Eq. (1) does not admit any classical symmetries but translations, consequently solution (28) is unobtainable by Lie classical symmetries, we observe that this is a noncharacteristic solution. We remark that most of the solutions obtained are characteristic solutions and so they are new solutions of the diffusion equation obtained when $k=0$ in these cases the solutions do not feel the influence of the diffusive interface.

4. Nonclassical symmetries and the singular manifold method

The complexity of determining Eq. (9), appears for many $\tau=0$ symmetries [6] and one advantage of the SMM is that provides non trivial solutions for (9). In a recent paper [11] Estévez and Gordoa have studied the Cahn–Hilliard equation (1) with $f(u)=u$ and $f(u)=u^2$ by using the SMM. In the following we compare these results with our results by using the nonclassical method:

- In [11] they claim that besides the trivial generator

$$\xi=0,\quad \tau=1,\quad \phi=0 \tag{29}$$

(which corresponds to a classical symmetry). For $f(u)=u^2$ the *only* infinitesimal generator of the nonclassical symmetries that reduce (1) to an ODE with the PP is

$$\xi=1,\quad \tau=0,\quad \phi=-\frac{1}{\sqrt{6k}}u^2 \tag{30}$$

The generator (30) yields to the similarity reduction

$$u=\frac{1}{k_1x+w(t)}, \tag{31}$$

where $w(t)$ satisfies the ODE

$$w'=0,$$

which satisfies the PP.
Nevertheless, it is easy to check that the following symmetry

$$\xi=1,\quad \tau=0,\quad \phi=\frac{it^{-1/2}}{2}, \tag{32}$$

satisfies Eq. (9) for the nonclassical symmetries with $\tau=0$ and yields the similarity reduction

$$u=\frac{ix}{2t^{1/2}}+w(t), \tag{33}$$

where $w(t)$ satisfies the ODE

$$2tw'+w=0,$$

which also satisfies the PP.

- In [11], for $f(u)=u$ they have got the following symmetry

$$\xi=1,\quad \tau=0,\quad \phi=\frac{2u}{x+x_0}. \tag{34}$$

They claim that, besides (29), the SMM allows to determine (34) and that these symmetries are the *only ones* in which the associated similarity reduction leads to an ODE of Painlevé type = 2E
Nevertheless, it is easy to check that the following symmetry

$$\xi=1,\quad \tau=0,\quad \phi=-\frac{u^{3/2}}{\sqrt{3k}}, \tag{35}$$

satisfies Eq. (9) for the nonclassical symmetries with $\tau=0$ and yields the similarity reduction

$$u=\frac{12k}{(x+w(t))^2}, \tag{36}$$

where $w(t)$ satisfies the ODE

$$w'=0,$$

which also satisfies the PP.

It is also easy to check that the infinitesimals (11) satisfy Eq. (9) for the nonclassical symmetries with $\tau=0$ = 2E These infinitesimals lead to the similarity reduction (12), where $w(t)$ satisfies (13), which is an ODE of Painlevé type.

Therefore, for the Cahn–Hilliard equation (1) with $f(u)=u$ and $f(u)=u^2$ by using the nonclassical method we have obtained three different symmetries that are respectively (32), (35) and (11) and that were not obtained in [11] by using the SMM.

We remark that although the generators (30),(32), (34), (35) do not satisfy the Lie classical determining equations, the corresponding solutions are group-invariant and can be derived from Lie classical symmetries. However solution (14), derived from (11) *is* *not invariant* under translations nor under the scaling group.

5. Concluding remarks

In this Letter we have seen a classification of symmetry reductions of a family of Cahn–Hilliard equation (1) using the classical Lie method of infinitesimals.

We have proved that for the parabolic type equation (1) the nonclassical method yields to symmetry reductions which are unobtainable by using the Lie classical method and the exact solutions obtained are not group invariant solutions. Consequently, in contradiction with the statement done in [20], we have proved that the nonclassical method is efficient for PDE's of the parabolic type.

We have obtained solutions by the nonclassical method that should also be obtainable by the direct method, these solutions were missing in [17,18].

We have discussed the symmetry reductions of this equation by using the nonclassical method with those derived in [11] by using the SMM. For this Cahn–Hilliard equation we have derived three nonclassical symmetries that reduce the equation to ODE's with the Painlevé property and were not obtained in [11] by the SMM. Therefore for this equation the nonclassical method is more general than the SMM and the SMM does not identify all the nonclassical symmetries that reduce the equation to ODE's with the PP.

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