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Similarity Reductions for a Nonlinear Diffusion Equation

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Abstract

Similarity reductions and new exact solutions are obtained for a nonlinear diffusion equation. These are obtained by using the classical symmetry group and reducing the partial differential equation to various ordinary differential equations. For the equations so obtained, first integrals are deduced which consequently give rise to explicit solutions. Potential symmetries, which are realized as local symmetries of a related auxiliary system, are obtained. For some special nonlinearities new symmetry reductions and exact solutions are derived by using the nonclassical method.

1 Introduction

Equation

$$u_t = (u^n)_{xx} + \frac{C}{x+\lambda}(u^n)_x, \tag{1}$$

with $n \in \mathbb{Q}\backslash\{0,1\}$ and $\lambda \in \mathbb{R}$, corresponds to nonlinear diffusion with convection. When $C = 0$ equation (1) becomes the one-dimensional porous medium equation

$$u_t = (u^n)_{xx}. \tag{2}$$

A complete group classification for (2) was derived by Ovsiannikov [23] and by Bluman [2, 4]. A classification for Lie-Bäcklund symmetries was obtained by Bluman and Kumei [5]. The main known exact solutions of nonlinear diffusion (2) are summarized by Hill [15]. In [15, 16, 17], Hill *et al* deduced a number of first integrals for stretching similarity solutions of the nonlinear diffusion equation, and of general high-order nonlinear evolution equations, by two different integration procedures. King [18] obtained approximate solutions to the porous medium equation (2).

The basic idea of any similarity solution is to assume a functional form of the solution which enables a PDE to be reduced to an ODE. The majority of known exact solutions of (2) turn out to be similarity solutions, even though originally they might have been derived, say by a separation of variable technique or as traveling wave solutions. For

$$u_t = (u^n)_{xx} + C\,(u^n)_x\,,$$

which is the Boussinesq equation of hydrology involved in various fields of petroleum technology and ground water hydrology, several exact solutions have been obtained by using isovector method [1].

More often than not the spatial dependent factors are assumed to be constant, although there is no fundamental reason to assume so. Actually, allowing for their spatial dependence enables one to incorporate additional factors into the study which may play an important role. For instance, in a porous medium this may account for intrinsic factors, like medium contamination with another material, or in plasma, this may express the impact that solid impurities arising from the walls have on the enhancement of the radiation channel. Knowing the importance of the effect of space-dependent parts on the overall dynamics of the nonlinear diffusion equation, a group classification for

$$u_t = (u^n)_{xx} + f(x)u^s u_x + g(x)u^m \tag{3}$$

was derived in [11], by studying those spatial forms which admit the classical symmetry group. Both the symmetry group and the spatial dependence was found through consistent application of the Lie-group formalism. The behaviour of the interface of a related problem with (1) has been studied by Okrasinski [22].

In this work we use the invariance of equation (1) under one-parameter group of transformations to reduce the PDE (1) to various ordinary differential equations. Most of the required theory and description of the method can be found in [6, 14, 19, 23, 24]. Following Hill [15, 16, 17], we have deduced some exact solutions of equation (1) by fully integrating the ODE's derived.

An obvious limitation of group-theoretic methods based in local symmetries, in their utility for particular PDE's, is that many of these equations do not have local symmetries. It turns out that PDE's can admit nonlocal symmetries whose infinitesimal generators depend on integrals of the dependent variables in some specific manner.

In [5, 6] Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by R{x,t,u}, in a conserved form, a related system denoted by S{x,t,u,v} with potentials as additional dependent variables, is obtained. Any Lie group of point transformations admitted by S{x,t,u,v} induces a symmetry for R{x,t,u}; when at least one of the generators of the group depends explicitly on the potential; then the corresponding symmetry is neither a point nor a Lie-Bäcklund symmetry. These symmetries of R{x,t,u} are called *potential* symmetries.

The nature of potential symmetries allows one to extend the uses of point symmetries to such nonlocal symmetries. In particular: Invariant solutions of S{x,t,u,v} yield solutions of R{x,t,u} which are not invariant solutions for any local symmetry admitted by R{x,t,u}. Potential symmetries for equation (3), when it can be written in a conserved form, have been recently derived [12].

In order to find potential symmetries of (1), we write this equation in the conserved form

$$D_xF - D_tG. \tag{4}$$

The associated auxiliary system S{x,t,u,v} is then given by

$$v_x = (x+\lambda)u, \qquad v_t = (x+\lambda)\left(u^n\right)_x + (C-1)u^n. \tag{5}$$

Suppose S{x,t,u,v} admits a local Lie group of transformations with infinitesimal generator

$$X_S = p(x,t,u,v)\frac{\partial}{\partial x} + q(x,t,u,v)\frac{\partial}{\partial t} + r(x,t,u,v)\frac{\partial}{\partial u} + s(x,t,u,v)\frac{\partial}{\partial v}. \tag{6}$$

This group maps any solution of S{x,t,u,v} to another solution of S{x,t,u,v} and hence induces a mapping of any solution of R{x,t,u} to another solution of R{x,t,u}. Thus (6) defines a symmetry group of R{x,t,u}. If

$$\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial v}\right)^2 + \left(\frac{\partial r}{\partial v}\right)^2 \neq 0 \tag{7}$$

then (6) yields a nonlocal symmetry of R{x,t,u}. Such nonlocal symmetry is called a *potential* symmetry of R{x,t,u} [5, 6].

Motivated by the fact that symmetry reductions for many PDE's are known that are not obtained by using the classical Lie group method, there have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole [3] introduced the nonclassical method to study the symmetry reductions of the heat equation and Clarkson and Mansfield [8] presented an algorithm for calculating the determining equations associated with the nonclassical method. The basic idea of the method is that the PDE (1) is augmented with the invariance surface condition

$$pu_x + qu_t - r = 0, \tag{8}$$

which is associated with the vector field

$$V = p(x,t,u)\frac{\partial}{\partial x} + q(x,t,u)\frac{\partial}{\partial t} + r(x,t,u)\frac{\partial}{\partial u}. \tag{9}$$

By requiring that both (1) and (8) are invariant under the transformation with infinitesimal generator (9) one obtains an overdetermined nonlinear system of equations for the infinitesimals $p(x,t,u)$, $q(x,t,u)$, $r(x,t,u)$. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is in general, larger than for the classical method as in this method one requires only the subset of solutions of (1) and (8) to be invariant under the infinitesimal generator (9). However, the associated vector fields do not form a vector space. These methods were generalized and called conditional symmetries by Fushchych and Nikitin [9] and also by Olver and Rosenau [20, 21] to include "weak symmetries", "side conditions" or "differential constraints".

The nonclassical symmetries of the nonlinear diffusion equation (3) with an absorption term and $f(x) = 0$, as well as new exact solutions were derived in [13]. In this work we obtain the special values of the parameter n such that nonclassical symmetries for (1) can be derived. We also report the reduction obtained as well as some new exact solutions.

2 Classical symmetries: Exact solutions

In this case, we find that the most general Lie group of point transformations admitted by (1) is:

1. For $n \neq 0, 1$ and $C \neq \dfrac{3n+1}{n+1}$ we obtain a three-parameter group $\mathcal{G}_3$. Associated with this Lie group is its Lie algebras $\mathcal{L}_3$, which can be respectively represented by the set of all the generators $\{V_i\}_{i=1}^{3}$. These generators are:

$$\frac{\partial}{\partial t}, \qquad V_2 = (x+\lambda)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \qquad V_3 = -t\frac{\partial}{\partial t} + \frac{u}{n-1}\frac{\partial}{\partial u}. \tag{10}$$

2. For $n \neq -1, 0, 1$ and $C = \dfrac{3n+1}{n+1}$, we obtain a four-parameter group $\mathcal{G}_4$. Its infinitesimal generators $\{V_i\}_{i=1}^{4}$ are:

$$\begin{aligned} &V_1' = V_1, \qquad V_2' = V_2, \qquad V_3' = V_3, \\ &V_4' = \frac{(x+\lambda)^{2-C}}{(C-2)(C-1)}\frac{\partial}{\partial x} - \frac{2(x+\lambda)^{1-C}}{(C-1)(n-1)}u\frac{\partial}{\partial u}. \end{aligned} \tag{11}$$

To ensure that an optimal set of reductions is obtained from the symmetries of (1), the optimal system is determined for $\lambda \neq 0$. The case $\lambda = 0$ was derived in [11]. In Table 1, we list the nontrivial optimal system $\{U_i\}$ with $i = 1, 2, 3, 4$. We also list the corresponding similarity variables and similarity solutions.

Table 1: Each row show the infinitesimal generators of the optimal system, the corresponding similarity variables and similarity solutions.

i	U_i	z_i	u_i
1	$V_2' + aV_3'$	$\dfrac{(x+\lambda)^{2-a}}{t}$	$h(z)(x+\lambda)^{\frac{a}{n-1}}$
2	$aV_1' + V_2' + 2V_3'$	$e^{-at}(x+\lambda)$	$h(z)(x+\lambda)^{\frac{2}{n-1}}$
3	$aV_3' + V_4'$	$c_1\,(n-1)\,(x+\lambda)^{C-1} + \log t$	$\dfrac{h(z)e^{c_1(x+\lambda)^{\frac{2n}{n+1}}}}{(x+\lambda)^{\frac{2}{n+1}}}$
4	$aV_1' + V_4'$	$\dfrac{a(C-2)(x+\lambda)^{C-1}}{t}$	$h(z)[(n-1)(x+\lambda)]^{\frac{2(2-C)}{n-1}}$

In Table 1, case $i = 1$, the constant $a \in \mathbb{Q} \setminus \{2\}$ and, in case $i = 3$, $c_1 = \dfrac{a}{n+1}$.

The ODE to which the PDE (1) is reduced by means of the infinitesimal generator U_1 is

$$\frac{dh}{dz}\left(\frac{1}{(a-2)^2 h^{n-1} n} - \frac{C_1+n+a-1}{(a-2)(n-1)z}\right) + \frac{ah(C_1-n+1)}{(a-2)^2(n-1)^2 z^2} + \frac{n-1}{h}\left(\frac{dh}{dz}\right)^2 + \frac{d^2h}{dz^2} = 0,$$

with

$$C_1 = Cn + an - C,$$

after taking $h = y^{\frac{1}{n}}$ it becomes

$$-\frac{dy}{dz}\left(\frac{C_1+n+a-1}{(a-2)(n-1)z} - \frac{y^{\frac{1}{n}-1}}{(a-2)^2 n}\right) + \frac{an(C_1-n+1)y}{(a-2)^2(n-1)^2 z^2} + \frac{d^2y}{dz^2} = 0.$$

In particular, the second order nonlinear differential equation for $y(z)$ obtained for $i=1$, admits first integrals for some values of a. As an example, we consider

$$a = 2 - 2n.$$

The first integral for $a = 2-2n$ is

$$-\frac{y}{z} + \frac{Cy}{2nz} - \frac{y}{2nz} + \frac{dy}{dz} + \frac{y^{\frac{1}{n}}}{4n^2} = c_1,$$

where c_1 is the integration constant. If this constant is zero we obtain for $a = 2-2n$

$$y = \frac{\left((1-n)\, z^{\frac{4n^2+k_2}{4n^2}} + \left(4k_1n^3 + k_1k_2n - k_1k_2\right) z^{\frac{k_2}{4n^3}}\right)^{\frac{n}{n-1}}}{\left(4n^3 + k_2n - k_2\right)^{\frac{n}{n-1}}\, z^{\frac{k_2}{4n^2-4n}}},$$

with

$$k_2 = (2c-2)\,n - 4n^2,$$

$$h = \frac{\left((1-n)\, z^{\frac{4n^2+k_2}{4n^2}} + \left(4k_1n^3 + k_1k_2n - k_1k_2\right) z^{\frac{k_2}{4n^3}}\right)^{\frac{1}{n-1}}}{\left(4n^3 + k_2n - k_2\right)^{\frac{1}{n-1}}\, z^{\frac{k_2}{4n^3-4n^2}}}.$$

Substituting in the similarity solution we obtain the exact solution

$$u = \frac{\left((n-1)\, z^{\frac{4n^2+k_2}{4n^2}} + \left(-4k_1n^3 - k_1k_2n + k_1k_2\right) z^{\frac{k_2}{4n^3}}\right)^{\frac{1}{n-1}}}{\left(-4n^3 - k_2n + k_2\right)^{\frac{1}{n-1}}\, z^{\frac{k_2}{4n^3-4n^2}}}.$$

When $C = 0$ this is the well-known dipole solution [15].

In particular, the second order nonlinear differential equation for $h(z)$ obtained for $i=2$, is

$$\frac{dh}{dz}\left(\frac{Cn+4n-C}{(n-1)z}+\frac{a}{h^{n-1}nz}\right)+\frac{2h\left(Cn+n-C+1\right)}{(n-1)^2z^2}+\frac{n-1}{h}\left(\frac{dh}{dz}\right)^2+\frac{d^2h}{dz^2}=0.$$

For the special value

$$C=-\frac{n+1}{n-1},$$

we integrated once to obtain

$$h^{n-1}\frac{dh}{dz}z+\frac{h^n\left(3n-1\right)}{(n-1)\,n}-\frac{h^n}{n}+\frac{ah}{n}=c_1.$$

After taking

$$h=y^{\frac{1}{n-1}}$$

it becomes

$$\frac{2y}{z}-\frac{a}{nz}+\frac{a}{z}+\frac{dy}{dz}=c_1,$$

where c_1 is the integration constant. If this constant is zero we obtain

$$y=\frac{c}{z^2}-\frac{a\,(n-1)}{2n},$$

so that

$$h=\left(\frac{ce^{2at}}{x^2}-\frac{a\,(n-1)}{2n}\right)^{\frac{1}{n-1}}x^{\frac{2}{n-1}}.$$

Substituting in the similarity solution we obtain the exact solution

$$u=\left(\frac{ce^{2at}}{x^2}-\frac{a\,(n-1)}{2n}\right)^{\frac{1}{n-1}}x^{\frac{2}{n-1}}.$$

The second order nonlinear ODE obtained, after taking $h(z)=y^{1/n}$, for $i=3$ is

$$-\frac{(n+1)^2\,y^{\frac{1}{n}-1}e^{-z}}{4{c_1}^2\,(n-1)^2\,n^3}\frac{dy}{dz}+\frac{d^2y}{dz^2}+\frac{2n}{n-1}\frac{dy}{dz}+\frac{n^2y}{(n-1)^2}=0,$$

where $c_1=\dfrac{a}{n+1}$.

The second order nonlinear ODE obtained, after taking $h(z)=y^{1/n}$, for $i=4$, is

$$\frac{d^2y}{dz^2}+\left(\frac{1}{2z}+c_1y^{1/n-1}\right)\frac{dy}{dz}=0,$$

where $c_1=\dfrac{(n-1)^{\frac{n-3}{n+1}}(n+1)^3}{16an^3}$.

3 Potential Symmetries

In order to find potential symmetries of (1), we write this equation in the conserved form (4), where

$$G = (x+\lambda)u,$$

$$F = (x+\lambda)\left(u^n\right)_x + (C-1)u^n.$$

The associated auxiliary system is given by (5). Besides $X_1 = \dfrac{\partial}{\partial t}$ and $X_2 = \dfrac{\partial}{\partial v}$ we obtain $X_3, \ldots, X_6$, given in Table 2.

Table 2: Each row show the infinitesimal generators, the corresponding similarity variables and similarity solutions.

(a)	p	q	r	s	n	C
X_3	$x+\lambda$	0	$\dfrac{2u}{n-1}$	$\dfrac{2nv}{n-1}$	$\neq 0,1$	arbitrary
X_4	$(x+\lambda)$	$2nt$	$-2u$	0	$\neq 0,1$	arbitrary
X_5	$k(x+\lambda)^{\frac{1-n}{1+n}}$	0	$-\dfrac{2k}{n+1}(x+\lambda)^{-\frac{2n}{n+1}}$	0	$\neq -1,0,1$	$\dfrac{3n+1}{n+1}$
X_6	$2(x+\lambda)v$	0	$-2(x+\lambda)^2u^2 - 2uv$	v^2	-1	$-\dfrac{1}{2}$

X_3, X_4 and X_5 project onto point symmetries of (1); while X_6 induces potential symmetries admitted by (1).

Solving the characteristic equation, we obtain the similarity variable $z = t$ and similarity solution $v = \sqrt{x+\lambda}E(t)$. In this case, $E(t)$ satisfies $E'(t) = 0$, so $E = \text{const}$ and we obtain the trivial solution

$$v = C_1\sqrt{x+\lambda}, \qquad u = \frac{C_1}{2(x+\lambda)^{\frac{3}{2}}}.$$

We must note that, although the infinitesimal p depends explicitly on v, the similarity variable does not depend on v.

4 Nonclassical symmetries

To apply the nonclassical method to (1) we require (1) and (8) to be invariant under the infinitesimal generator (9). In the case $q \neq 0$, without loss of generality, we may set $q(x,t,u) = 1$. The nonclassical method applied to (1) give rise to four determining equations for the infinitesimals.

$$\frac{\partial^2 p}{\partial u^2}u - n\frac{\partial p}{\partial u} + \frac{\partial p}{\partial u} = 0,$$

$$-\left(n\frac{\partial^2 r}{\partial u^2}-2n\frac{\partial^2 p}{\partial u\partial x}+2\frac{nC}{x+\lambda}\frac{\partial p}{\partial u}\right)u^{n-1}$$

$$-(n-1)\,n\frac{\partial r}{\partial u}u^{n-2}+(n-1)\,nru^{n-3}-2p\frac{\partial p}{\partial u}=0,$$

$$-\left(2n\frac{\partial^2 r}{\partial u\partial x}-n\frac{\partial^2 p}{\partial x^2}+\frac{nC}{x+\lambda}\frac{\partial p}{\partial x}+\frac{-nC}{(x+\lambda)^2}p\right)u^{n+1}$$

$$-2\,(n-1)\,n\frac{\partial r}{\partial x}u^n+\left(2\frac{\partial p}{\partial u}r-2p\frac{\partial p}{\partial x}-\frac{\partial p}{\partial t}\right)u^2+(n-1)\,pru=0,$$

$$-\left(n\frac{\partial^2 r}{\partial x^2}+\frac{nC}{x+\lambda}\frac{\partial r}{\partial x}\right)u^n+\left(\frac{\partial r}{\partial t}+2\frac{\partial p}{\partial x}r\right)u-(n-1)\,r^2=0,$$

where $f(x)=\dfrac{nC}{x+\lambda}$. Solutions of this system depend in a fundamental way on the values of n. By solving the determining equations we obtain

$$p=p_2(x,t)u^n+p_1(x,t).$$

We can distinguish now the following: if $n\in\left\{0,-1,-\frac{1}{2}\right\}$ we recover the classical symmetries, and if $n\notin\left\{0,-1,-\frac{1}{2}\right\}$, we obtain that

$$r=\mathrm{a}_2u^{n+2}+\mathrm{a}_3u^{n+1}+\mathrm{a}_1u^{n+1}+\frac{\mathrm{r}_2}{u^{n-1}}+\mathrm{a}_4u^2+\mathrm{r}_1u,$$

where

$$a_1=-\frac{Cp_2}{n(x+\lambda)},\qquad a_2=-\frac{2\mathrm{p}_2{}^2}{(n+1)\,(2n+1)},\qquad a_3=\frac{1}{n}\frac{\partial p_2}{\partial x},\qquad a_4=-\frac{2\mathrm{p}_1\mathrm{p}_2}{n+1}$$

and p_1, p_2, r_1, and r_2 are related by two conditions. After considering the special values for n for which new symmetries different from Lie classical symmetries can be obtained, we can now distinguish the following:

- for $n\neq\frac{1}{2}$ we recover the classical symmetries,
- for $n=\frac{1}{2}$ it follows that $p_2=0$ and p_1, r_1, and r_2 are related by the following conditions

$$-\mathrm{p}_1\mathrm{r}_1-4\mathrm{p}_1\frac{\partial \mathrm{p}_1}{\partial x}-2\frac{\partial \mathrm{p}_1}{\partial t}=0,$$

$$-\mathrm{p}_1\mathrm{r}_2-\frac{\partial \mathrm{r}_1}{\partial x}+\frac{\partial^2 \mathrm{p}_1}{\partial x^2}-2\frac{nC}{x+\lambda}\frac{\partial p_1}{\partial x}+2\frac{nC}{(x+\lambda)^2}p_1=0,$$

$$2\frac{\partial \mathrm{r}_1}{\partial t}+\mathrm{r}_1{}^2+4\frac{\partial \mathrm{p}_1}{\partial x}\mathrm{r}_1=0,$$

$$2\frac{\partial r_2}{\partial t} + 2r_1r_2 + 4\frac{\partial p_1}{\partial x}r_2 - \frac{\partial^2 r_1}{\partial x^2} - 2\frac{nC}{x+\lambda}\frac{\partial r_1}{\partial x} = 0,$$

$$-\frac{\partial^2 r_2}{\partial x^2} - 2\frac{nC}{x+\lambda}\frac{\partial r_2}{\partial x} + {r_2}^2 = 0.$$

We do not solve the above equations in general, but consider some special solutions:

1. Choosing $p_1 = k$, $r_1 = 0$ and $C = 2$, then $r_2 = \dfrac{2}{(x+\lambda)^2}$ and we obtain the nonclassical ansatz

$$z = x - kt, \qquad u = \left(h(z) - \frac{1}{k(x+\lambda)}\right)^2,$$

where $k \neq 0$ and $h(z)$ satisfies the following ODE

$$h'' + 2khh' = 0; \tag{12}$$

whose solutions are

$$h(z) = \frac{k_4 \tanh(k_4(z+k_2))}{k}, \quad \text{if} \quad kk_3 > 0 \quad \text{and} \quad k_4 = \sqrt{kk_3},$$

$$h(z) = -\frac{k_4 \tan(k_4(z+k_2))}{k}, \quad \text{if} \quad kk_3 < 0 \quad \text{and} \quad k_4 = \sqrt{-kk_3}.$$

This leads to the exact solutions

$$u(x,t) = \left(\frac{k_4 \tanh(k_4(z+k_2))}{k} - \frac{1}{k(x+C)}\right)^2 \quad \text{if} \quad kk_3 > 0,$$

$$u(x,t) = \left(\frac{k_4 \tan(k_4(z+k_2))}{k} - \frac{1}{k(x+C)}\right)^2, \quad \text{if} \quad kk_3 < 0.$$

2. Choosing $p_1 = p_1(x)$, $r_1 = r_1(x)$ and $r_2 = r_2(x)$ we obtain

$$r(x,t,u) = -4p'u + \left[\frac{5p_1''}{p_1} - \frac{Cp_1'}{(x+\lambda)p_1} + \frac{C}{(x+\lambda)^2}\right]\sqrt{u},$$

where p_1 must satisfy the following equations

$$\frac{C}{2(x+\lambda)} = \frac{3p_1'}{2p_1} - \frac{p_1''}{2p_1'} - \frac{c_1}{2p_1p_1'},$$

$$-\frac{2C}{x+\lambda}\left[2p_1^2p_1''' + 4p_1p_1'p_1'' - (p_1')^3\right] + \frac{cp_1}{(x+\lambda)^2}\left[Cp_1p_1'' + 8p_1p_1'' + 2(p_1')^2\right]$$
$$-\frac{C}{(x+\lambda)^3}(3C-2)p_1^2p_1' + \frac{3}{(x+\lambda)^4}(C-2)Cp_1^3 - 5p_1^2p_1''''$$
$$+10p_1p_1'p_1''' + 30p_1(p_1'')^2 - 10(p_1')^2p_1'' = 0.$$

We observe that these condition are satisfied for $p_1 = k_1(x+\lambda)^{\frac{C-1}{2}}$ if $C \in \left\{\frac{5}{3}, -1, 3\right\}$.

- For $C = 3$, $p_1 = k_1(x+\lambda)$ we recover classical symmetries.
- For $C = \frac{5}{3}$, we obtain

$$p_1 = k_1(x+\lambda)^{1/3}, \qquad r = -\frac{4k_1 u}{3(x+a)^{2/3}}.$$

Hence a nonclassical ansatz is

$$z = \frac{3k_1}{2}(x+\lambda)^{2/3} - kt, \qquad u = (x+\lambda)^{-4/3}h(z),$$

where $h(z)$ satisfies the following ODE

$$-2hh'' + (h')^2 + -4kk_1^2 h^{3/2} h' = 0,$$

whose solutions are

$$h(z) = \frac{k_2}{4k_1^2 k}\tan(k_1 k_4(z+k_3)) \quad \text{if } kk_2 > 0, \quad k_4 = \sqrt{kk_2} \quad \text{and} \quad k_1 < 0$$

$$h(z) = \frac{k_2}{4k_1^2 k}\tanh(k_1 k_4(z+k_3)) \quad \text{if } kk_2 < 0, \quad k_4 = \sqrt{-kk_2} \quad \text{and} \quad k_1 > 0$$

- For $C = -1$, we obtain

$$p_1 = \frac{k_1}{x+\lambda}, \qquad r = \frac{4(k_1 u + 2\sqrt{u})}{(x+\lambda)^2}.$$

Hence a nonclassical reduction is

$$z = \frac{x^2 + 2\lambda x}{k_1} - t, \qquad u = \frac{1}{k_1^2}(h(z)(x+\lambda)^2 - 2)^2,$$

where $h(z)$ satisfies (12) with $k = k_1$.

3. Choosing $p_2 = r_2 = 0$ and $p_1 = p_1(x)$, we obtain $r_1 = -4p_1'$ and $f = \dfrac{5p_1' - 2k_1}{2p_1}$, where p_1 satisfies

$$p_1^2 p_1'' + 2p_1(p_1')^2 - 2k_1 p_1 p_1' - k_2 p_1 = 0,$$

which is a classical reduction [11].

5 Concluding Remarks

In this paper we have used the invariance of (1) under group of transformations to reduce (1) to ODEs. We desired to minimize the search for group-invariant solutions to that of finding non-equivalent branches of solutions, consequently we have constructed all the invariant solutions with respect to the one-dimensional optimal system of subalgebras, as well as all the ODEs to which (1) is reduced. For the equations so obtained, first integrals have been deduced which give rise to explicit solutions. Potential symmetries as well as nonclassical symmetries were used to obtain new solutions of (1). The new solutions are unobtainable by Lie classical symmetries.

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