Classical and Nonclassical Symmetries of a Generalized Boussinesq Equation

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Abstract

We apply the Lie-group formalism and the nonclassical method due to Bluman and Cole to deduce symmetries of the generalized Boussinesq equation, which has the classical Boussinesq equation as an special case. We study the class of functions $f(u)$ for which this equation admit either the classical or the nonclassical method. The reductions obtained are derived. Some new exact solutions can be derived.

The Boussinesq equation arises in several physical applications, the first one was propagation of long waves in shallow water [3]. There have been several generalizations of the Boussinesq equation such as the modified Boussinesq equation, or the dispersive water wave.

Another generalized Boussinesq equation is

$$u_{tt} - u_{xx} + (f(u) + u_{xx})_{xx} = 0, \tag{1}$$

which has the classical Boussinesq equation as an special case for $f(u) = \dfrac{u^2}{2} + u$. Recently conditions for the finite-time blow-up of solutions of (1) have been investigated by Liu [8].

In this work we classify the Lie symmetries of (1) and we study the class of functions $f(u)$ for which this equation is invariant under a Lie group of point transformations. Most of the required theory and description of the method can be found in [2, 10, 11].

Motivated by the fact that symmetry reductions for many PDE's are known that are not obtained using the classical Lie group method, there have been several generalizations of the classical Lie group method for symmetry reductions. Clarkson and Kruskal [4] introduced an algorithmic method for finding symmetry reductions, which is known as the direct method. Bluman and Cole [1] developed the nonclassical method to study the symmetry reductions of the heat equation. The basic idea of the method is to require that both the PDE (1) and the surface condition

$$\Phi \equiv p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial t} - r = 0, \tag{2}$$

must be invariant under the infinitesimal generator. These methods were generalized and called conditional symmetries by Fushchych *et al* [5] and also by Olver and Rosenau [6, 7] to include 'weak symmetries', 'side conditions' or 'differential constraints'.

We consider the classical Lie group symmetry analysis of equation (1). Invariance of equation (1) under a Lie group of point transformations with infinitesimal generator

$$V = p(x,t,u)\frac{\partial}{\partial x} + q(x,t,u)\frac{\partial}{\partial t} + r(x,t,u)\frac{\partial}{\partial u} \tag{3}$$

leads to a set of twelve determining equations for the infinitesimals. For totally arbitrary $f(u)$, the only simmetries are the group of space and time translations which are defined by the infinitesimal generators

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}. \tag{4}$$

We obtain travelling wave reductions

$$z = x - \lambda t, \qquad u = h(z), \tag{5}$$

where $h(z)$, after integrating twice with respect to z, satisfies

$$h'' + (\lambda^2 - 1)h + f(h) = k_1 z + k_2, \tag{6}$$

with k_1 and k_2 arbitrary constants. The only functional forms of $f(u)$, with $f(u) \neq$ const. and $f(u)$ nonlinear, which have extra symmetries are given in Table 1

Table 1: Symmetries for the generalized Boussinesq equation.

i	$f(u)$	V_3^i
1	$d(au+b)^n + u + c$	$x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + \frac{2}{a(1-n)}(au+b)\frac{\partial}{\partial u}$
2	$d\log(au+b) + u + c$	$x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + \frac{2}{a}(au+b)\frac{\partial}{\partial u}$
3	$de^{(au+b)} + u + c$	$x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \frac{2}{a}\frac{\partial}{\partial u}$

We observe that equation (6) with $f(h) = d(ah+b)^n + kh$ can be solved. Setting $\lambda^2 = 1 - k$ and $k_1 = 0$ the solution is

- For $m = n + 1$ and $n \neq -1$,

$$\pm\left(\frac{am}{2}\right)^{1/2}\int(-a(k_2 h + k_3)m - d(ah+b)^m)^{-1/2}dh = z + k_4.$$

- For $n = -1$,

$$\pm \left(\frac{a}{2}\right)^{1/2} \int \left(-a(k_2 h + k_3) - d \log (ah + b)\right)^{-1/2} dh = z + k_4.$$

- For $n = 2$, depending upon the choice of the constants, this equation is solvable; for $k_1 \neq 0$ in terms of the first Painlevé equation, and elliptic or elementary functions if $k_1 = 0$.

- For $n = 3$, setting $k_1 = k_2 = 0$, the equation is solvable in terms of the Jacobi elliptic functions.

In Table 2 we list the corresponding similarity variables and similarity solutions.

Table 2: Each row gives the functions $f(u)$ for which (1) can be reduced to an ODE, as well as the corresponding similarity variables and similarity solutions.

i	V_3^i	$f(u)$	z_i	u_i
1	V_3^1	$f = d(au + b)^n + ku + c$	$\dfrac{x}{\sqrt{t}}$	$t^{\frac{1}{1-n}} h(z) - \dfrac{b}{a}$
2	V_3^2	$f = d\log(au + b) + u + c$	$\dfrac{x}{\sqrt{t}}$	$th(z) - \dfrac{b}{a}$
3	V_3^3	$f = de^{(au+b)} + u + c$	$\dfrac{x}{\sqrt{t}}$	$-\dfrac{1}{a} \log(th(z))$

In the Table 3 we show the ODE's to which PDE (1) is reduced by

Table 3: Symmetries for the generalized Boussinesq equation with $k = nda^n$.

V_3^i	ODEi
V_3^1	$h'''' + \left(\dfrac{z^2}{4} + kh^{n-1}\right) h'' + k(n-1)h^{n-2}(h')^2 + \left(\dfrac{z}{n-1} + \dfrac{3z}{4}\right) h' + \dfrac{nh}{(n-1)^2} = 0$
V_3^2	$4h^2 h'''' + 4d(hh'' - (h')^2) + h^2(z^2 h'' - zh') = 0$
V_3^3	$4gg''' + z^2(g')^2 + 2z^2 - zg + k_1 z + k_2 - de^{-g'} = 0$

- ODE1 for $n = 2$, multiplied by z, can be integrated once and we obtain

$$\left(\frac{z^3}{4} + h\,k\,z\right) h' + h\,z^2 + zh''' - \frac{h^2\,k}{2} - h'' = 0.$$

- ODE1 for $n = 3$, can be integrated once and we obtain

$$\left(\frac{z^2}{4} + h^2 k\right) h' + \frac{3\,h\,z}{4} + h''' = 0.$$

- ODE1 for $n = -1$, integrating once, we obtain

$$\left(\frac{z^2}{4} + \frac{k}{h^2}\right) h' - \frac{h\,z}{4} + h''' = 0.$$

- ODE3 with $h = e^{g'}$ has been obtained after integrating twice with respect to z.

In the nonclassical method one requires only the subset of solutions of (1) and (2) to be invariant under the infinitesimal generator (3). In the case $q \neq 0$ we may set $q(x, t, u) = 1$ in (2 without loss of generality. The nonclassical method applied to (1) gives rise to a set of eight nonlinear determining equations for the infinitesimals. The solutions for these equations depend on the function $f(u)$. We can distinguish the following cases: For

$$f(u) = du^2 + bu + c,$$

the solution is

$$p = -d(p_1(t)x + p_2(t)),$$

$$r = p_1(p_1' + 2p_1^2)x^2 + (p_1p_2' + p_2p_1' + 4p_1^2p_2)x + 2dp_1u + p_2p_2' + 2p_1p_2^2 + (1 - b)p_1,$$

where $p_1(t) = \dfrac{h'(t)}{2h}$, $p_2(t) = k_1p_1 \displaystyle\int \frac{h(t)dt}{(h'(t))^2} + k_2p_1$, and $h(t)$ satisfies

$$(h'(t))^2 = k_3h^3 + k_4 \tag{7}$$

Here $k_1, \ldots, k_4$ are arbitrary costants. Equation (7) is solvable in terms of the Weierstrass elliptic functions if $k_3k_4 \neq 0$, and in terms of elementary functions otherwise. Solving (7) for $h(t)$ yields the six canonical symmetry reductions derived by Clarkson for the classical Boussinesq equation using the nonclassical method, and by Clarkson and Kruskal using the direct method.

For any other function $f(u)$ listed in Table 1 the same symmetries, as were obtained by the classical method, appear.

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