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Location of a moving service facility

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Abstract. In this paper we consider the general question in the field of mathematics of whether some properties or algorithms that hold in finite dimension spaces also hold in function spaces. We answer this question concerning the very well-known Weiszfeld algorithm for the Weber problem. In order to do that, we consider the Weber problem with trajectories (functions of time) instead of points in a finite-dimensional space. This is in fact the problem of locating a moving service facility. Properties are proved assuring that the problem is well-established and that an optimal solution exists if L^p $1 \leq p \leq +\infty$ spaces are considered. An extension of Weiszfeld's algorithm is proposed to solve this kind of problem and it is shown that under some assumptions it presents global convergence properties. Moreover, an example is included showing that this extension is not trivial because the natural pointwise extension of Weiszfeld's algorithm does not have to converge to an optimal solution of the considered problem while the new algorithm does.

Key words: Location, Weber problem, Weiszfeld algorithm

1 Introduction

An important area of development of the Operations Research is the new adaptation of tools of Mathematics to classical problems of this field. Within the field of Location Theory one of the most widely studied problems is the well-known Weber problem. In the basic model we deal with a set A of given points (the demand points) and we want to locate a new facility x^* minimizing the weighted sum of distances to the points in A.

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A large number of authors have studied this problem because it is an useful way to model real-world applications (see for instance the book of Francis et al. (1992) or the paper of Wesolowsky (1993)). The most extensively used method to solve this problem is the so called Weiszfeld algorithm. Weiszfeld algorithm is an iterative method based in a fixed point equation obtained from a necessary optimality condition for this problem. Several papers have dealt with the convergence of this algorithm and with its improvements and extension (see for instance Chandrasekaran and Tamir (1989), Drezner (1992) and Brimberg and Love (1993)). However, there was still a general question concerning this algorithm that had not been considered. It corresponds with the general question in the field of mathematics of whether it is possible to apply similar tools in finite and function spaces. One of the aims of this paper is to show that Weiszfeld algorithm can be reformulated to deal as well with location problems in L^p spaces of functions. Apart from the importance itself of providing the same kind of tools for these two different frameworks (finite and function spaces) this paper also shows that this extension is not trivial. In a first analysis, one could think that the formulation of Weber problems in L^p spaces of functions should lead to algorithms whose application consists of a pointwise extension of the corresponding in finite dimension spaces. Nevertheless, as we show in the paper this is not the case. In the last section we provide examples where the pointwise application of the classical Weiszfeld algorithm produces worse solutions than the obtained applying the proposed algorithm.

In order to study these questions we deal with an extension of the Weber problem that we call the Dynamic Weber problem. It consists of substituting the points in the set A by trajectories (functions of time), so that we replace the environmental framework from a finite-dimensional space to a infinite-dimensional one. In fact, this problem consists of locating a moving service facility with respect to moving service demands.

Although this approach has not been previously addressed in the literature, several attempts can be found dealing with earlier versions of the problem of locating moving service facility. The most direct reference corresponds with the paper of Abdel-Malek (1985). He studies a very particular case of location of a moving facility under squared Euclidean distance. Wesolowsky (1973), studies a Weber problem where he imposed that initial conditions (the weights and the demand points) varied in a fixed number of epochs. Wesolowsky (1975), generalizes the classical location-allocation problem allowing the cost functions being different at different fixed epochs. Recently, Drezner and Wesolowsky (1991) generalize the Weber problem to the case where the weights are allowed to change a finite number of times. The goal is to find an optimal solution being a step function.

Apart from looking for the answer of our previously considered question there are two more reasons which justify this new approach to the Weber problem. The first reason is connected to the nature of this approach. Indeed, although it is a variational problem it does not fit the requirements of Euler-Lagrange condition: 1) we are interested in general solutions non necessarily smooth, and 2) the objective function of this problem is

$$\sum_{a \in A} w_a \left(\sum_{k=1}^{N} \int_I |x_k(t) - a_k(t)|^p dt \right)^{1/p}$$

which cannot be transformed into the usual one $\int_I F(t, x, x')\,dt$. Therefore, in order to solve the problem "ad hoc" procedures have to be developed.

The second reason relies on the properties of the optimal solution of this problem and justifies the mathematical foundations of the proposed algorithm. As we have already mentioned, the optimal solutions of this problem do not have to coincide with the pointwise solution (see Brimberg and Love (1993) for details on the convergence of classical Weiszfeld's algorithm). That is, the solution obtained when solving the problem optimally for each time epoch t in the interval I.

Regarding to its applicability, this model and the proposed algorithm can be used to solve location problems as well as to perform non-parametric estimation of the mean, median or more complex functions of stochastic processes or time series. On the other hand, as well as the classical Weber problem allows us to compute sample 1-principal points of random variables (see Flury (1990)), this new approach leads us to the computation of 1-principal functions of stochastic processes. In addition, it should be noted that the dynamic approach adapts better than the static one to model certain location situations. For instance, the location of a trajectory of a moving service facility with respect to a set of fixed routes or corridors.

This paper is organized as follows. In Section 2 the Dynamic Weber problem is formulated and several preliminary properties are shown. Section 3 is devoted to extend the well-known Weiszfeld's algorithm to this kind of dynamic problems. Section 4 proves some preliminary properties and Section 5 proves the global convergence of the proposed methodology. In addition, it also includes an illustrative example of application of our algorithm which shows that the pointwise solutions are not necessarily optimal for this problem. The paper ends with some conclusions.

2 The model

Throughout this section, we introduce a location problem which consists of a generalization of the well-known Weber problem. We consider the normed spaces $Y_p := L^p(I, \mathbb{R}^N)$ and $X_p := L^p(I, \mathbb{R})$, being I a finite interval. That is, $x \in Y_p$ means that

$$x : I \longrightarrow \mathbb{R}^N$$
$$t \mapsto x(t) = (x_1(t), \ldots, x_N(t))$$

where each $x_k(\cdot)$ belongs to X_p for all $k = 1, \ldots, N$. The norms in these spaces are defined for any $x \in Y_p$ by

$$\|x\|_p := \left(\sum_{k=1}^{N} |x_k|_p^p \right)^{1/p}$$

and for any $x_k \in X_p$ by

$$|x_k|_p := \left(\int_I |x_k(t)|^p\,dt \right)^{1/p}.$$

Finally, in order to obtain a better readability of this paper we denote by $m(B)$ the Lebesgue measure of the measurable set B, a.e. stands for almost everywhere, and $\langle \cdot, \cdot \rangle$ denotes the scalar product defined as follows:

$$\langle \cdot, \cdot \rangle : Y_p \times Y_r \longrightarrow \mathbb{R}$$

$$(x, y) \mapsto \langle x, y \rangle = \int_I \sum_{k=1}^{N} x_k(t) y_k(t)\, dt$$

where $\frac{1}{p} + \frac{1}{r} = 1$.

Therefore, given a finite set of given trajectories $A \subset Y_p$ the problem consists of looking for a new trajectory x^*, solving the following optimization problem

$$\inf_{x \in Y_p} F_p(x) = \sum_{a \in A} \omega_a \|x - a\|_p \tag{1_p}$$

where ω_a are constants greater than 0.

It should be noted that Y_p is a infinite dimensional space, and the difficulty when dealing with optimization in infinite dimensional spaces are caused by the fact that their topologies are more complex than the topology of $\mathbb{R}^N$. The differences are well-known in existence theory since in these spaces the Existence Theorem of Weirstrass, which uses compactness arguments, is not as useful as in finite theory. However in our case this gap can be avoided using results on the connection between weak and strong topologies, which in fact are consequence of the well known theorem of Mazur.

Before describing algorithms solving Problem (1_p), we want to establish certain properties assuring that this problem is well-stated and that it has an optimal solution.

For the sake of completeness we give an easy proof of the following existence theorem.

Theorem 2.1. *For any $p \in (1, +\infty)$ we have*

$$\inf_{x \in Y_p} F_p(x) = \min_{x \in Y_p} F_p(x)$$

Proof: It is straightforward to see that any optimal solution of Problem (1_p) must belong to $C_p = \{y \in Y_p : \|y\|_p \leq 2 \max_{a \in A} \|a\|_p\}$.

On the other hand, the objective function F_p is a continuous convex function defined on the bounded set C_p. Using Proposition 38.12 in Ekeland et al. (1979) we have that there exists the minimum of F_p within C_p which proves the result. □

For $p = 1$, Y_1 is not reflexive and this theorem cannot be applied. To assure existence in this case, we have to embed Y_1 in the space $NBV(I, \mathbb{R}^N)$, the normalized space of functions of bounded variation. Since, $NBV(I, \mathbb{R}^N)$ can be uniquely identified with the dual of $\mathscr{C}(I, \mathbb{R}^N)$ then Problem (1_1) also has optimal solution in $NBV(I; \mathbb{R}^N)$. In conjunction with the previous results of existence, we conclude this section studying the uniqueness of solution for Problem (1_p). In order to do that we introduce the concept of non collinearity.

Definition 2.1. *The set A of demand points is called* **non collinear** *if it contains three elements a, a', and a'' such that does not exist $\lambda \in [0, 1]$ verifying $a = \lambda a' + (1 - \lambda)a''$.*

Then we have the following result.

Theorem 2.2. *If the set A of demand points is non collinear then there exists a unique optimal solution to Problem (1_p), $\forall p \in (1, +\infty)$.*

Proof: Let us consider the function $F(x) = \sum_{a \in A} \omega_a d_a(x)$ where $d_a(x) = \|x - a\|_p$. We will prove that $F = F(x)$ is a strictly convex function when A is non collinear.

If F is not strictly convex there exist $\theta \in (0, 1)$, $x, y \in Y_p$ with $x \neq y$ such that

$$d_a(\theta x + (1 - \theta)y) = \theta d_a(x) + (1 - \theta)d_a(y) \quad \forall a \in A$$

which implies that exist $0 \leq \lambda_a$ such that $x(t) - a(t) = \lambda_a(y(t) - a(t))$ *a.e.*. Since $x \neq y$, it follows that $\lambda_a \neq 1$. Hence,

$$\frac{1}{1 - \lambda_a} x(t) - \frac{\lambda_a}{1 - \lambda_a} y(t) = a(t) \ a.e. \quad \text{and} \quad \forall a \in A$$

which contradicts that A is non collinear. Hence F is strictly convex. Once F is strictly convex the result follows. □

3 Weiszfeld's dynamic algorithm

The objective function F_p is convex because the norm $\| \cdot \|_p$ is a convex function for $p \geq 1$. As a result of the convexity of F_p we know that necessary conditions are also sufficient for optimality (see e.g. Bazaraa et al. (1979)). It is worth notice that the general optimality condition is $\frac{\partial F_p}{\partial x_k}(x; h) \geq 0 \ \forall h \in Y_p$ $k = 1, \ldots, n$ $p > 1$ where $\frac{\partial F}{\partial x_k}(x; h)$ stands for the directional derivative of F_p at x in the direction of h.

However, we cannot use this expression to develop an interative algorithm to solve the problem. Nevertheless, for those solutions not equal everywhere to any demand function $\frac{\partial F}{\partial x_k}(x; h)$ is a Gateaux differential and hence linear continuous functional on Y_p, therefore x is a solution of (1_p) iff

$$\frac{\partial F_p}{\partial x_k}(x; h) = 0 \quad k = 1, \ldots, n \quad \forall h \in Y_p \quad p > 1. \tag{1}$$

It should be noted that the above expression cannot be used for $p = 1$ because $F_p(\cdot)$ is not differentiable for $p = 1$.

The derivative in (1) can be written for a particular k, $1 \le k \le N$, as

$$\sum_{a \in A} \omega_a \left(\|x-a\|_p^{1-p} \int_I |x_k(t)-a_k(t)|^{p-2}(x_k(t)-a_k(t))h_k(t)\,dt \right) = 0$$

$$\forall h = (h_1, \ldots, h_N) \in Y_p.$$

$$\sum_{a \in A} \frac{\omega_a}{\|x-a\|_p^{p-1}} \langle |x_k - a_k|^{p-2}(x_k - a_k), h_k \rangle = 0 \quad \forall h_k \in X_p.$$

Using the completeness of X_p, the following expression holds,

$$\sum_{a \in A} \frac{\omega_a}{\|x-a\|_p^{p-1}} |x_k(t)-a_k(t)|^{p-2}(x_k(t)-a_k(t)) = 0 \ a.e.$$

Isolating the unknown variable we obtain that the optimal solution must verify the following equation in order to be an optimal solution of Problem (1_p).

$$x_k(t) = \sum_{a \in A} \frac{\frac{\omega_a |x_k(t)-a_k(t)|^{p-2}}{\|x-a\|_p^{p-1}}}{\sum_{a \in A} \frac{\omega_a |x_k(t)-a_k(t)|^{p-2}}{\|x-a\|_p^{p-1}}} a_k(t) \quad \forall k = 1, \ldots, N$$

Thus, we obtain an iterative process by means of the fixed point equation, $\psi(x) = x$; where $\psi(x) = (\psi_1(x), \ldots, \psi_N(x))$ is given by

$$\psi_k(x^q)(t) = \sum_{a \in A} \frac{\frac{\omega_a |x_k^q(t)-a_k(t)|^{p-2}}{\|x^q-a\|_p^{p-1}}}{\sum_{a \in A} \frac{\omega_a |x_k^q(t)-a_k(t)|^{p-2}}{\|x^q-a\|_p^{p-1}}} a_k(t) \quad \forall k = 1, \ldots, N. \tag{2}$$

The main difference of this expression with respect to the scheme for the finite dimensional case is that in its present form $\psi(x)$ is a function of t. With this in mind, the iterative scheme is

$$x_k^{q+1}(t) = \sum_{a \in A} \frac{\frac{\omega_a |x_k^q(t)-a_k(t)|^{p-2}}{\|x^q-a\|_p^{p-1}}}{\sum_{a \in A} \frac{\omega_a |x_k^q(t)-a_k(t)|^{p-2}}{\|x^q-a\|_p^{p-1}}} a_k(t) \quad k = 1, \ldots, N \tag{3}$$

From this definition we can make the following observations:

1. Case $p \ge 2$.
   If $x^q = a$ a.e., for some $a \in A$ then x^{q+1} is not well-defined and consequently we modify the definition of the iterate x^{q+1}, being $x^{q+1}(t) = a(t)\ \forall t$. So that, x^{q+1}, is finally defined in the following way

$$x^{q+1} = \begin{cases} \psi(x^q), & \text{if } x^q - a \neq 0 \ a.e. \text{ for all } a \in A \\ a, & \text{if } x^q - a = 0 \ a.e., \text{ for some } a \in A \end{cases}$$

2. Case $1 \le p < 2$. In this case we define the following set

$$A_k(x) = \{t \in I : \exists a \in A, |x_k(t) - a_k(t)| = 0\} \tag{4}$$

If there exists $A_k(x^q) \subset I$, with $m(A_k(x^q)) > 0$ so that $x_k^q(t) = a_k(t)\ \forall t \in A_k(x^q)$, we define $x_k^{q+1}(t) = a_k(t)$ for all $t \in A_k(x^q)$.
Thus, $x_k^{q+1}(t)$ is defined in the following way

$$x_k^{q+1}(t) = \begin{cases} a_k(t), & \forall t \in A_k(x^q) \\ \psi_k(x^q)(t), & \forall t \in I \backslash A_k(x^q) \end{cases}$$

Remark 3.1. The iterations defined by the Weiszfeld dynamic algorithm does not coincide with the function generated pointwise from the classical Weiszfeld algorithm applied to every point in I. This fact can easily be shown looking at the expression of the iterates (3). For $p > 1$, that formula depends for each $t \in I$ on the norm of $\|x^q - a\|$ for all $a \in A$. This dependence does not appear on the pointwise approach because in this case for each $t \in I$ it depends on $|x^q(t) - a(t)|_p^{p-1}$. It should be noted that in this case $|\cdot|_p$ stands for the norm l_p in $\mathbb{R}^n$. For further details see the Example in Section 5.2. There, we show that the solution obtained by solving the problem in every point of the interval I is worse that the solution of the problem obtained iterating in Y_p.

4 Properties of Weiszfeld's dynamic algorithm

In this section we present some preliminary properties of the iterative scheme we have previously introduced in Section 3. In order to study the convergence of this scheme we consider the following sets,

$$H_k = \{x \in Y_p : \exists A_k(x) \text{ with } m(A_k(x)) > 0\}$$

where $A_k(x)$ was defined in (4). Then, we introduce the set

$$S_p = \begin{cases} \bigcup_k H_k, & \text{if } 1 \le p < 2 \\ \hat{A}, & \text{if } p \ge 2 \end{cases}$$

where $\hat{A}$ is a set of functions with the same cardinality that A such that for each $\hat{a} \in \hat{A}$ there exists a unique $a \in A$ with $\hat{a} = a$ almost everywhere.

It is straightforward to see that this iterative approach is actually a modified gradient descent method. However, the global convergence depends on the range of admissible values for the parameter p. Indeed, provided that $x^q \neq a$ a.e. for all $a \in A$,

$$x_k^{q+1}(t) = x_k^q(t) - \sum_{a \in A} \frac{\frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}}}{\sum_{a \in A} \frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}}} (x_k^q(t) - a_k(t))$$

then $x^{q+1} = (x_1^{q+1}, \ldots, x_N^{q+1})$ verifies

$$x^{q+1}(t) = x^q(t) - S^{q+1}(t)\nabla F_p(x^q)(t)$$

where $S^{q+1}(t)$ is a diagonal $N \times N$ matrix being the $k - th$ entry

$$S_k^{q+1}(t) = \frac{1}{\sum_{a \in A} \frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}}}.$$

Definition 4.1. *A function $x \in Y_p$ is said to be regular with respect to Problem (1_p) if $x \notin S_p$.*

Proposition 4.1. *Let x^* denote an optimal solution of Problem (1_p) with $p > 1$.*

a) *If $x^q = x^*$ a.e. then $x^{q+1} = x^*$ a.e.*
b) *$x^q \notin S_p$ and $x^{q+1} = x^q$ a.e. then $x^q = x^*$ a.e.*

Proof: The proof runs parallel to the one given in the finite dimensional case, Brimberg and Love (1993), and it is left to the reader. □

Proposition 4.2. (Descent property) *If $1 \le p \le 2$, and x^{q+1} and x^q do not coincide almost everywhere then $F(x^{q+1}) < F(x^q)$.*

Proof: Assuming x^q is given we know that

$$x_k^{q+1}(t) = \sum_{a \in A} \frac{\frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}}}{\sum_{a \in A} \frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}}} a_k(t)$$

Let us define $V(x^q) = \{k : x^q \notin H_k\}$. For each fixed $y_k \in L^p(I, \mathbb{R})$ and $k \in V(x^q)$ consider

$$g_k(y_k)(t) = \sum_{a \in A} \frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}} (y_k(t) - a_k(t))^2$$

g_k as a function of y_k is strictly convex and

$$\frac{\partial}{\partial y_k} g_k(y_k)(t) = 2 \sum_{a \in A} \frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}} (y_k(t) - a_k(t))$$

Therefore, its minimum is achieved at

$$y_k(t) = \sum_{a \in A} \frac{\frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}}}{\sum_{a \in A} \frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}}} a_k(t)$$

and as soon as t varies y_k coincides almost everywhere with x_k^{q+1} $\forall k$. Therefore, the function g_k is strictly convex and it reaches its minimum at x_k^{q+1}. As x_k^{q+1} is the minimum, then:

$$g_k(x_k^{q+1})(t) < g_k(x_k^q)(t) = \sum_{a \in A} \frac{\omega_a |x_k^q(t) - a_k(t)|^p}{\|x^q - a\|_p^{p-1}} \ a.e. \tag{5}$$

On the other hand, for $k \notin V(x^q)$ there exists a set $A_k(x^q)$, with $x_k^q(t) = a_k(t)$ $\forall t \in A_k(x^q)$ for some $a \in A$. Thus, if $t \in A_k(x^q)$ $x_k^{q+1}(t) = x_k^q(t)$. In this case we define for each t

$$h_k(y)(t) = \begin{cases} \sum_{a \in A} \frac{\omega_a |y_k(t) - a_k(t)|^p}{\|x^q - a\|_p^{p-1}}, & t \in A_k(x^q) \\ g_k(y)(t), & t \notin A_k(x^q) \end{cases} \tag{6}$$

Now, for all $k \notin V(x^q)$ since $x_k^{q+1}(t) = x_k^q(t)$ whenever $t \in A_k(x^q)$ we obtain

$$h_k(x_k^{q+1})(t) = h_k(x_k^q)(t) \quad t \in A_k(x^q),$$

and using (5) and (6) for any $t \notin A_k(x^q)$

$$h_k(x_k^{q+1})(t) < h_k(x_k^q)(t) \quad t \notin A_k(x^q) \tag{7}$$

Therefore, we have

$$\begin{aligned} &\int_I \left(\sum_{k \in V(x^q)} g_k(x_k^{q+1})(t) + \sum_{k \notin V(x^q)} h_k(x_k^{q+1})(t) \right) dt \\ &\quad < \int_I \left(\sum_{k \in V(x^q)} g_k(x_k^q)(t) + \sum_{k \notin V(x^q)} h_k(x_k^q)(t) \right) dt \\ &\quad = \sum_{a \in A} \sum_{k=1}^{N} \omega_a \|x^q - a\|_p^{1-p} \int_I |x_k^q(t) - a_k(t)|^p \, dt \\ &\quad = \sum_{a \in A} \omega_a \|x^q - a\|_p \\ &\quad = F(x^q) \end{aligned} \tag{8}$$

The strict inequality previously obtained is due to the strict convexity of $g_k(\cdot)$, and the fact that it reaches its minimum at x_{k+1} and (7). Now, we proceed to get the thesis of this theorem. In order to obtain this inequality we do the following,

$$\int_I \left(\sum_{k\in V(x^q)} g_k(x_k^{q+1})(t) + \sum_{k\notin V(x^q)} h_k(x_k^{q+1})(t) \right) dt$$

$$= \int_I \left(\sum_{a\in A} \sum_{k\in V(x^q)} \frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}} (x_k^{q+1}(t) - a_k(t))^2 \right) dt$$

$$+ \int_{t\notin A_k(x^q)} \left(\sum_{a\in A} \sum_{k\notin V(x^q)} \frac{\omega_a |x_k^q(t) - a_k(t)|^{p-2}}{\|x^q - a\|_p^{p-1}} (x_k^{q+1}(t) - a_k(t))^2 \right) dt$$

$$+ \int_{t\in A_k(x^q)} \left(\sum_{a\in A} \sum_{k\notin V(x^q)} \frac{\omega_a |x_k^{q+1}(t) - a_k(t)|^{p}}{\|x^q - a\|_p^{p-1}} \right) dt \tag{9}$$

If $k \notin V(x^q)$ and $t \in A_k(x^q)$ then we have that $x_k^q(t) = x_k^{q+1}(t)$.
Thus, considering the previous equation (9) we obtain;

$$\int_I \left(\sum_{k\in V(x^q)} g_k(x_k^{q+1}(t)) + \sum_{k\notin V(x^q)} h_k(x_k^{q+1}(t)) \right) dt$$

$$= \int_I \left(\sum_{a\in A} \sum_{k=1}^{N} \omega_a \|x^q - a\|_p^{1-p} |x_k^q(t) - a_k(t)|^{p-2} (x_k^{q+1}(t) - a_k(t))^2 \right) dt$$

Now, bounding the second part of this equation, we obtain the following expression

$$\int_I \sum_{a\in A} \sum_{k=1}^{N} \omega_a \|x^q - a\|_p^{1-p} |x_k^q(t) - a_k(t)|^{p-2} (x_k^{q+1}(t) - a_k(t))^2 \, dt \tag{10}$$

$$\geq \int_I \sum_{a\in A} \sum_{k=1}^{N} \omega_a \|x^q - a\|_p^{1-p}$$

$$\times \left\{ \frac{p-2}{p} |x_k^q(t) - a_k(t)|^p + \frac{2}{p} |x_k^{q+1}(t) - a_k(t)|^p \right\} dt$$

$$= \sum_{a\in A} \omega_a \left(1 - \frac{2}{p}\right) \|x^q - a\|_p$$

$$+ \frac{2}{p} \sum_{a\in A} \omega_a \|x^q - a\|_p^{1-p} \sum_{k=1}^{N} \int_I |x_k^{q+1}(t) - a_k(t)|^p \, dt \tag{11}$$

$$\geq \sum_{a \in A} \omega_a \left(1 - \frac{2}{p}\right) \|x^q - a\|_p$$

$$+ \frac{2}{p} \sum_{a \in A} \omega_a \Big[(1-p)\|x^q - a\|_p + p\|x^{q+1} - a\|_p\Big]$$

$$= -F(x^q) + 2F(x^{q+1}) \tag{12}$$

where the same inequality (see Chapter 1 in Beckenbach (1965)) has been employed in (10) and (11). Therefore, from (8) and (12), we get the following inequality

$$-F(x^q) + 2F(x^{q+1}) < F(x^q) \quad 1 \leq p \leq 2$$

Hence, we obtain

$$F(x^{q+1}) < F(x^q) \quad 1 \leq p \leq 2.$$

Thus, we have proved that our algorithm gives a descent sequence for $1 \leq p \leq 2$. □

In the previous proof we have used the inequalities (10) and (11). To use the inequality (10) it is necessary that $\frac{p-2}{2} \leq 2$, that is, $p \leq 6$. To use the inequality (11) it is necessary that $1 - p \leq 0$, that is, $p \geq 1$.

Under the same hypotheses that in Proposition 4.2 the following result holds.

Corollary 4.1. *The sequence given by* $\{F(x^q)\}_{q \in \mathbb{N}}$ where $\{x^q\}_{q \in \mathbb{N}}$ *is the sequence generated by the mapping* ψ, *is convergent for* $p \in (1, 2]$.

Proposition 4.3. *For any starting point* x_0 *and* $p \in (1, 2]$, *the sequence originated from the algorithm, contains at least a subsequence weakly convergent.*

Proof: Since the sequence verifies

$$\|x^q\|_p \leq \sum_{a \in A} \|a\|_p \quad \forall q$$

then it is bounded. Now using that every bounded set in Y_p is weakly sequentially compact, see Daniel (1971), it follows that the sequence generated by the mapping ψ contains a subsequence weakly convergent. □

5 The convergence of Weiszfeld's dynamic algorithm

In this section we study the convergence of the proposed scheme for the Dynamic Weber problem. We will show that under mild hypotheses for $p = 1$ and $p = 2$ the scheme converges to an optimal solution of Problem (1_p). Additional hypothesis over the set of demand functions are needed to prove convergence for $p \in (1, 2)$, which shows the difference with the finite-dimensional case. For $p > 2$ it may happen that the step-size is too large causing F_p to increase between iterations

5.1 CASE $p = 1$ and $p = 2$

Lemma 5.1. *The algorithm given by the scheme $x^q = \psi(x^{q-1})$ where ψ was defined in (2) is convergent for $p = 1$ and $p = 2$.*

Proof: Case $p = 1$

This is a special case because the norm in the space Y_1 is non differentiable. Therefore, the fixed point equation obtained to develop the Weiszfeld algorithm is not a necessary condition for optimality. Despite that, we consider the scheme of the Weiszfeld algorithm obtained for $p > 1$, even in the case $p = 1$. That is, the scheme is given by the following expression

$$x_k^{q+1}(t) = \sum_{a \in A} \frac{\omega_a |x_k^q(t) - a_k(t)|^{-1}}{\sum_{a \in A} \omega_a |x_k^q(t) - a_k(t)|^{-1}} a_k(t) \qquad k = 1, \ldots, N$$

This sequence coincides with the sequence generated by the Weiszfeld algorithm for the pointwise case and it is well-known, Brimberg and Love (1993), that this scheme is convergent in the pointwise case. Let $x^*(t)$ be the pointwise limit. Then, we can apply the dominated convergence theorem to the sequence $\{x^q - x^*\}_{q \geq 1}$ to deduce that x^* is also the limit of the considered sequence in the strong topology of the space Y_1.

Case $p = 2$

The sequence generated by the algorithm for $p = 2$ is given by,

$$x^{q+1}(t) = \sum_{a \in A} \frac{\frac{\omega_a}{\|x^q - a\|_2}}{\sum_{a \in A} \frac{\omega_a}{\|x^q - a\|_2}} a(t)$$

Therefore, it follows that $\{x^q\} \subset \mathit{aff}\{A\}$, where $\mathit{aff}(A)$ is the affine manifold generated by the elements in A. Since in our case the variety is generated by a finite number of elements then $\mathit{dim}(\mathit{aff}\{A\}) < \infty$.

On the other hand, by Proposition 4.3 $\{x^q\}_{q \in \mathbb{N}}$ is weakly convergent and it is included in a linear variety of finite dimension then it contains a subsequence strongly convergent.

We can assume without loss of generality that $\psi(x^k) \neq x^k$ for any k, where ψ was defined in (2). Otherwise, we would have that the whole sequence converges to x^k in a finite number of steps.

We will prove the convergence by contradiction. Let us assume that the sequence $\{x^q\}$ has two accumulation points $p_1 \neq p_2$. We can consider a ball B_1, centered at p_1 such that $p_2 \notin B_1$. In addition, we can choose a subsequence $\{x^{n_k}\}_{k \geq 1}$ verifying:

1. $x^{n_k} \longrightarrow p_1$
2. $x^{n_k+1} = \psi(x^{n_k}) \notin B_1$ for all $k \geq 1$.

Otherwise p_2 would not be an accumulation point different from p_1.

The sequence $\{\psi(x^{n_k})\}_{k\geq 1}$ has a limit because under our hypotheses ψ is a continuous function. Moreover, this limit is $\lim_{k\to\infty} \psi(x^{n_k}) = \psi(p_1)$. Since the whole sequence $\{\psi(x^{n_k})\}_{k\geq 1}$ does not belong to B_1 then $\psi(p_1)$ cannot belong to $int(B_1)$. Therefore $\psi(p_1) \notin B_1$, (in particular that means that $p_1 \notin A$).

However, we have that

$$x^{n_k} \neq \psi(x^{n_k}) \quad \text{and} \quad \psi(x^{n_k}) \neq x^{n_{k+1}}$$

Hence applying Proposition 4.1, we get

$$F(x^{n_k}) > F(\psi(x^{n_k})) = F(x^{n_k+1}) > F(x^{n_{k+1}})$$

and taking limit when k goes to infinity

$$F(p_1) \geq F(\psi(p_1)) \geq F(p_1)$$

what contradicts that $F(p_1) \neq F(\psi(p_1))$. Therefore, we have proved the thesis of this theorem. □

Theorem 5.1. *Let $\{x^q\}_{q\in\mathbb{N}}$ be a regular sequence generated by the algorithm for $p = 1$ or 2 whose limit does not coincide with a demand function a.e.. Then $\{x^q\}_{q\in\mathbb{N}}$ converges to an optimal solution of Problem (1_p).*

Proof: Since for $p = 1$ the iterates of Weiszfeld's adapted algorithm are the same that for the pointwise case (see Lemma 5.1) the thesis follows directly from the convergence results in Brimberg and Love (1993). Now for $p = 2$, again using Lemma 5.1 the sequence generated by the algorithm converges under these hypotheses. Let x^* be the limit. Then we have,

$$x^* = \lim_{q\to\infty} x^q = \lim_{q\to\infty} x^{q+1} = \psi(\lim_{q\to\infty} x^q) = \psi(x^*)$$

As we assume that $x^* \notin A$ then $x^* - \psi(x^*) = 0$ is equivalent to the fact that the Gateaux differential of F at x^* in the direction of h, for any $h \in Y_2$ equals zero. That is to say,

$$\nabla F(x^*, h) = 0 \quad \forall h \in Y_2$$

Therefore x^* is an optimal solution. □

Concerning the regularity of the optimal solutions generated by our method for $p = 2$, we want to point out that the solutions have the same properties that the demand functions because they belong to the affine manifold spanned by these demand functions.

5.2 CASE $p \in (1, 2)$

In the previous subsection has been proved the convergence of Weiszfeld's dynamic algorithm for $p = 1$ and $p = 2$. The proof for $p = 2$ is based on the fact that we can obtain a convergent subsequence from the sequence obtained by the adaptation of Weiszfeld's algorithm. However, if $p \in (1, 2)$ we do not have this result. For this reason, we impose additional conditions on the set of demands functions to obtain the convergence of the algorithm for this case.

Firstly, we consider the Sobolev space $W^{1,p}(I, \mathbb{R}^N)$. It is defined, Brezis (1983), as follows

$$W^{1,p}(I, \mathbb{R}^N) = \left\{ x \in Y_p : \exists g \in Y_p \text{ such that } \int_I x(t)\phi'(t)\,dt = -\int_I g(t)\phi(t)\,dt \ \forall \phi \in C_c^1(I) \right\}$$

where $C_c^1(I)$ is the space of functions continuously differentiable with compact support. We denote $g = x'$, because if x is differentiable and its derivative belongs to Y_p then the function g is its derivative.

Recall that $W^{1,p}$ is a Banach space with the norm defined for any $u \in W^{1,p}$ as

$$\|u\|_{1,p} = \|u\|_p + \|u'\|_p$$

In order to improve the readability of the paper we include without proof several properties that hold in these spaces that will be used to prove the strong convergence results. The proofs of these properties and further details on Sobolev spaces can be found in the book of Brezis (1983), which will be our reference for this subject.

Lemma 5.2. *The following statements hold*

i) *Let* $u, v \in W^{1,p}(I, \mathbb{R}^N)$ *then* $uv \in W^{1,p}(I, \mathbb{R}^N)$
ii) *There exists a compact embedding from* $W^{1,p}(I, \mathbb{R}^N)$ *into* $L^p(I, \mathbb{R}^N)$.
iii) $W^{1,p}(I, \mathbb{R}^N)$ *is a reflexive Banach space.*

The existence of a compact embedding is a very important fact because it implies that if a sequence converges in the weak topology of $W^{1,p}(I, \mathbb{R}^N)$ then also converges in the strong topology of Y_p. Since $W^{1,p}(I, \mathbb{R}^N)$ is a reflexive Banach space we have that every bounded sequence has a convergence subsequence in its weak topology.

Lemma 5.3. *If the set A of demand functions is included in* $W^{1,p}(I, \mathbb{R}^N)$, *the starting function* $x^o \in W^{1,p}(I, \mathbb{R}^N)$ *and* $x^{q-1} \notin S_p$ *then* $x^q = \psi(x^{q-1})$ *is included in* $W^{1,p}(I, \mathbb{R}^N)$.

Proof: Recall that ψ is defined in (2) as

$$\psi(x)(t) = \sum_{a \in A} \varphi_{a,x}(t)a(t)$$

where

$$\varphi_{a,x}(t) = \frac{\omega_a \|x - a\|^{1-p} |x(t) - a(t)|^{p-2}}{\sum_{a \in A} \omega_a \|x - a\|^{1-p} |x(t) - a(t)|^{p-2}}$$

Thus, to prove that the sequence generated by the algorithm is included in $W^{1,p}(I,\mathbb{R}^N)$ it suffices to prove that $\psi(x)$ and $\psi'(x)$ belong to Y_p for all $x \in W^{1,p}(I,\mathbb{R}^N)\backslash S_p$.

Since $0 \le \varphi_{a,x}(t) \le 1$ for all t, then $\psi(x)$ is bounded by the function $\sum_{a\in A} a$ which belongs to Y_p. Hence, $\psi(x) \in Y_p$ $\forall x \in W^{1,p}(I,\mathbb{R}^N)\backslash S_p$. Now, we have to prove that $\psi'(x) \in Y_p$ $\forall x \in W^{1,p}(I,\mathbb{R}^N)\backslash S_p$. We know that

$$\psi'(x)(t) = \sum_{a\in A} \varphi'_{a,x}(t)a(t) + \sum_{a\in A} \varphi_{a,x}(t)a'(t).$$

Since $0 \le \varphi_{a,x}(t) \le 1$ and $a \in W^{1,p}(I,\mathbb{R}^m)$ we have that $\sum_{a\in A} \varphi_{a,x}(t)a'(t) \in Y_p$. Therefore, since all the demand functions are bounded in order to prove that $\sum_{a\in A} \varphi'_{a,x}(t)a(t) \in Y_p$ we only have to prove that $\sum_{a\in A} \varphi'_{a,x}(t) \in Y_p$. To this end, we compute $\varphi'_{a,x}(\cdot)$.

In order to simplify the notation we introduce the following functions

$$h_{a,x}(t) := |x(t) - a(t)|^{p-2}$$

$$q_{a,x} := \omega_a \|x - a\|_p^{1-p}$$

$$c_x(t) := \left(\sum_{a\in A} q_{a,x}h_{a,x}(t)\right)^2$$

Therefore, we can write down $\varphi'_{a,x}$ in the following way

$$\varphi'_{a,x}(t) = \frac{q_{a,x}h'_{a,x}(t)\sum_{b\in A} q_{b,x}h_{b,x}(t) - q_{a,x}h_{a,x}(t)\sum_{b\in A} q_{b,x}h'_{b,x}(t)}{c_x(t)}.$$

Using this expression it is straightforward to see that $\sum_{a\in A} \varphi'_{a,x}(t) = 0$. From this result we deduce that $\sum_{a\in A} \varphi'_{a,x}(t) \in Y_p$. Hence, using Lemma 5.2

$$\psi(x)(t) = \sum_{a\in A} \varphi_{a,x}(t)a(t) \in W^{1,p}(I,\mathbb{R}^N) \ \forall x \in W^{1,p}(I,\mathbb{R}^N)\backslash S_p.$$

Thus, the proof is complete. □

In the following, we study the convergence of the proposed algorithm for the dynamic Weber problem. First of all, it should be noted that the sequence generated by the algorithm is bounded. Then, it contains a subsequence weakly convergent in Y_p. However, this result is not enough and we look for additional conditions which assure the strong convergence of the sequence. We will prove the global convergence of this scheme for $p \in (1,2)$ provided that the starting function belongs to the Sobolev space $W^{1,p}(I,\mathbb{R}^N)$.

Theorem 5.2. *Assume that every demand function $a \in A$ belongs to $W^{1,p}(I,\mathbb{R}^N)$, the starting point of the algorithm also belongs to $W^{1,p}(I,\mathbb{R}^N)$, the sequence generated by the algorithm and its limit is a regular function for $p \in (1,2)$. Then, this sequence strongly converges to an optimal solution of Problem (1_p) for $p \in (1,2)$.*

Proof: We have already proved that the sequence generated by the algorithm is bounded. By Lemma 5.3 it is included in the Sobolev space $W^{1,p}(I, \mathbb{R}^N)$. Thus, using Lemma 5.2, ii) we have that there exists a convergent subsequence. By Proposition 4.2 the sequence generated by the algorithm is descent. Then using the same arguments that in the case $p = 2$ we obtain that the whole sequence is convergent and its limit is the optimal solution of Problem (1_p). □

In what follows an example is included illustrating the use of Weiszfeld's algorithm. Moreover, it shows that the pointwise application of classical Weiszfeld's algorithm does not work with the dynamic Weber problem. Let us denote by $\chi_{(a,b)}(t)$ the indicator function of the interval (a, b), that is

$$\chi_{(a,b)} = \begin{cases} 1, & \text{if } t \in (a,b) \\ 0, & \text{other case} \end{cases}$$

Example. Let us consider for $N = 2$ the space $Y_2 = L^2(0,5) \times L^2(0,5)$. In this space, we consider the demand functions

$$a_1(t) = (0,0)\chi_{(0,2)}(t) + (5,4)\chi_{(2,5)}(t)$$

$$a_2(t) = (4,0)\chi_{(0,2)}(t) + (1,2)\chi_{(2,5)}(t)$$

$$a_3(t) = (2,4)\chi_{(0,2)}(t) + (7,3)\chi_{(2,5)}(t)$$

and weights

$$\omega_1 = \omega_2 = \tfrac{2}{5} \quad \omega_3 = \tfrac{1}{5}.$$

We use to solve this example the proposed Weiszfeld adapted algorithm with starting function

$$x_o(t) = (2, 0.5)\chi_{(0,2)}(t) + (4, 3.5)\chi_{(2,5)}(t).$$

After 139 iterations the optimal solution is found. Table 1 shows the iterations of the algorithm. The column *It.* gives the number of iterations; *Functions* gives the iterates and *Objective* the objective value of the problem for the corresponding iteration.

A total of 139 iterations were necessary to obtain an optimal solution. On the left-hand side of this table the 28 first iterations are shown. On the right-hand side the last 28 iterations are included. Note that for this example an optimal solution is $(0.614988, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$ and the optimal objective value is 5.29298.

On the other hand, we also solve the problem point-wisely. That is to say, we solve the problem using Weiszfeld's algorithm applied to every point in the interval $[0, 5]$. Since we are considering demand functions with only two different steps, this is equivalent to solve two different classical Weber problems. The first one having demand points $(0, 0)$, $(4, 0)$ and $(2, 4)$ and the second one $(5, 4)$, $(1, 2)$ and $(7, 3)$. Using as starting points $(2, 0)$ and $(4, 3)$ respectively, Table 2 and 3 show the iterations of these two problems.

Table 1. Iterations of Weiszfeld's adapted algorithm

It.	Functions	Objective
1	$(1.3358, 0.515323)\chi_{(0,2)}(t) + (4.17952, 3.3321)\chi_{(2,5)}(t)$	5.33666
2	$(1.22076, 0.492286)\chi_{(0,2)}(t) + (4.27153, 3.38962)\chi_{(2,5)}(t)$	5.3233
3	$(1.12804, 0.46992)\chi_{(0,2)}(t) + (4.34188, 3.43598)\chi_{(2,5)}(t)$	5.31443
4	$(1.05272, 0.448851)\chi_{(0,2)}(t) + (4.39613, 3.47364)\chi_{(2,5)}(t)$	5.30842
5	$(0.991041, 0.429736)\chi_{(0,2)}(t) + (4.4387, 3.50448)\chi_{(2,5)}(t)$	5.30428
6	$(0.940075, 0.41278)\chi_{(0,2)}(t) + (4.47271, 3.52996)\chi_{(2,5)}(t)$	5.30137
7	$(0.897583, 0.397911)\chi_{(0,2)}(t) + (4.50033, 3.55121)\chi_{(2,5)}(t)$	5.29928
8	$(0.861848, 0.384934)\chi_{(0,2)}(t) + (4.52309, 3.56908)\chi_{(2,5)}(t)$	5.29777
9	$(0.831556, 0.373619)\chi_{(0,2)}(t) + (4.54206, 3.58422)\chi_{(2,5)}(t)$	5.29665
10	$(0.805691, 0.363746)\chi_{(0,2)}(t) + (4.55806, 3.59715)\chi_{(2,5)}(t)$	5.29582
11	$(0.783464, 0.355114)\chi_{(0,2)}(t) + (4.57165, 3.60827)\chi_{(2,5)}(t)$	5.29519
12	$(0.764254, 0.347549)\chi_{(0,2)}(t) + (4.58329, 3.61787)\chi_{(2,5)}(t)$	5.29471
13	$(0.747566, 0.340902)\chi_{(0,2)}(t) + (4.59334, 3.62622)\chi_{(2,5)}(t)$	5.29434
14	$(0.733005, 0.335046)\chi_{(0,2)}(t) + (4.60204, 3.6335)\chi_{(2,5)}(t)$	5.29406
15	$(0.720247, 0.329875)\chi_{(0,2)}(t) + (4.60963, 3.63988)\chi_{(2,5)}(t)$	5.29384
16	$(0.70903, 0.325299)\chi_{(0,2)}(t) + (4.61627, 3.64549)\chi_{(2,5)}(t)$	5.29366
17	$(0.699136, 0.321239)\chi_{(0,2)}(t) + (4.6221, 3.65043)\chi_{(2,5)}(t)$	5.29352
18	$(0.690385, 0.31763)\chi_{(0,2)}(t) + (4.62724, 3.65481)\chi_{(2,5)}(t)$	5.29342
19	$(0.682626, 0.314416)\chi_{(0,2)}(t) + (4.63179, 3.65869)\chi_{(2,5)}(t)$	5.29333
20	$(0.67573, 0.31155)\chi_{(0,2)}(t) + (4.63582, 3.66214)\chi_{(2,5)}(t)$	5.29326
21	$(0.669589, 0.308989)\chi_{(0,2)}(t) + (4.6394, 3.66521)\chi_{(2,5)}(t)$	5.29321
22	$(0.664111, 0.306698)\chi_{(0,2)}(t) + (4.64259, 3.66794)\chi_{(2,5)}(t)$	5.29316
23	$(0.659217, 0.304646)\chi_{(0,2)}(t) + (4.64543, 3.67039)\chi_{(2,5)}(t)$	5.29313
24	$(0.654837, 0.302806)\chi_{(0,2)}(t) + (4.64797, 3.67258)\chi_{(2,5)}(t)$	5.2931
25	$(0.650913, 0.301155)\chi_{(0,2)}(t) + (4.65024, 3.67454)\chi_{(2,5)}(t)$	5.29308
26	$(0.647394, 0.299671)\chi_{(0,2)}(t) + (4.65228, 3.6763)\chi_{(2,5)}(t)$	5.29306
27	$(0.644234, 0.298336)\chi_{(0,2)}(t) + (4.6541, 3.67788)\chi_{(2,5)}(t)$	5.29304
28	$(0.641393, 0.297135)\chi_{(0,2)}(t) + (4.65574, 3.6793)\chi_{(2,5)}(t)$	5.29303
112	$(0.614993, 0.285898)\chi_{(0,2)}(t) + (4.6709, 3.6925)\chi_{(2,5)}(t)$	5.29298

Table 1 (continued)

It.	Functions	Objective
113	$(0.614993, 0.285898)\chi_{(0,2)}(t) + (4.67091, 3.6925)\chi_{(2,5)}(t)$	5.29298
114	$(0.614992, 0.285898)\chi_{(0,2)}(t) + (4.67091, 3.6925)\chi_{(2,5)}(t)$	5.29298
115	$(0.614992, 0.285897)\chi_{(0,2)}(t) + (4.67091, 3.6925)\chi_{(2,5)}(t)$	5.29298
116	$(0.614992, 0.285897)\chi_{(0,2)}(t) + (4.67091, 3.6925)\chi_{(2,5)}(t)$	5.29298
117	$(0.614991, 0.285897)\chi_{(0,2)}(t) + (4.67091, 3.6925)\chi_{(2,5)}(t)$	5.29298
118	$(0.614991, 0.285897)\chi_{(0,2)}(t) + (4.67091, 3.6925)\chi_{(2,5)}(t)$	5.29298
119	$(0.61499, 0.285897)\chi_{(0,2)}(t) + (4.67091, 3.6925)\chi_{(2,5)}(t)$	5.29298
120	$(0.61499, 0.285897)\chi_{(0,2)}(t) + (4.67091, 3.6925)\chi_{(2,5)}(t)$	5.29298
121	$(0.61499, 0.285897)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
122	$(0.61499, 0.285897)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
123	$(0.61499, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
124	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
125	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
126	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
127	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
128	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
129	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
130	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
131	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
132	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
133	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
134	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
135	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
136	$(0.614989, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
137	$(0.614988, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
138	$(0.614988, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298
139	$(0.614988, 0.285896)\chi_{(0,2)}(t) + (4.67091, 3.69251)\chi_{(2,5)}(t)$	5.29298

Table 2. Iterations of Weiszfeld's algorithm for the points in (0,2)

It.	Functions	Objective
1	(2., 0.503666)	2.34922
2	(2., 0.514076)	2.34919
3	(2., 0.515972)	2.34919
4	(2., 0.51632)	2.34919
5	(2., 0.516383)	2.34919
6	(2., 0.516395)	2.34919
7	(2., 0.516397)	2.34919
8	(2., 0.516398)	2.34919
9	(2., 0.516398)	2.34919
10	(2., 0.516398)	2.34919
11	(2., 0.516398)	2.34919
12	(2., 0.516398)	2.34919
13	(2., 0.516398)	2.34919
14	(2., 0.516398)	2.34919

The solution obtained after the application of this procedure is $(2, 0.516398)\chi_{(0,2)} + (5,4)\chi_{(2,5)}$ and the objective value evaluated at this function is 5.73320.

The comparison of this value with 5.29298 (the objective value of the previously obtained solution) shows that our adaptation of Weiszfeld's algorithm cannot be avoided by a pointwise application of the classical algorithm.

6 Conclusions

This paper shows an original application of classical tools of functional analysis to a well-known problem within the field of the Operations Research. The paper solves the problem of locating a moving service facility and it extends Weiszfeld's algorithm to a general formulation of the Weber Problem on L^p spaces and showing its global convergence for the cases of $p \in [1,2]$. For $p = 1$ and $p = 2$ the proofs are extensions of the proofs given in the finite dimensional case. For $p \in (1,2)$ additional hypotheses need to be assumed and a new proof is given. Although similarities can be found with the pointwise formulation this paper shows that the direct application of Weiszfeld's algorithm to the considered extension of the problem does not assure convergence to an optimal solution of the problem. This fact is due to the different topo-

Table 3. Iterations of Weiszfeld's algorithm for the points in (2,5)

It.	Functions	Objective	It.	Functions	Objective
1	(4.44796, 3.47534)	2.32396	30	(4.99952, 3.99928)	2.23613
2	(4.61555, 3.59042)	2.29595	31	(4.99961, 3.99942)	2.23612
3	(4.72927, 3.68265)	2.2776	32	(4.99969, 3.99953)	2.23611
4	(4.80511, 3.75471)	2.2656	33	(4.99975, 3.99962)	2.2361
5	(4.85635, 3.81)	2.25759	34	(4.9998, 3.9997)	2.2361
6	(4.89194, 3.85215)	2.25208	35	(4.99984, 3.99976)	2.23609
7	(4.91742, 3.88433)	2.24818	36	(4.99987, 3.9998)	2.23609
8	(4.93613, 3.90904)	2.24536	37	(4.99989, 3.99984)	2.23608
9	(4.95015, 3.92814)	2.24326	38	(4.99991, 3.99987)	2.23608
10	(4.96082, 3.94302)	2.24169	39	(4.99993, 3.9999)	2.23608
11	(4.96904, 3.95467)	2.24049	40	(4.99994, 3.99992)	2.23608
12	(4.97544, 3.96385)	2.23956	41	(4.99996, 3.99993)	2.23607
13	(4.98045, 3.9711)	2.23884	42	(4.99996, 3.99995)	2.23607
14	(4.98439, 3.97687)	2.23827	43	(4.99997, 3.99996)	2.23607
15	(4.98752, 3.98146)	2.23783	44	(4.99998, 3.99996)	2.23607
16	(4.99, 3.98512)	2.23748	45	(4.99998, 3.99997)	2.23607
17	(4.99198, 3.98804)	2.2372	46	(4.99998, 3.99998)	2.23607
18	(4.99356, 3.99039)	2.23697	47	(4.99999, 3.99998)	2.23607
19	(4.99483, 3.99227)	2.23679	48	(4.99999, 3.99999)	2.23607
20	(4.99584, 3.99378)	2.23665	49	(4.99999, 3.99999)	2.23607
21	(4.99665, 3.99499)	2.23654	50	(4.99999, 3.99999)	2.23607
22	(4.99731, 3.99597)	2.23645	51	(4.99999, 3.99999)	2.23607
23	(4.99783, 3.99675)	2.23637	52	(5., 3.99999)	2.23607
24	(4.99825, 3.99738)	2.23631	53	(5., 3.99999)	2.23607
25	(4.99859, 3.99789)	2.23626	54	(5., 4.)	2.23607
26	(4.99887, 3.9983)	2.23623	55	(5., 4.)	2.23607
27	(4.99909, 3.99863)	2.2362	56	(5., 4.)	2.23607
28	(4.99926, 3.9989)	2.23617	57	(5., 4.)	2.23607
29	(4.99941, 3.99911)	2.23615	58	(5., 4.)	2.23607

logical structure induced by the norm in the space Y_p. An easy example is solved showing this counterintuitive result.

Finally, with respect to the regularity of the solution provided by the algorithm a few words should be added. In our general formulation any function of Y_p can be considered as demand function and consequently the solution may be a general function of this space. Nevertheless, as soon as we choose more regular demand functions (for instance in $W^{1,p}(I, \mathbb{R}^N)$ which is a subspace of $\mathscr{C}(I, \mathbb{R}^N)$) the final solution has the same properties. In this sense, we provide tools for the application of our algorithm in the general case although, of course, it can be applied in simpler cases.

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