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Classical point symmetries of a porous medium equation

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Abstract. The Lie-group formalism is applied to deduce symmetries of the porous medium equation $u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x$. We study those spatial forms that admit the classical symmetry group. The reduction obtained from the optimal system of subalgebras are derived. Some new exact solutions can be obtained.

1. Introduction

The quasi-linear parabolic equation

$$u_t = [\Phi(u,x)]_{xx} + F(u,u_x,x) \tag{1}$$

serves as a simple mathematical model for various physical problems. Perhaps its most common use, at the present time, is to describe the flow of liquids in porous media, or the transport of thermal energy in plasma. In both cases the most commonly employed form for F is

$$F(u,u_x,x) = f(x)u^s u_x + g(x)u^m\,. \tag{2}$$

The first term on the right-hand side of (2) is of a convective nature. In the theory of an unsaturated porous medium, the convective part represents the effect of gravity. The second term on the right-hand side describes volumetric absorption, that in the case of plasma is caused by radiation, to which plasma is transparent.

While more often that not the spatial-dependent factors in (2) are assumed to be constant, there is no fundamental reason to assume so. Actually, allowing for their spatial dependence enables one to incorporate additional factors into the study which may play an important role. For instance, in a porous medium this may account for intrinsic factors, like medium contamination with another material, or in a plasma, this may express the impact that solid impurities arising from the walls have on the enhancement of the radiation channel.

The importance of the effect of space-dependent parts on the overall dynamics of (1) is well known. Thus, the model equation to be considered here is

$$u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x \tag{3}$$

with $n \neq 0$.

When $f(x) = 0$ and $g(x) = 0$ equation (3) becomes

$$u_t = (u^n)_{xx}\,. \tag{4}$$

A complete group classification for the nonlinear heat equation (4) was derived by Ovsiannikov [47–49] by considering the PDE as a system of PDEs, and by Bluman [8, 11]. A classification for Lie–Backlund symmetries was obtained by Bluman and Kumei [9].

The basic idea of any similarity solution is that an assumed functional form of the solution enables a PDE to be reduced to an ODE. The majority of known exact solutions of (4) turn out to be similarity solutions, even though originally they might have been derived, say by the separation of variables technique, or as travelling wave solutions. The main known exact solutions of nonlinear diffusion (4) are summarized by Hill [28]. In [28–30], Hill *et al* have deduced a number of first integrals for stretching similarity solutions of the nonlinear diffusion equation, and of general high-order nonlinear evolution equations, by two different integration procedures.

King [37] obtained approximate solutions to the porous medium equation (4), integral results for the multi-dimensional nonlinear diffusion equation [38], and determined [36] new results by generalizing known instantaneous source and dipole solutions of N-dimensional radially nonlinear diffusion equations. He also applied generalized Backlund transformations and obtained a number of equivalence transformations, that derive links between a large number of different types of nonlinear diffusion equations [39, 42]. By using local and non-local symmetries, some exact solutions, which are not similarity solutions of (4) for special values of n [40], were obtained [41].

Nonlinear diffusion with absorption arises in many areas of science and engineering. It occurs in the spatial diffusion processes where the physical structure of the medium changes with concentration. The same PDE also arises in the context of nonlinear heat conduction with a source term. For example, materials undergoing heating by microwave radiation exhibit thermal conductivities and body heating which are strongly dependent on temperature. Here, we suppose that the diffusivity and absorption term have a power-law dependence on concentration $u(x, t)$ such that the basic equation is

$$u_t = (u^n)_{xx} + g(x)u^m \tag{5}$$

where n and m are constants. For $g(x) =$ constant exact solutions and first integrals are obtained by Hill in [31], with the technique of separation of variables and the use of invariant one-parameter group transformations to reduce the governing PDE to various ODEs. For two of the equations so obtained, first integrals were deduced which subsequently give rise to a number of explicit, simple solutions. Nonlinear diffusion with absorption is characterized by phenomena such as 'blow-up', 'extinction', and 'waiting time' behaviour. The indices n and m encompasses a wide range of this physical behaviour. For example, Kalashnikov [33] has shown that $u(x, t) \equiv 0$ for all x, after a finite time, provided that $n > 1$ and $0 < m < 1$, a phenomenon referred to as 'extinction'.

A well known exact solution of (5) applying for $m = 2 - n$ is due to Kersner [35].

For $m = 1$, Gurtin and MacCamy [26] proposed a transformation that reduces (5), with $g(x) =$ constant and $m = 1$, to (4). However, in general, the background details necessary to obtain solutions of (5), with $m = 1$, via this transformation and (4) are about the same as those required to obtain the solutions directly from (5). In [23] Galaktionov presented a technique of 'separation of variables' for constructing new exact solutions of the nonlinear heat conduction equations with a source, which are reduced to equations with quadratic nonlinearities. Most of the solutions thus constructed are not invariant under point-transformation groups and Lie–Backlund groups. The proposed method was first implemented in [6] to construct an exact solution of equation (3) with $f(x) = 0$, $g(x) \equiv C > 0$ and $m = n$. In [24] a method is proposed to obtain exact blow-up solutions for nonlinear heat conduction equations with source.

Several references for the classification of Lie and Lie–Backlund symmetries for heat equations in homogeneous and non-homogeneous medium, are also listed in [32].

Equation (3) for $g(x) \equiv 0$ adopts the following form:

$$u_t = (u^n)_{xx} + f(x)u^s u_x\,. \tag{6}$$

For $s = 1$ we obtain a particular case of the generalized Hopf equation. Lie symmetries for this equation were obtained by Katkov [34]. When $n = 1$ in (6) we obtain the Burgers equation. Non-local symmetries and Lie–Backlund symmetries for this equation are well known [34, 45, 2, 32].

The generalized diffusion equation

$$T_t = (D_1(T)T_x)_x + a(D_2(T))_x + b(x,t)D_3(T)$$

where $T(x,t)$ denotes the temperature at a point, a an arbitrary constant, D_1 D_2 and D_3 are arbitrary functions of temperature T and $b(x,t)$ is another arbitrary function of x and t, which has been analysed via an isovector approach, and some new exact solutions have been obtained by Bhutani [7]. We recover some of the results obtained by him when $D_i(u)$ $i = 1, 2, 3\ldots,$ have a power-law dependence.

The one-dimensional reaction–diffusion process, governed by a system of nonlinear differential equations with arbitrary source functions,

$$a_t = D_1 a_{xx} + A(a,b,x,t) \qquad b_t = D_2 b_{xx} + B(a,b,x,t)$$

where x and t are space and time coordinates, a and b are the reaction–diffusion variables, $A(a,b,x,t)$ and $B(a,b,x,t)$ are arbitrary nonlinear functions describing the kinetics of the process, and $D_1 \neq 0$ and $D_2 \neq 0$ are diffusion constants, is studied with an isovector method. Similarity solutions and nonlinear ODEs are provided for fairly general forms of the source functions by Suhubi [56].

Classical and non-classical symmetries of the nonlinear equation (5), with $n = 1$, are considered by Clarkson and Mansfield [18] by using the method of differential Grobner bases, and by Arrigo *et al* [4] in constructing several new exact solutions.

In this paper we solve a group-classification problem for (3), by studying those spatial forms which admit the classical symmetry group. Both the symmetry group and the spatial dependence will be found through consistent application of the Lie-group formalism.

The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. The machinery of Lie-group theory provides a systematic method to search for these special group-invariant solutions. For PDEs with two independent variables, as in equation (3), a single group reduction transforms the PDE into ODEs, which are generally easier to solve than the original PDE. Most of the required theory and description of the method can be found in [10, 27, 55, 45, 49].

In general, if a differential equation admits a Lie group $\mathcal{G}_r$ and its Lie algebra $\mathcal{L}_r$ is of dimension $r > 1$, one could, in principle, consider invariant solutions based on one-, two-, etc, dimensional subalgebras of $\mathcal{L}_r$. However, there are an infinite number of subalgebras, e.g. one-dimensional subalgebras. This problem becomes manageable by recognizing that if two subalgebras are similar, i.e. they are connected to each other by a transformation from the symmetry group (with Lie algebra $\mathcal{L}_r$), then their corresponding invariant solutions are connected to each other by the same transformation. Therefore, it is sufficient to put all similar subalgebras of a given dimension, say s, into one class and select a representative from each class. The set of all these representatives, of all these classes, is called an *optimal system of order s* (Ovsiannikov [48, 49]). In order to find all invariant solutions with respect to s-dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order s. The set of invariant solutions obtained in this way is called an *optimal* *system of invariant solutions*. Of course, the form of these invariant solutions depends on the choice of the representatives.

Since equation (3) has two independent variables, we only consider one-parameter subgroups. We have already seen that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. Although in general this latter problem can still be quite complicated, for one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. The construction of the one-dimensional optimal system appears in Ovsiannikov [49] using a global matrix for the adjoint transformation. Olver [45], uses a slightly different technique, which we will follow, that is: we construct a table showing the separate adjoint actions of each element in $\mathcal{L}_r$ as it acts on all other elements, this construction is done easily by summing the Lie series [49]. We then consider a general element $\vec{v}$ in $\mathcal{L}_r$ and ask whether it can be transformed into a new element of a simpler form by subjecting it, iteratively, to various adjoint transformations. For further details and proofs see Olver [45] and Coggeshall [20].

The structure of the work is as follows. In section 2 we study the Lie symmetries of (3) for $n \neq 1$, and in section 3 for $n = 1$. In each section we consider different cases and subcases depending on $f(x)$, $g(x)$, n, m and s. For each subcase we list the functions $f(x)$ and $g(x)$ for which we obtain the Lie group of point transformations admitted by the corresponding equation, its Lie algebra as well as the corresponding optimal system. We also report the reduction obtained from the optimal system of subalgebras. In the appendix we list the commutator tables and adjoint tables corresponding to the ten different Lie algebras obtained, as well as the different choices for functions $f(x)$, $g(x)$ and constants n, m, and s, for which (3) is invariant under a Lie group of point transformations, as well as their infinitesimal generators.

2. Lie symmetries for $n \neq 1$

For $n \neq 1$, equation (3) is invariant under a Lie group of point transformations with infinitesimal generator

$$X = p(x,t,u)\frac{\partial}{\partial x} + q(x,t,u)\frac{\partial}{\partial t} + r(x,t,u)\frac{\partial}{\partial u} \tag{7}$$

if and only if

$$p = p(x,t) \qquad q = q(t) \qquad r(x,t,u) = \frac{2p_x - q_t}{n-1}u \tag{8}$$

where p, q, f and g are related by the following conditions:

$$p_t(1-n)u - n(3n+1)p_{xx}u^n + [[q_t(s+1-n) + p_x(n-1)]f + f_x p(1-n)]u^{s+1} = 0 \tag{9}$$

$$(2p_{tx} - q_{tt})u - 2np_{xxx}u^n - 2fp_{xx}u^{s+1} + [[q_t(m-n) + 2p_x(1-m)]g + g_x p(1-n)]u^m = 0\,. \tag{10}$$

Solutions of this system depend in a fundamental way on the values of n, m, s and on the functions $f(x)$ and $g(x)$ so we can distinguish the following cases depending on $f(x)$, $g(x)$, n, m and s.

2.1. *Case I:* $f(x) = 0$, $g(x) = 0$

The nonlinear diffusion equation (4)

$$u_t = (u^n)_{xx}$$

arises in many areas in science and engineering, and the majority of the references cited here contain numerous additional references to the various applications of (4). For example, equation (4) describes the motion of a thin liquid film spreading under gravity, the flow in thin-saturated regions in porous media and the percolation of gas through a porous medium. A brief account of these particular applications and the original references can be found in Lacey *et al* [43]. The evolution of the density $u = u(x,t)$ of an ideal gas flowing isentropically in a homogeneous porous medium is governed by this equation, which corresponds to the case of pure diffusion, a self-similar solution for this equation has been obtained by Aronson and Gravelau [3]. Gilding [25] seeks solutions of (4) with $n > 1$, in the form $u(x,t) = a(t)f(z)$, $z = b(t)[x + \lambda(t)]$. In [29], Hill summarizes the main known exact solutions of this equation, and proposes a new integration procedure for some of the second-order differential equations that (4) is reduced to.

In this case, we find that the most general Lie group of point transformations admitted by (4) is for *n arbitrary*, a four-parameter group $\mathcal{G}_4$, and for $n = -\frac{1}{3}$ a five-parameter group $\mathcal{G}_5$. Associated with these Lie groups are their Lie algebras $\mathcal{L}$, which can be respectively represented by the set of all the generators $\{V_i\}_{i=1}^4$ and $\{V_i\}_{i=1}^5$. These generators are:

$$\begin{aligned} &V_1 = \frac{\partial}{\partial x} \qquad V_2 = \frac{\partial}{\partial t} \qquad V_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \\ &V_4 = -t\frac{\partial}{\partial t} + \frac{u}{n-1}\frac{\partial}{\partial u} \qquad V_5 = x^2\frac{\partial}{\partial x} - 2xu\frac{\partial}{\partial u}\,. \end{aligned} \tag{11}$$

In order to construct the one-dimensional optimal system U_i, following Olver, we construct a table showing the separate adjoint actions of each element in $\mathcal{L}$ as it acts on all other elements. This construction is done easily by summing the Lie series. The commutator table, and its adjoint table appears respectively in tables A1 and A2, that are in the appendix. For $n \neq -\frac{1}{3}$, only the first four rows and columns must be considered. In table 1, we list the non-trivial optimal system $\{U_i\}$ with $i = 1, \ldots, 7$, for $n = -\frac{1}{3}$; and $i = 1, \ldots, 4$, for *n arbitrary*, as well as the corresponding similarity variables and similarity solutions. In table 2 we list the ODEs to which PDE (4) is reduced.

In particular, the second-order nonlinear differential equation for $y(z)$, obtained for $i = 1$, is shown to admit first integrals for two values of c, namely,

$$c = 2 - 2n \qquad c = 1 - n$$

and this first integral is

$$\frac{\mathrm{d}y}{\mathrm{d}z} - \frac{ky}{z} + \frac{y^{1/n}}{(c-2)^2} = c_1$$

where $k = \frac{cn+n+c-1}{(c-2)(n-1)}$, and c_1 is the constant then of integration. If this constant is zero integrating, we obtain for $c = 1 - n$, the point source solution and, for $c = 2 - 2n$, the dipole solution [28].

2.2. *Case II:* $f(x) = 0$

In this case equation (3) is the same as (5),

$$u_t = (u^n)_{xx} + g(x)u^m$$

Table 1. Each row shows the infinitesimal generators of the optimal system, the corresponding similarity variables and similarity solutions. In this table we write $d^2 = -ac$ if $ac < 0$, and $\delta = 4ac - b^2$. The constants a, b, c are arbitrary.

n	i	U_i	z_i	u_i	
arbitrary	1	$V_3 + cV_4$	$\frac{1}{t\,x^{c-2}}$	$h(z)\,x^{\frac{c}{n-1}}$	
arbitrary	2	$aV_2 + V_3 + 2V_4$	$e^{-a\,t}\,x$	$h(z)\,x^{\frac{2}{n-1}}$	
arbitrary	3	$aV_1 + V_4$	$t\,e^{a\,x}$	$h(z)\,t^{\frac{1}{1-n}}$	
arbitrary	4	$aV_1 + cV_2$	$c\,x - b\,t$	$h(z)$	
$-\frac{1}{3}$	5.1	$aV_1 + bV_4 + cV_5$	$t^{1/b}\exp\left(\frac{\arctan(\sqrt{c}\,x/\sqrt{a})}{\sqrt{a}\,\sqrt{c}}\right)$	$\frac{h\exp\left(-3\,b\,\arctan(\sqrt{c}\,x/\sqrt{a})/4\sqrt{a}\,\sqrt{c}\right)}{(c\,x^2+a)^{3/2}}$	$ac > 0$
$-\frac{1}{3}$	5.2	$aV_1 + bV_4 + cV_5$	$\frac{t^{1/b}(c\,x-d)^{1/2d}}{(c\,x+d)^{1/2d}}$	$\frac{h(c\,x+d)^{3b/8d}}{(c\,x-d)^{3b/8d}(c\,x^2+a)^{3/2}}$	$ac < 0$
$-\frac{1}{3}$	6.1	$aV_1 + bV_2 + cV_5$	$\frac{\arctan(c\,x/\sqrt{a\,c})}{\sqrt{a\,c}} - \frac{t}{b}$	$\frac{h}{(c\,x^2+a)^{3/2}}$	$ac > 0$
$-\frac{1}{3}$	6.2	$aV_1 + bV_2 + cV_5$	$\frac{1}{2d}\log\frac{cx-d}{cx+d} - \frac{t}{b}$	$\frac{h}{(c\,x^2+a)^{3/2}}$	$ac < 0$
$-\frac{1}{3}$	7.1	$aV_1 + bV_3 + cV_5$	$\frac{\exp\left(2\,\arctan\left((2\,c\,x+b)/\sqrt{\delta}\right)/\sqrt{\delta}\right)}{t^{1/2b}}$	$\frac{h\exp\left(3\,b\,\arctan\left(2\,c\,x+b/\sqrt{\delta}\right)/\sqrt{\delta}\right)}{(c\,x^2+b\,x+a)}^{3/2}$	$\delta > 0$
$-\frac{1}{3}$	7.2	$aV_1 + bV_3 + cV_5$	$\frac{(2\,c\,x-d+b)^{1/d}}{t^{1/2b}(2\,c\,x+d+b)^{1/d}}$	$\frac{h(2\,c\,x-d+b)^{3b/2d}}{(2\,c\,x+d+b)^{3b/2d}(c\,x^2+b\,x+a)^{3/2}}$	$\delta < 0$

Table 2. Each row shows the ODEs to which PDE (4) is reduced by U_i, after making $h(z) = y^{1/n}$.

n	i	$A_i(y, y', y'') = 0$
arbitrary	1	$-\frac{dy}{dz}\left(\frac{c\,n+n+c-1}{(c-2)(n-1)\,z} - \frac{y^{1/n-1}}{(c-2)^2\,n}\right) + \frac{c\,n(c\,n-n+1)\,y}{(c-2)^2(n-1)^2\,z^2} + \frac{d^2y}{dz^2} = 0$
arbitrary	2	$\frac{dy}{dz}\left(\frac{a\,y^{1/n-1}}{n\,z} + \frac{4\,n}{(n-1)\,z}\right) + \frac{2\,n(n+1)\,y}{(n-1)^2\,z^2} + \frac{d^2y}{dz^2} = 0$
arbitrary	3	$-\frac{dy}{dz}\left(\frac{y^{1/n-1}}{a^2\,n\,z} - \frac{1}{z}\right) + \frac{y^{1/n}}{a^2(n-1)\,z^2} + \frac{d^2y}{dz^2} = 0$
arbitrary	4	$c^2\,n\,y\,\frac{d^2y}{dz^2} + b\,y^{1/n}\,\frac{dy}{dz} = 0$
$-\frac{1}{3}$	5.1, 5.2	$\frac{dy}{dz}\left(\frac{3}{b\,y^4\,z^{b+1}} + \frac{b}{2\,z} + \frac{1}{z}\right) + \frac{a\,c\,y}{z^2} + \frac{b^2\,y}{16\,z^2} + \frac{d^2y}{dz^2} = 0$
$-\frac{1}{3}$	6.1, 6.2	$\frac{3\,dy/dz}{b\,y^4\,z^{b+1}} + \frac{b\,dy/dz}{2\,z} + \frac{dy/dz}{z} + \frac{a\,c\,y}{z^2} + \frac{b^2\,y}{16\,z^2} + \frac{d^2y}{dz^2} = 0$
$-\frac{1}{3}$	7.1, 7.2	$\frac{dy}{dz}\left(-\frac{3z^{2b-1}}{2by^4} - \frac{b}{z} + \frac{1}{z}\right) + \frac{acy}{z^2} + \frac{d^2y}{dz^2} = 0$

which corresponds to nonlinear diffusion with absorption.

It arises in the spatial diffusion of biological populations [44], and in a number of chemical diffusion processes where the physical structure of the medium changes with concentration. The same equation also appears in the context of nonlinear heat conduction with a source term. For example, materials undergoing heating by microwave radiation exhibit thermal conductivities and body-heating which are strongly dependent on temperature. It is known, that the addition of absorption to purely diffusive cases causes a deep qualitative change in the process of diffusion, for instance, the thermal front exhibits a quite different behaviour than for the pure diffusive case [54].

2.2.1. Case II.a $f(x) = 0$, *m arbitrary.* In this case, we find that the most general Lie group of point transformations admitted by (5) is, for *n arbitrary*, the four-parameter group

$\mathcal{G}_4$, and for $n = -\frac{1}{3}$ the five-parameter group $\mathcal{G}_5$, both of them obtained for case I. Their Lie algebras, are represented by the set of all the generators $\{V_i\}_{i=1}^4$ and $\{V_i\}_{i=1}^5$, respectively. These generators are listed in (11). The commutator table appears in table A1 and the adjoint table in table A2 in the appendix.

Here $\lambda = c(n-1)^\beta$, $\beta = \frac{k_4(m-n)}{(1-n)k_3} - 2$, $p(x) = k_5x^2 + k_3x + k_1$, $\varphi(x) = \arctan \frac{p'(x)}{\sqrt{d}}$, and $\phi(x) = \frac{2k_5x - \sqrt{-d} + k_3}{2k_5x + \sqrt{-d} + k_3}$, with $d = 4k_1k_5 - k_3^2$ and $k_6 = 2k_3 - k_4$.

Table 3. Each row shows the functions $g(x)$ for which equation (5) can be reduced to by some of the generators V_i, as well as those generators.

n	(i)	$g(x)$		V_i
arbitrary	1	$\lambda(k_3x + k_1)^\beta$		V_1, V_2, V_3, V_4
arbitrary	2	arbitrary		V_2
arbitrary	3	$c\exp\left(\frac{k_4(n-m)x}{k_1(n-1)}\right)$		V_1, V_2, V_4
arbitrary	4	c_2		V_1, V_2
$-\frac{1}{3}$	5	$p(x)^{\frac{3}{2}(m-1)}\exp\left(-\frac{(3m+1)k_6}{2\sqrt{d}}\varphi(x)\right)$	$d > 0$	V_1, V_2, V_3, V_4, V_5
$-\frac{1}{3}$	6	$p(x)^{\frac{3}{2}(m-1)}\Phi(x)^{-\frac{(3m+1)k_6}{2\sqrt{-d}}}$	$d < 0$	V_1, V_2, V_3, V_4, V_5

The one-dimensional optimal systems, for $i = 1, \ldots, 5$, are the ones obtained in case I. Hence, the similarity variables and solutions appear in table 1, and the ODEs to which (5) is reduced are of the following form:

$$A_i(y, y', y'') + B_i(y) = 0$$

where $A_i(y, y', y'')$ are listed in table 2, and $B_i(y)$ are the terms listed in the last column of table 4. In the other cases, the optimal system can be obtained from table 1 by considering the generators U_i, whose 'components' V_i appear in table 3. For instance, for $i = 4$ the optimal system is obtained by considering only V_1 and V_2.

2.2.2. Case II.b: $f(x) = 0$, $m = 1$. In this case considering $g(x) =$ constant, we find that, besides the group obtained in case II.a, for *n arbitrary*, the most general Lie group of point transformations admitted by (5) is the four-parameter group $\mathcal{G}'_4$, and for $n = -\frac{1}{3}$ is the five-parameter group G'_5. Their infinitesimal generators are:

$$\begin{aligned} &V'_1 = V_1 \qquad V'_2 = V_2 \qquad V'_3 = x\frac{\partial}{\partial x} + \frac{2}{(n-1)}u\frac{\partial}{\partial u} \\ &V'_4 = \mathrm{e}^{k(1-n)t}\frac{\partial}{\partial t} + k\mathrm{e}^{k(1-n)t}u\frac{\partial}{\partial u} \qquad V'_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}\,. \end{aligned} \tag{12}$$

The commutator table and the adjoint table appear in tables A3 and A4, respectively, from the appendix.

In table 6, we list the ODEs to which (5) is reduced, after making $h = y^{\frac{1}{1+d}}$, where $d = n - 1$.

2.2.3. Case II.c: $f(x) = 0$, $m = n$. In this case we find that, besides the group obtained for *m arbitrary* in case II.a, the most general Lie group of point transformations admitted for *g arbitrary* by (5) is, for *n arbitrary* a two-parameter group $\mathcal{G}_2^1$, and for $n = -\frac{1}{3}$, a three-parameter group $\mathcal{G}_3^1$. Associated with these Lie groups are the Lie algebras, which can be represented by the set of all the generators $\{V_i^1\}_{i=1}^2$, and $\{V_i^1\}_{i=1}^3$, respectively, where

$$V_1^1 = V_4 \qquad V_2^1 = V_2 \qquad V_3^1 = p(x)\frac{\partial}{\partial x} - \frac{3p'(x)}{2}u\frac{\partial}{\partial u} \tag{13}$$

where $p(x)$ satisfies

$$2p(x)p''(x) - (p'(x))^2 + 4g(x)p(x)^2 = k\,. \tag{14}$$

The commutator table and adjoint table are A5 and A6. In table 7 we list the optimal system $\{U_i^1\}$ $i = 1, 2, \ldots,$ obtained for $n = -\frac{1}{3}$, as well as the corresponding similarity variables and similarity solutions. If $n \neq -\frac{1}{3}$, then the only subgroups allowed by (5) are V_1^1 and V_2^1, and the optimal system comprises them. The similarity variable, is in both cases $z = x$, with the corresponding similarity variables $u = t^{\frac{1}{1-n}}h(x)$ and $u = h(x)$. In table 8 we list the ODEs to which PDE (5) is reduced.

Frequently the second-order differential equations obtained can be easily integrated and yield to exact solutions. For instance, choosing $k = 0$ in $A_2(y, y'y'') = 0$, and after

Table 4. Each row shows for each of the infinitesimal generators of the optimal system, the corresponding function $g(x)$ for which (4) can be reduced to an ODE, and the terms $B_i(y)$ of these ODEs. Here $\phi(x) = 3ck_1x + k_2$, $\varphi_1(x) = \frac{(3bs+4b)\arctan(\sqrt{c}\,x/\sqrt{a})}{4\sqrt{a}\sqrt{c}}$, and $\varphi_2(x) = -\frac{3bm\arctan\left((2cx+b)/\sqrt{(4ac-b^2)}\right)}{\sqrt{4ac-b^2}}$ and $\alpha(t) = -\sqrt{a}\,t - b\arctan\left(x/\sqrt{a}\right)/\sqrt{a}\,b$.

n	i	$g(x)$	$B_i(y)$	
arbitrary	1	$k_2x^{\frac{c(n-m)}{n-1}-2}$	$-\frac{k_2ny^m/n}{z^2}$	
arbitrary	2	$k_2nx^{\frac{2(1-m)}{n-1}}$	$k_2y^{\frac{m-n+1}{n}}/z^2$	
arbitrary	3	$-a^2k_2n\,\mathrm{e}^{\frac{(n-m)ax}{n-1}}$	$k_2z^{\frac{n+m-2}{1-n}}y^{\frac{m-n+1}{n}}$	
arbitrary	4	k_2	$k_2ny^{\frac{n+m}{n}}$	
$-\frac{1}{3}$	5.1	$\frac{(12ck_1x+3bk_1)(cx^2+a)^{3s/2}\mathrm{e}^{\varphi_1(x)}}{4}$	$\frac{k_2}{3y^{3s-1}}$	$ac > 0$
$-\frac{1}{3}$	5.2	$\frac{(cx+d)^{\frac{3bds}{8ac}+\frac{3bd}{8ac}}(4\phi(x)+3bk_1)(cx^2+a)^{3s/2+3/2}}{4(cx-d)^{\frac{3bds}{8ac}+\frac{3bd}{8ac}}}$	$\frac{k_2}{3y^{3s-1}}$	$ac < 0$
$-\frac{1}{3}$	6.1	$-(3ck_1x+k_2)\left(cx^2+a\right)^{3s/2+3/2}$	$\frac{k_2}{y^{3s+7}}$	$ac > 0$
$-\frac{1}{3}$	6.2	$-(3ck_1x+k_2)\left(cx^2+a\right)^{3s/2+3/2}$	$\frac{k_2}{y^{3s+7}}$	$ac < 0$
$-\frac{1}{3}$	7.1	$-3ck_1x\left(cx^2+bx+a\right)^{3m/2}\mathrm{e}^{\varphi_2(x)}$	$-\frac{k_2}{3y^{3s-1}}$	$4ac-b^2 > 0$
$-\frac{1}{3}$	7.2	$\frac{-(2cx+d+b)^{3b(s+1)/2d}\phi(x)(cx^2+bx+a)^{3(s+1)/2}}{(2cx-d+b)^{3b(s+1)/2d}}$	$-\frac{k_2}{3y^{3s-1}}$	$4ac-b^2 < 0$

Table 5. Each row show the infinitesimal generators of the optimal system, the corresponding similarity variables and solutions. Here $d = n - 1$.

n	U_i	z_i	u_i
arbitrary	$aV_2 + V_3' + bV_4'$	$\frac{x}{(a\,\mathrm{e}^{kdt}+b)^{1/akd}}$	$\frac{h\,\mathrm{e}^{kt}}{a^{2/akd^2}(a\,\mathrm{e}^{kdt}+b)^{(akd-2)/akd^2}}$
arbitrary	$aV_1 + bV_2 + cV_4'$	$\frac{\mathrm{e}^{x/a}}{(b\,\mathrm{e}^{dkt}+c)^{1/bdk}}$	$\frac{h\,\mathrm{e}^{kt}}{(b\,\mathrm{e}^{dkt}+c)^{1/d}}$
$-\frac{1}{3}$	$aV_1 + bV_2 + cV_4' + V_5'$	$\left(c\,\mathrm{e}^{4kt/3}+b\right)^{3/4bk}\mathrm{e}^{\alpha(t)}$	$8h\left(c\,\mathrm{e}^{4kt/3}+b\right)^{3/4}\left(\frac{a}{x^2+a}\right)^{3/2}$

Table 6. Each row shows the ODEs to which PDE (5) is reduced by U_i, $d = n - 1$.

n	U_i	$A'_i(y, y', y'') = 0$
arbitrary	U'_1	$a^{2/a\,d\,k}\, d\, \frac{\mathrm{d}y}{\mathrm{d}z}\, z + d\,(d+1)\, y^{\frac{d}{d+1}}\, \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + a^{2/a\,d\,k}\,(d+1)\,(a\,d\,k - 2)\, y = 0$
arbitrary	U'_2	$\frac{\mathrm{d}y}{\mathrm{d}z}\Big(\frac{a^2}{(d+1)\, y^{\frac{d}{d+1}}\, z} + \frac{1}{z}\Big) + \frac{a^2\, b\, k}{y^{\frac{d}{d+1}-1}\, z^2} + \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = 0$
$-\frac{1}{3}$	U'_3	$y^4\, \frac{\mathrm{d}^2 y}{\mathrm{d}z^2}\, z^2 + \big(y^4 - 48\, a^2\big)\, \frac{\mathrm{d}y}{\mathrm{d}z}\, z + a\, y^5 + 16\, a^2\, b\, k\, y = 0$

Table 7. Each row show the infinitesimal generators of the optimal system, the corresponding similarity variables and similarity solutions.

i	U_i^1	z_i	u_i
1	$V_1^1 + c V_3^1$	$t \exp\big(\frac{1}{c} \int \frac{1}{p(x)} \mathrm{d}x\big)$	$\frac{h}{p(x)^{3/2}} \exp\big(-\frac{3}{4c} \int \frac{1}{p(x)} \mathrm{d}x\big)$
2	$a V_2^1 + c V_3^1$	$\frac{a}{c} \int \frac{1}{p(x)} \mathrm{d}x - t$	$\frac{h}{p(x)^{3/2}}$

Table 8. Each row show the ODEs to which PDE (5) is reduced by U_i, $h = y^{-3}$.

U_i^1	$A_i(y, y', y'') = 0$
U_1	$\frac{3\,\mathrm{d}y/\mathrm{d}z}{2\,z} + \frac{3\,c^2\,\mathrm{d}y/\mathrm{d}z}{y^4\, z^2} + \frac{y}{16\, z^2} + \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = 0$
U_2	$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{3\,c^2\,\mathrm{d}y/\mathrm{d}z}{a^2\, y^4} + \frac{3\,c^2\,k\,y}{4\,a^2} = 0$

integrating once we obtain,

$$\frac{\mathrm{d}y}{\mathrm{d}z} - \frac{3c^2}{a^2 y^3} - c_1 = 0\,.$$

By choosing the constant of integration, $c_1 = 0$, an exact solution is

$$u = \frac{a^{3/2}}{2\sqrt{2}\, c^{3/4}\, p(x)^{3/2} \big(c\,t - c\,c_2 - a\, \int \frac{1}{p(x)} \mathrm{d}x\big)^{3/4}}\,.$$

It is of interest to observe that although in the present case for $g =$ constant equation (5) does not admit the stretching-type similarity solution, i.e. a solution arising from invariance under the stretching group [46], although it does admit that type of solution for $g(x) = c/x^2$.

2.3. Case III: $g(x) = 0$.

Equation (3) becomes (6),

$$u_t = (u^n)_{xx} + f(x) u^s u_x$$

which corresponds to nonlinear diffusion with convection. When $f(x) =$ constant, we arrive at the Boussinesq equation of hydrology, which is involved in various fields of petroleum technology and ground water hydrology. Several exact solutions of this equation have been obtained by using an isovector method [7].

It is a well known fact that non-negative solutions, u, of (6) may give rise to interfaces (or free boundaries) separating regions where $u > 0$ from those where $u = 0$. These fronts are relevant in the physical problems modelled and their occurrence is essentially due to slow diffusion ($n > 1$) or to convective phenomenon dominating over diffusion ($s < n - 1$). In this case, if $s \leqslant 0$ and $s < n - 1$, there is a great contrast with pure diffusion phenomenon [1].

2.3.1. Case III.a: s arbitrary. In this case, we find that the most general Lie group of point transformations admitted by (6) is the four-parameter group $\mathcal{G}_4$ obtained for case I, as well as for case II.a. Its Lie algebra, can be represented by the set of all generators $\{V_i\}_{i=1}^4$, these generators are listed in (11). The commutator table and the adjoint table appear in the four first rows and columns of tables A1 and A2 from the appendix. In the following table we list the functions $f(x)$ for which equation (6) can be reduced by some of the generators V_i as well as these generators.

Table 9. Each row shows the functions $f(x)$ for which equation (6) can be reduced to by some of the generators V_i, as well as these generators.Where $\alpha = \frac{k_4(n-s-1)}{k_3(n-1)} - 1$ and $\gamma = c(n-1)^\alpha$.

n	(i)	f	V_i
arbitrary	1	$\gamma(k_3x + k_1)^\alpha$	V_1, V_2, V_3, V_4
arbitrary	2	arbitrary	V_2
arbitrary	3	$c_1 \exp\left(\frac{k_4(n-s-1)x}{k_1(n-1)}\right)$	V_1, V_2, V_4
arbitrary	4	c_1	V_1, V_2
$2s+1$	5	c_1	V_1, V_2, V_3

The one-dimensional optimal system was obtained in case I, the similarity variables and solutions are listed in table 1, and the ODEs to which (6) is reduced are of the following form:

$$A_i(y, y', y'') + C_i(y, y') = 0$$

where $A_i(y, y', y'')$ appear in table 2, and $C_i(y, y')$ are the terms listed in the last column of table 10.

Table 10. Each row shows the infinitesimal generators of the optimal system, the corresponding function $f(x)$ for which (6) can be reduced to an ODE, and the $C_i(y, y')$ terms of those ODEs.

i	U_i	$f(x)$	$C_i(y, y')$
1	$V_3 + cV_4$	$-k_1 x^{\frac{c(n-s-1)}{n-1}-1}$	$-\frac{k_1 y^{\frac{s-n+1}{n}} y'}{z} + \frac{k_1 c n y^{\frac{s+1}{n}}}{((c-2)n-c+2)z^2}$
2	$aV_2 + V_3 + 2V_4$	$k_1 n x^{\frac{2s-n+1}{1-n}}$	$\frac{k_1 y^{\frac{s-n+1}{n}}}{z} y'$
3	$aV_1 + V_4$	$-a\, k_1\, n \exp\left(-\frac{(a s - a n + a)x}{n-1}\right)$	$\frac{\mathrm{k1}\, y^{\frac{s-n+1}{n}}}{z^{\frac{s}{n-1}}} y'$
4	$aV_1 + cV_2$	k_1	$k_1 y^{\frac{s+1}{n}} y'$

If $u_0(x) = \delta(x)$, where $\delta(x)$ is the Dirac measure, we have a similarity solution of the form $u(x,t) = t^{-N} H(z)$, where the similarity variable is $z = xt^{-N}$, $N^{-1} = n + 1$, and the corresponding ODE is

$$n(n+1)(H^n)_{zz} + (c_1 z^{s-n} H^{s+1-n} + z)(H^n)_z + nH = 0\,.$$

If we choose $f =$ constant, integrating once we obtain

$$(n+1)(H^n)_z + H(cH^n + z) = 0$$

Table 11. Each row show the functions f(x) for which equation (5) can be reduced by some of the generators V_i, as well as these generators.

n	V_i^1	f
arbitrary	V_1^1, V_2^1	arbitrary
arbitrary	V_1^1, V_2^1, V_3^1	$\frac{c}{p(x)} - \frac{n(3n+1)p'(x)}{(n-1)p(x)}$

Table 12. Each row show the ODEs to which (6) can be reduced by U_i^1, $h = 1/y^3$.

U_i^1	$A_i(y)$
U_1^1	$\frac{d^2y}{dz^2} + \left(\frac{3}{2z} - \frac{3c^2}{z} + \frac{3c^2}{z^2y^4}\right)\frac{dy}{dz} + \left(\frac{9c^2k}{8} - \frac{3c^2}{4} + \frac{1}{16}\right)\frac{y}{z^2} = 0$
U_2^1	$\frac{d^2y}{dz^2} - \left(\frac{3c^2}{a^2y^4} + \frac{3c^2}{a}\right)\frac{dy}{dz} + \frac{c^2y^4}{a} + \frac{c^2}{a^2} = 0$

recovering a result obtained by Rosenau [53].

2.3.2. Case III.b: $s = n - 1$. In this case we find that besides the group obtained for *s arbitrary*, the most general Lie group of point transformations admitted by (6), is the three-parameter group $\mathcal{G}_3^1$ obtained for case II.c, when $n = -\frac{1}{3}$. Its infinitesimal generators $\{V_i\}_{i=1}^3$, are listed in (13), its commutator table and adjoint table are, respectively, in tables A5 and A6. In table 11, we show the functions $f(x)$ for which equation (6) can be reduced by some of the generators V_i.

Where in table 11 $p(x)$ satisfies

$$n(1-n)p(x)p''(x) + 2n^2(p'(x))^2 + (1-n)6cp'(x) = K\,. \tag{15}$$

For $n = -\frac{1}{3}$, we obtain the one-dimensional optimal system $\{U_i^1\}$. $i = 1, 2$. Hence, the similarity variables and solutions are given in table 7, and the ODEs to which (6) is reduced, after making $h = 1/y^3$, are listed in table 12.

2.3.3. Case III.c: $s = 0$, $n = -\frac{1}{3}$. In this case, besides the group obtained for *s arbitrary* we must distinguish between:

(i) For $f(x) =$ constant, the most general Lie group of point transformations obtained is $\mathcal{G}_5''$, its infinitesimal generators are $\{V_i''\}_{i=1}^5$.

$$\begin{aligned} &V_1'' = V_1 \qquad V_2'' = V_2 \qquad V_3'' = (x+kt)^2\frac{\partial}{\partial x} - 3(x+kt)u\frac{\partial}{\partial u} \\ &V_4'' = (x+kt)\frac{\partial}{\partial x} - \frac{3u}{2}\frac{\partial}{\partial u} \qquad V_5'' = -kt\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{3u}{4}\frac{\partial}{\partial u} \end{aligned} \tag{16}$$

and its commutator and adjoint table appear in tables A7 and A8, respectively.

(ii) For $f(x) = k_1 x + k_2$, the group obtained is $\mathcal{G}_5'''$. Its infinitesimal generators are:

$$
\begin{aligned}
V_1''' &= \mathrm{e}^{k_1 t}(x + \frac{k_2}{k_1})^2 \frac{\partial}{\partial x} - 3\mathrm{e}^{k_1 t}(x + \frac{k_2}{k_1}) u \frac{\partial}{\partial u} \\
V_2''' &= -\mathrm{e}^{-2k_1 t}(k_1 x + k_2) \frac{\partial}{\partial x} + \mathrm{e}^{-2k_1 t} \frac{\partial}{\partial t} \\
V_3''' &= -(k_1 x + k_2) \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{3k_1}{k_2} u \frac{\partial}{\partial u} \\
V_4''' &= (x + \frac{k_2}{k_1}) \frac{\partial}{\partial x} - \frac{3}{2} u \frac{\partial}{\partial u} \\
V_5''' &= \mathrm{e}^{-k_1 t} \frac{\partial}{\partial x}
\end{aligned}
\qquad (17)
$$

and its commutator and adjoint table appear in tables A9 and A10.

In table 13, we list the one-dimensional optimal system $\{U_i''\}$, $i = 1, 2, 3$, obtained for $f(x) =$ constant, the optimal system U_i''' $i = 1, 2, \ldots,$ obtained for $f(x) = k_1 x + k_2$, as well as their similarity variables and similarity solutions.

Table 13. Each row show the infinitesimal generators U_i'' and U_i''' of the optimal systems, as well as their similarity variables and similarity solutions, $d = \sqrt{a\,b\,k}$.

i	U_i	z_i	u_i
1	$aV_2 + bV_3'' + cV_5''$	$\frac{1}{d}\arctan\left(\frac{bx}{d} + \frac{dt}{a}\right) - \frac{1}{c}\log(ct + a)$	$h\,(c\,t + a)^{3/4}\cos^3 \frac{d(c\,z + \log(c\,t + a))}{c}$
2	$aV_2'' + V_4'' + cV_5''$	$(c\,t + a)^{1/c}\,z - k\,t - a\,k$	$h\,(c\,t + a)^{\frac{3c-6}{4c}}$
3	$aV_1'' + V_3'' + cV_5''$	$\sqrt{a}\tan\left(\frac{\sqrt{a}\,z}{c} + \frac{\sqrt{a}\log t}{c}\right) - k\,t$	$h\,t^{3/4}\cos^3 \frac{\sqrt{a}(z + \log t)}{c}$
1	$V_3''' + cV_4'''$	$\frac{\mathrm{e}^{-t}(k_1 x + k_2)^{1/c}}{k_1^{1/c}}$	$h\,\mathrm{e}^{-3ct/2}$
2	$aV_1''' + V_3''' + cV_5'''$	$-k_1 t + \frac{2k_1}{k}\arctan\left(\frac{2a\mathrm{e}^{k_1 t}(k_1 x + k_2)}{kk_1} + \frac{k_1}{k}\right)$	$\frac{h\exp(3k_1 t/2)}{\sec^3((k\,z + k\,k_1\,t)/2\,k_1)}$

Table 14. Each row show the ODEs to which (6) can be reduced respectively by U_i'', or U_i''', $h = 1/y^3$.

U_i'', U_i'''	$A_i''(y) = 0$	$f(x)$
U_1''	$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{3\,d^4\,\mathrm{d}y/\mathrm{d}z}{b^2\,y^4} + d^2\,y - \frac{3\,c\,d^4}{4\,b^2\,y^3} = 0$	k
U_2''	$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{3z}{y^4}\frac{\mathrm{d}y}{\mathrm{d}z} - \frac{3c}{4y^3} + \frac{3}{2y^3} = 0$	k
U_3''	$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{3\,a^2\,\mathrm{d}y/\mathrm{d}z}{c^2\,y^4} + \frac{a\,y}{c^2} - \frac{3\,a^2}{4\,c^2\,y^3} = 0$	k
U_1'''	$\frac{\mathrm{d}y}{\mathrm{d}z}\left(-\frac{3\,c(k_1 + c)\,z^{2c-1}}{y^4} - \frac{c-1}{z}\right) + \frac{3\,c^3\,z^{2c-2}}{2\,y^3} + \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = 0$	$k_1 x + k_2$
U_2'''	$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{3\,k^4\,\mathrm{d}y/\mathrm{d}z}{16\,a^2\,k_1\,y^4} + \frac{k^2\,y}{4\,k_1^2} + \frac{k^4\,k_3}{16\,a^2\,k_1^2\,y^3} - \frac{3\,k^4}{32\,a^2\,k_1\,y^3} = 0$	$k_1 x + k_2$

In table 14 we list the ODEs to which (6) is reduced by $\{U_i''\}$, after making $h = 1/y^3$.

2.4. Case IV: $f(x) \neq 0$ and $g(x) \neq 0$.

$$u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x\,.$$

Table 15. Each row shows the functions $f(x)$ and $g(x)$ for which equation (3) can be reduced by some of the generators V_i, as well as those generators. Here $\alpha = \frac{k_4(n-s-1)}{k_3(n-1)} - 1$, $\gamma = c(n-1)^\alpha$, $\lambda = c(n-1)^\beta$, $\beta = \frac{k_4(m-n)}{(1-n)k_3} - 2$. $\varphi(x) = \arctan\frac{2k_5x+k_3}{\sqrt{d}}$, $\phi(x) = \frac{2k_5x-\sqrt{-d}+k_3}{2k_5x+\sqrt{-d}+k_3}$, $\phi_1(x) = c(3k_5x+1)$ with $d = 4k_5k_1 - k_3^2$, $k_6 = 2k_3 - k_4$, and $p(x) = k_5x^2 + k_3x + k_1$.

n	m	(i)	f	g		V_i
arbitrary	arbitrary	1	$\gamma(k_3x+k_1)^\alpha$	$\lambda(k_3x+k_1)^\beta$		V_1, V_2, V_3, V_4, V_5
arbitrary	arbitrary	2	arbitrary	arbitrary		V_2
arbitrary	arbitrary	3	$c\exp\left(\frac{k_4(n-s-1)x}{k_1(n-1)}\right)$	$c\exp\left(\frac{k_4(n-m)x}{k_1(n-1)}\right)$		V_1, V_2, V_4
arbitrary	arbitrary	4	c_1	c_2		V_1, V_2
$-\frac{1}{3}$	$s+1$	5	$cp(x)^{\frac{3s+2}{2}}\exp\left(-\frac{k_6(3s+4)\varphi(x)}{2\sqrt{d}}\right)$	$\phi_1(x)p(x)^{3s/2}\exp\left(-\frac{k_6(3s+4)\varphi}{2\sqrt{d}}\right)$	$d>0$	V_1, V_2, V_3, V_4, V_5
$-\frac{1}{3}$	$s+1$	6	$cp(x)^{\frac{3s+2}{2}}\phi(x)^{-\frac{k_6(3s+4)}{4\sqrt{-d}}}$	$\phi_1(x)p(x)^{3s/2}\phi^{-\frac{k_6(3s+4)}{4\sqrt{-d}}}$	$d<0$	V_1, V_2, V_3, V_4, V_5

This equation corresponds to porous media with sources, or thermal evolution with sources and convection. This equation exhibits a wide variety of wave phenomena, some of which were studied, for $f(x) = \text{constant}$ and $g(x) = \text{constant}$, by Rosenau and Kamin [53].

2.4.1. Case IV.a: s arbitrary. In this case, we find that the most general Lie group of point transformations admitted by (3) is

(i) For *m, n arbitrary* the four-parameter group $\mathcal{G}_4$ obtained for case I, as well as for case II.a. and case III.a.
(ii) For $m = s+1$, $n = -\frac{1}{3}$, the five-parameter group $\mathcal{G}_5$, obtained for case I and case II.b.

Associated with these Lie groups are their Lie algebras, which can be represented by the set of all the generators $\{V_i\}_{i=1}^4$, and $\{V_i\}_{i=1}^5$, respectively. These generators are listed in (11). In order to construct the one-dimensional optimal system, we need the commutator table and the adjoint table that appear in tables A1 and A2. In table 15, we list the functions $f(x)$ and $g(x)$, for which equation (3) can be reduced to ODEs by some generators, as well as these generators V_i.

The one-dimensional optimal system for case IV.a, is the one obtained in case I, hence, the similarity variables and similarity solutions are listed in table 1. The ODEs to which (3) is reduced are of the following form:

$$A_i(y, y', y'') + B_i(y) + C_i(y, y') = 0$$

where $A_i(y, y', y'')$, $B_i(y)$ and $C_i(y, y')$ are listed in tables 2, 4 and 10, respectively.

For $f = c_1$ and $g = c_2$ we find a travelling-waves solution $u = H(z)$ where $z = x - ct$ and H satisfies

$$nH'' + \frac{n(n-1)}{H}H'^2 + k_3H^{s-n+1}H' + cH^{1-n}H' + k_2H^{m-n+1} = 0\,.$$

If $u_0(x) = \delta(x)$, where $\delta(x)$ is the Dirac measure, we have a similarity solution of the form $u(x,t) = t^{-N}H(z)$, where the similarity variable is $z = xt^{-N}$, $N^{-1} = n+1$, and the corresponding ODE is

$$n(n+1)(H^n)_{zz} + (c_1z^{s-n}H^{s+1-n} + z)(H^n)_z + nH + c_2z^{m-n-2}H^m = 0\,.$$

2.4.2. Case IV.b: $s = 0$, $n = -\frac{1}{3}$. In this case besides the group obtained for *s arbitrary*, we find that the most general Lie group of point transformations admitted by (3) is:

Table 16. Each row show the functions $f(x)$ and $g(x)$ for which equation (3) can be reduced by some of the generators V_i'', or V_i''' as well as those generators.

m	$f(x)$	$g(x)$	i	U_i'', U_i'''
arbitrary	k	$c(k_4x+k_1)^{\frac{1}{3}(3m-7)}$	1	V_1'', V_2'', $V_4''+V_5''$
arbitrary	k_1x+k_2	$c(k_1x+k_2)^{3(m-1)}$	2	V_1''', $k_1V_4'''+V_3'''$
arbitrary	k_1x+k_2	$c2^{3(m-1)/2}(k_1x+k_2)^{3(m-1)/2}$	3	V_3''', V_4'''
arbitrary	k_1x+k_2	$c(k_1x+k_2)^{-2}$	4	V_2''', $k_1V_4'''+V_3'''$
$-\frac{1}{3}$	k_1x+k_2	$c(k_1x+k_2)^{-2}$	5	V_2''', V_3''', V_4'''
$\frac{1}{3}$	k_1x+k_2	$c(k_1x+k_2)^{-2}$	6	V_1''', V_2''', $k_1V_4'''+V_3'''$
arbitrary	k_1x+k_2	c	7	V_5''', $k_1V_4'''+V_3'''$
1	k_1x+k_2	c	8	V_1''', V_3''', V_4, V_5'''

Table 17. Each row shows the functions $f(x)$ and $g(x)$ for which equation (3) can be reduced by some of the generators V_i'', or V_i''' as well as these generators.

n	m	V_i^1	$f(x)$	$g(x)$
arbitrary	arbitrary	V_1^1, V_2^1	arbitrary	arbitrary
arbitrary	arbitrary	V_1^1, V_2^1, V_3^1	$\frac{c_1}{p(x)}-\frac{n(3n+1)p'(x)}{(n-1)p(x)}$	$c_2\,p(x)^{\frac{2(m-1)}{1-n}}\exp\left(\frac{k_1(n-m)\int\frac{dx}{p(x)}}{1-n}\right)$
arbitrary	n	V_1^1, V_2^2, V_3^3	$\frac{c_1}{p(x)}-\frac{n(3n+1)p'(x)}{(n-1)p(x)}$	$\phi(x)$

(i) For $f(x)=$ constant, the five-parameter group $\mathcal{G}_5''$.
(ii) For $f(x)=k_1x+k_2$, the five-parameter group $\mathcal{G}_5'''$.

Both groups were obtained for case III.c, their Lie algebras, can be represented by the set of generators $\{V_i''\}_{i=1}^5$, and $\{V_i'''\}_{i=1}^5$, respectively. These generators are listed in (16) and (17), their commutator and adjoint tables appear in A7–A10. In table 16, we list the different functions $f(x)$ and $g(x)$, for which equation (3) can be reduced to ODEs, and the corresponding generators $\{V_i''\}$ and $\{V_i'''\}$.

2.4.3. Case IV.c: $s=n-1$. In this case, we find that the most general Lie group of point transformations admitted by (3) is the three-parameter group $\mathcal{G}_3^1$ obtained for case II.c, as well as case III.b when $n=-\frac{1}{3}$. Its infinitesimal generators $\{V_i^1\}_{i=1}^3$ appear in (13). Its commutator table and the adjoint table are shown in tables A5 and A6, respectively. In table 17 we list the different choices for $f(x)$ and $g(x)$, and the generators (V_i^1) allowed by equation (3) for these choices.

Where in table 17, $p(x)$ satisfies (15) and $\phi(x)$ can be obtained from

$$-2np'''(x)-2f(x)p''(x)+(2p'(x)(1-m)+k_1(m-n))\phi(x)+p(1-n)\phi'(x)=0$$

In table 18 we write the ODEs to which (3) is reduced, here $h=\frac{1}{y^3}$.

3. Lie symmetries for $n=1$

In this case equation (3) becomes

$$u_t=u_{xx}+g(x)u^m+f(x)u^su_x\,. \tag{18}$$

Table 18. Each row shows the ODEs to which PDE (3) is reduced by U_i^1, here $h(z) = y^{-3}$.

U_i^1	$A_i(y, y', y'')$
U_1^1	$\frac{dy}{dz}\left(-\frac{3c^2}{z}+\frac{3}{2z}+\frac{3c^2}{y^4z^2}\right)+y\left(\frac{c^2c_1}{z^2}+\frac{9c^2k}{8z^2}-\frac{3c^2}{4z^2}+\frac{1}{16z^2}\right)+\frac{d^2y}{dz^2}=0$
U_2^1	$\frac{d^2y}{dz^2}+\left(-\frac{3c^2}{a^2y^4}-\frac{3c^2}{a}\right)\frac{dy}{dz}+\left(\frac{c^2c_1}{a^2}+\frac{9c^2k}{8a^2}\right)y=0$

Table 19. Each row shows the functions $f(x)$ and $g(x)$for which equation (18) can be reduced by some of the generators V_i^2, as well as those generators. Here $\alpha = -\left(\frac{k_4s}{k3}+1\right)$, and $\beta = -\frac{k_4}{k_3}(m-1)-2$.

(i)	$f(x)$	$g(x)$	V_i^2
1	$c_1(k_3x+k_1)^\alpha$	$c_2(k_3x+k_1)^\beta$	$V_1^2, V_2^2, V_3^2, V_4^2$
2	arbitrary	arbitrary	V_2^2
3	$\exp\left(-\frac{k_4s}{k_1}\right)$	$\exp\left(-\frac{k_4(m-1)}{k_1}\right)$	V_1^2, V_2^2, V_4^2

Equation (18) is invariant under a Lie group of point transformations with infinitesimal generator (7)

$$X = p(x,t,u)\frac{\partial}{\partial x}+q(x,t,u)\frac{\partial}{\partial t}+r(x,t,u)\frac{\partial}{\partial u}$$

if and only if

$$p=\frac{q_tx}{2}+p_1(t) \qquad q=q(t) \qquad r=r_1(x,t)u+r_2(x,t) \tag{19}$$

where p, q, r and f are related by the following conditions:

$$\left[(f'x+f)\frac{q_t}{2}+fr_1s+f'p_1\right]u^s+fr_2su^{s-1}+\frac{q_{tt}x}{2}+2r_{1x}+p_{1t}=0 \tag{20}$$

$$\left(\left(\frac{g'x}{2}\right)q_t+g(m-1)r_1+g'p_1\right)u^m+fr_{1x}u^{s+1}+gmr_2u^{m-1}+fr_{2x}u^s+(r_{1xx}-r_{1t})u$$
$$+(r_{2xx}-r_{2t})=0\,. \tag{21}$$

We can then consider the following case.

3.1. *Case V: $s \neq 0$ and $s \neq 1$*

In this case, we find that the most general Lie group of point transformations admitted by (18) is a four-parameter group $\mathcal{G}_4^2$. Associated with this Lie group is the Lie algebra, which can be represented by the set of all the generators $\{V_i^2\}_{i=1}^4$

$$V_1^2=V_1 \qquad V_2^2=V_2 \qquad V_3^2=V_3 \qquad V_4^2=u\frac{\partial}{\partial u}\,. \tag{22}$$

The commutator table and the adjoint table appear in tables A11 and A12, respectively. In table 19, we list the different choices for $f(x)$ and $g(x)$ and the generators $\{V_i^2\}$ allowed, for these choices by equation (18).

In table 20 we list the one-dimensional optimal system $\{U_i^2\}$ $i=1,2,\ldots,$ as well as the corresponding similarity variables and similarity solutions.

Table 20. Each row shows the infinitesimal generators of the optimal system, the corresponding functions $f(x)$ and $g(x)$ for which (18) can be reduced to an ODE, as well as the corresponding similarity variables and solutions.

i	U_i^2	z_i	u_i	$f(x)$	$g(x)$
1	$V_3^2 + cV_4^2$	$\frac{x^2}{t}$	$x^c h(z)$	$\frac{c_1}{2\,x^{cs+1}}$	$-\frac{c_2}{x^{cm-c+2}}$
2	$aV_1^2 + V_2^2 + cV_4^2$	$x - at$	$h(z)$	c_1	c_2

Table 21. Each row show the ODEs to which PDE (18) is reduced by U_i^2.

U_i^2	$A_i(z) = 0$
U_1^2	$4z^2\frac{d^2h}{dz^2} + (z + c_1h^s + 4c + 2)z\frac{dh}{dz} - c_2h^m - \frac{cc_1h^{s+1}}{2} + c(c-1)h = 0$
U_2^2	$\frac{d^2h}{dz^2} + (c_1h^s + a)\frac{dh}{dz} + c_2h^m = 0$

3.2. Case VI.: $s = 1$

In this case we find that.

(i) For $f(x) = \frac{b}{x+2a}$, and $g(x) = k - \frac{2}{(x+2a)^2}$, we obtain the two-parameter group $\mathcal{G}_2^3$. Its infinitesimal generators are $\{V_i^3\}_{i=1}^2$,

$$V_1^3 = k_1\frac{e^{k_1t}(x+2a)}{2}\frac{\partial}{\partial x} + e^{k_1t}\frac{\partial}{\partial t} - e^{k_1t}\frac{k_1^2(x+2a)^2}{2b}\frac{\partial}{\partial u} \qquad V_2^3 = V_2\,. \tag{23}$$

(ii) For $f(x) = c(x+2a)^{(2b+1)}$, and $g(x) = \frac{d}{(x+2a)^2}$, we obtain the group $\mathcal{G}_2^4$. Its infinitesimal generators are $\{V_i^4\}_{i=1}^2$,

$$V_1^4 = (x+2a)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - (b+1)u\frac{\partial}{\partial u} \qquad V_2^4 = V_4\,. \tag{24}$$

(iii) For $f(x) = c$, and $g(x) = 0$, the five-parameter group G_5^7. Its infinitesimal generators are $\{V_i^7\}_{i=1}^5$.

$$\begin{aligned} &V_1^7 = V_1 \qquad V_2^7 = V_2 \qquad V_3^7 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (tu + x)\frac{\partial}{\partial u} \\ &V_4^7 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u} \qquad V_5^7 = t\frac{\partial}{\partial x} - \frac{u}{c}\frac{\partial}{\partial u}\,. \end{aligned} \tag{25}$$

Their respective commutator and adjoint tables are in the appendix A13–A16. In table 22 we list the one-dimensional optimal systems $U_i^3U_i^4$, with $i = 1, 2, \ldots,$ and U_i^7 $i = 1, \ldots, 7$, as well as the corresponding similarity variables and solutions.

The ODEs to which (18) is reduced appear in table 23.

If $c = -1$ when $f(x) =$ constant then we get the classical Burgers equation. Lie symmetries of this equation are known, non-classical symmetries for this equation have been obtained by Pucci [50] and Arrigo *et al* [5].

3.3. Case VII.: $s = 0$

Equation (3) adopts the parabolic normal form

$$u_t = u_{xx} + g(x)u + f(x)u_x \tag{26}$$

Table 22. Each row show the infinitesimal generators of the optimal system, the corresponding functions $f(x)$ and $g(x)$ for which (18) can be reduced to an ODE, as well as the corresponding similarity variables and solutions.

i	U_i^3	z_i	u_i	$f(x)$	$g(x)$
1	$aV_1^3 + V_2^3$	$\frac{e^{k_1 t}}{(x+2a)^2}$	$h(z) - \frac{k_1(x+2a)^2}{2b}$	$\frac{b}{x+2a}$	$k - \frac{2}{(x+2a)^2}$
2	V_2^3	x	$h(x)$	$\frac{b}{x+2a}$	$k - \frac{2}{(x+2a)^2}$
1	V_1^4	$\frac{(x+2a)^2}{t}$	$\frac{h}{t^{b+1}}$	$c(x+2a)^{(2b+1)}$	$\frac{d}{(x+2a)^2}$
2	V_2^4	x	$h(x)$	$c(x+2a)^{(2b+1)}$	$\frac{d}{(x+2a)^2}$

Table 23. Each row show the ODEs to which PDE (18) is reduced by U_i^3 or U_i^4.

U_i^3	$B_i(y) = 0$
U_1^3	$2z^2 h_{zz} + (3 - bh)zh_z - h = 0$
U_2^3	$h_{zz} + gh + fhh_z = 0$
U_1^4	$4z^2 H_{zz} + (2cz^{b+2}H + z^2 + 2z)H_z + (b+1)zH + dH = 0$
U_2^4	$h_{zz} + gh + fhh_z = 0$

which is a Fokker–Planck equation. In this case

$$r_1 = -\frac{q_{tt}x^2}{8} - \frac{p_{1t}x}{2} + r_3(t) - f\left(\frac{q_t x}{4} + \frac{p_1}{2}\right) \tag{27}$$

if $r_2(x,t) = 0$, and the following conditions must be satisfied:

$$-\frac{q'''(t)}{8}x^2 - \frac{p_1''(t)}{2}x + r_3' + \frac{q''(t)}{4} + (E+g)'(x)\left(\frac{q'(t)}{2}x + p_1(t)\right) + (E+g)(x)q'(t) = 0 \tag{28}$$

where

$$E(x) = \frac{f'}{2} + \frac{f^2}{4}\,. \tag{29}$$

If we differentiate (28) three times with respect to x, as E is a function only of x, we deduce that it must be

$$(E+g)'''(x) = \frac{c}{(x+\lambda)^5}$$

integrating three times with respect to x, we then get that $E(x)$ adopts the following form:

$$(E+g)(x) = c_1x^2 + c_2x + c_3 + \frac{c_4}{(x+\lambda)^2} \tag{30}$$

where c_1, c_2, c_3, c_4 and λ are constants. Substituting (30) into (28) we obtain that the following conditions must be satisfied:

$$\begin{aligned} &q'''(t) - 16c_1q'(t) = 0 \qquad && p_1'''(t) - (3c_2q'(t) + 4c_1p_1(t)) = 0 \\ &r_3'(t) + \frac{q''(t)}{4} + c_3q'(t) + c_2p_1(t) = 0 \qquad && 8c_4(\lambda q'(t) - 2p_1(t)) = 0\,. \end{aligned} \tag{31}$$

There are two cases in which (28) can be solved and the group is non-trivial

$$\text{Case A:} \qquad (E+g)(x) = Kx^2 + bx + c \tag{32}$$

$$\text{Case B}: \qquad (E+g)(x) = Kx^2 + c + \frac{d}{x^2}\,. \tag{33}$$

In these cases a six-parameter and a four-parameter Lie group are admitted, respectively. This classification was obtained for $f = 0$ by Ovsiannikov [49]. Lie point symmetries for (26), when $f'(x) = 0$ have been considered by Hill [27], Bluman [11] presented a detailed analysis of a boundary problem, and potential symmetries have been found by Pucci and Saccomandi [51].

3.4. Case VIII.: $s = 0$ $m \neq 1$

In this case equation (3) adopts the following form:

$$u_t = u_{xx} + g(x)u^m + f(x)u_x \tag{34}$$

and the following conditions must be satisfied:

$$r_{1xx} + f r_{1x} - r_{1t} = 0 \tag{35}$$

$$g'\left(\frac{q'(t)}{2}x + p_1(t)\right) + g[(m-1)r_1 + q'(t)] = 0\,. \tag{36}$$

If we substitute (27) into (35) and (36), we obtain (28) and

$$\frac{f(x)}{2}(m-1) - \frac{g'(x)}{g(x)} = -\frac{(m-1)[q''(t)x^2 + 4p_1(t)x - 8r_3(t)] - 8q'(t)}{4(q'(t)x + 2p_1(t))} \tag{37}$$

the left-hand side of equation (37) depends only on x. So by differentiating with respect to t, we obtain that the following conditions must be satisfied:

$$\begin{aligned} &q'(t)q'''(t) - q''(t)^2 = 0 \\ &p_1(t)q'''(t) - 3p_1'(t)q''(t) + 2p_1''(t)q'(t) = 0 \\ &q'(t)r_3(t) - q''(t)r_3(t) - p_1(t)p_1''(t) + p_1'(t)^2 = 0 \\ &(m-1)(p_1(t)r_3'(t) - p_1'(t)r_3) + p_1(t)q''(t) - p_1'(t)q'(t) = 0\,. \end{aligned} \tag{38}$$

We can distinguish:

(i) If $q''(t) \neq 0$ then from (38) we obtain:

$$\begin{aligned} &q = \frac{k_2}{k_1}e^{k_1 t} + k_3 \qquad p_1 = k_4 e^{k_1 t} + k_5 e^{k_1 t/2} \\ &r_3 = k_6 e^{k_1 t} - \frac{k_1 k_4 k_5}{2k_2} e^{k_1 t/2} \\ &k_6 = -\frac{k_1 k_4^2(m-1) + 2k_2^2}{2k_2(m-1)}\,. \end{aligned} \tag{39}$$

(ii) If $q''(t) = 0$, then from (38) we obtain

$$q = k_2 t + k_3 \qquad p_1 = k_5 \qquad r_3 = k_6\,. \tag{40}$$

If we consider that in equation (28) E depends only on x we obtain, as in the previous case, that $E(x)$ adopts the form (30). Substituting (30) into (28) we obtain that conditions (31) must be satisfied. There are two cases in which (28) can be solved and the group is non-trivial:

$$\text{Case A}: \qquad E_1(x) = ax^2 + bx + c$$

$$\text{Case B}: \qquad E_2(x) = ax^2 + c + \frac{d}{x^2}\,.$$

Table 24. Each row shows the infinitesimal generators of the optimal system, the corresponding functions $E(x)$ and $g(x)$ for which (18) can be reduced to an ODE, as well as their similarity variables and solutions.

U_i	z_i	u_i	$E(x)$	$g(x)$
$V_1^5 + bV_3^5$	$\mathrm{e}^{-k_1 t/2}\, x + b\,\mathrm{e}^{-k_1 t}$	$h \exp\left(-\frac{k_1 x^2}{8} + \frac{k_1 t}{1-m} - \frac{1}{2}\int f(x)\,\mathrm{d}x\right)$	$ax^2 + c$	$k \exp\left(\frac{k_1 x^2}{8} + \frac{1}{2}\int f(x)\,\mathrm{d}x\,(m-1)\right)$
V_2^5	x	$h(x)$	$c_1^2/4$	c_2
$V_1^6 + kV_4^6$	$\frac{x^2}{t}$	$h \exp(-\frac{1}{2}\int f(x)\,\mathrm{d}x)\, x^{\frac{k-8}{4m-4}}$	cx^{-2}	$\frac{4\,c_1 \exp\left(\frac{1}{2}(m-1)\int f(x)\,\mathrm{d}x\right)}{x^{k/4}}$
$V_2^6 + kV_3^6 + k_5 V_4^6$	$x - kt$	$h \exp\left(\frac{k_5 x}{8m-8} - \frac{1}{2}\int f(x)\,\mathrm{d}x\right)$	c	$-\frac{4\,c_1 \exp(\frac{1}{2}\int f(x)\,\mathrm{d}x\,(m-1))}{x^{k/4}}$

Table 25. Each row shows the ODEs to which PDE (18) is reduced by U_i^5 or U_i^6.

U_i^5	$A_i(z) = 0$
U_1^5	$b\,\frac{\mathrm{d}h}{\mathrm{d}z}\, k_1 + h^m\, k + \frac{\mathrm{d}^2 h}{\mathrm{d}z^2} = 0$
U_2^5	$h_{zz} + gh + fhh_z = 0$
U_1^6	$\frac{\frac{\mathrm{d}h}{\mathrm{d}z}\, z(m\,z - z + 2\,m + k - 10)}{4\,(m-1)} - \frac{h(k-8)(4\,m - k + 4)}{64\,(m-1)^2} + \mathrm{c}\, h^m + \frac{\mathrm{d}^2 h}{\mathrm{d}z^2} - \frac{c\,h}{4} = 0$
U_2^6	$\frac{h\,{k_5}^2}{64(m-1)^2} - \frac{\mathrm{d}h}{\mathrm{d}z}\left(-\frac{k_5}{4(m-1)} - k\right) - \mathrm{c}\, h^m + \frac{\mathrm{d}^2 h}{\mathrm{d}z^2} - c\,h = 0$

(i) If $q''(t) \neq 0$, substituting (39) and $E(x)$ into (28) we find, that the most general Lie group of point transformations admitted by (34) is for $E(x) = ax^2 + c$ and $g(x) = \exp(\frac{m-1}{2}(\int f(x)\,\mathrm{d}x + \frac{k_1 x^2}{4}))$ a three-parameter group $\mathcal{G}_3^5$. Associated with this Lie groups is the Lie algebra, which can be represented by the generators $\{V_i^5\}_{i=1}^3$:

$$
\begin{aligned}
V_1^5 &= \frac{\mathrm{e}^{k_1 t} x}{2}\frac{\partial}{\partial x} + \frac{\mathrm{e}^{k_1 t}}{k_1}\frac{\partial}{\partial t} - \mathrm{e}^{k_1 t}\left(\frac{k_1 x^2}{8} + \frac{x f(x)}{4} - \frac{1}{1-m}\right)\frac{\partial}{\partial u} \\
V_2^5 &= V_2 \qquad V_3^5 = \frac{\mathrm{e}^{k_1 t}}{2}\frac{\partial}{\partial x} - \mathrm{e}^{\frac{k_1 t}{2}}\left(\frac{k_1 x}{4} + \frac{f(x)}{2}\right)\frac{\partial}{\partial u}
\end{aligned}
\quad . \qquad (41)
$$

(ii) If $q''(t) = 0$, substituting (40) and $E(x)$ into (28); we obtain for $E(x) = c_1(k_1 x + 2k_3)^{-2}$ and $g(x) = c\exp(\frac{m-1}{2}\int f(x)\,\mathrm{d}x)(k_1 x + 2k_3)^{\frac{k_4}{2k_1}}\, \mathcal{G}_3^6$.
Associated with this Lie group are the Lie algebras, which can be represented by the generators $\{V_i^6\}_{i=1}^3$:

$$
V_1^6 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \left(\frac{x f(x)}{4} - \frac{1}{1-m}\right)\frac{\partial}{\partial u} \qquad V_2^6 = V_2 \qquad V_3^6 = \frac{\partial}{\partial x} - \frac{f(x)}{2}\frac{\partial}{\partial u}\, . \qquad (42)
$$

Here $a = \frac{k_1^2}{16}$ and $c = \frac{k_1(5-m)}{4(m-1)}$. In table 24 we list the infinitesimal generators of the one-dimensional optimal system, the corresponding functions $E(x)$ and $g(x)$, as well as their similarity variables and solutions.

In the table 25 we list the ODEs to which (34) is reduced.

For case VIII, if $f(x) = 0$ and $g(x) = \text{constant}$, we recover the solutions obtained by Clarkson [18].

3.5. Concluding remarks

In this paper we have classified the Lie symmetries of the quasi-linear parabolic equation (3). Recognizing the importance of the space-dependent parts on the overall dynamics of (1), we have studied those spatial forms which admit the classical symmetry group. In general, the groups that leave (3) invariant depend on several parameters, to each one-parameter subgroup there will correspond a family of group-invariant solutions. We desired to minimize the search for group-invariant solutions to that of finding non-equivalent branches of solutions, which leads to the concept of optimal systems of group-invariant solutions, from which, every other solution can be derived. To obtain the one-dimensional optimal systems of solutions, following Olver, we have looked for the one-dimensional optimal systems of subalgebras. We then constructed all the invariant solutions with respect to the one-dimensional optimal system of subalgebras, as well as all the ODEs to which (3) is reduced. We have found ten different Lie algebras depending on m, n, s, $f(x)$ and $g(x)$, whose commutator tables and adjoint tables are listed in the appendix. We also list the different choices for functions $f(x)$, $g(x)$ and constants n, m, and s, for which equation (3) is invariant under a Lie group of point transformations, as well as their infinitesimal generators.

Lie symmetries provide only the beginning of a systematic solution technique for equation (3). In a forthcoming paper, methods based on non-local symmetries introduced by Bluman [10, 12–15] as well as 'non-classical symmetries' due to Bluman and Cole [11], will be used to obtain new solutions to (3). The new solutions being unobtainable by the method of Lie classical symmetries. We will construct non-local symmetries (potential symmetries) which are realized as local symmetries of a related auxiliary system of differential equations, by using potential symmetries we can also linearize (3) by an explicit non-invertible mapping. We will also study special techniques which may allow us to unfold new solutions.

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Appendix

Table A1. Each row shows the function $f(x)$, $g(x)$ and the constants n, m and s, for which equation (3) can be reduced by some of the generators V_i, as well as those generators.

V_i^l	Case	$f(x)$	$g(x)$	n	m	s
V_i	I	0	0	arbitrary, $-\frac{1}{3}$		
V_i	II.a	0	$\lambda(k_3x+k_1)^\beta$	arbitrary	arbitrary	
V_i	II.a	0	arbitrary	arbitrary	arbitrary	
V_i	II.a	0	$c\exp\left(\frac{k_4(n-m)x}{k_1(n-1)}\right)$	arbitrary	arbitrary	
V_i	II.a	0	c_2	arbitrary	arbitrary	

Table A1. Continued.

V_i^l	Case	$f(x)$	$g(x)$	n	m	s
V_i	II.a	0	$p(x)^{\frac{3}{2}(m-1)} \exp\left(-\frac{(3m+1)k_6}{2\sqrt{-d}}\varphi(x)\right)$	$-\frac{1}{3}$	arbitrary	
V_i	II.a	0	$p(x)^{\frac{3}{2}(m-1)} \Phi^{-\frac{(3m+1)k_6}{2\sqrt{-d}}\varphi(x)}$	$-\frac{1}{3}$	arbitrary	
V_i	II.b	0	c	arbitrary, $-\frac{1}{3}$	1	arbitrary
V_i	II.c	0	arbitrary	arbitrary, $-\frac{1}{3}$	n	
V_i	III.a	$\gamma(k_3x+k_1)^\alpha$	0	arbitrary		$\neq n-1, 0$
V_i	III.a	arbitrary	0	arbitrary		$\neq n-1, 0$
V_i	III.a	$c_1 \exp\left(\frac{k_4(n-s-1)x}{k_1(n-1)}\right)$	0	arbitrary		$\neq n-1, 0$
V_i	III.a	c_1	0	arbitrary		$\neq n-1, 0$
V_i	III.a	c_1	0	arbitrary		$\frac{n-1}{2}$
V_i^1	III.b	arbitrary	0	arbitrary, $-\frac{1}{3}$		$n-1$
V_i''	III.c	k	0	$-\frac{1}{3}$		0
V_i'''	III.c	k_1x+k_2	0	$-\frac{1}{3}$		0

Table A2. Each row shows the function $f(x)$, $g(x)$ and the constants n, m and s, for which equation (3) can be reduced by some of the generators V_i.

V_i^l	Case	$f(x)$	$g(x)$	n	m	s
V_i	IV.a	$\gamma(k_3x+k_1)^\beta$	$\lambda(k_3x+k_1)^\beta$	arbitrary	arbitrary	arbitrary
V_i	IV.a	arbitrary	arbitrary	arbitrary	arbitrary	arbitrary
V_i	IV.a	$c_1 \exp\left(\frac{k_4(n-s-1)x}{k_1(n-1)}\right)$	$c_2 \exp\left(\frac{k_4(n-m)x}{k_1(n-1)}\right)$	arbitrary	arbitrary	arbitrary
V_i	IV.a	c_1	c_2	arbitrary	arbitrary	arbitrary
V_i	IV.a	$cp(x)^{\frac{3x+2}{2}} \exp\left(-\frac{k_6(3s+4)\varphi}{2\sqrt{-d}}\right)$	$\phi_1(x)p(x)^{3s/2} \exp\left(-\frac{k_6(3s+4)\varphi}{2\sqrt{-d}}\right)$	$-\frac{1}{3}$	$s+1$	arbitrary
V_i	IV.a	$cp(x)^{\frac{3x+2}{2}} \phi^{-\frac{k_6(3s+4)}{4\sqrt{-d}}}$	$\phi_1(x)p(x)^{3s/2}\phi^{-\left(\frac{k_6(3s+4)}{4\sqrt{-d}}\right)}$	$-\frac{1}{3}$	arbitrary	arbitrary
V_i'''	IV.b	k	$c(k_4x+k_1)^{\frac{1}{3}(3m-7)}$	$-\frac{1}{3}$	arbitrary	0
V_i'''	IV.b	k_1x+k_2	$c(k_1x+k_2)^{3(m-1)}$	$\frac{1}{3}$	arbitrary	0
V_i'''	IV.b	k_1x+k_2	$c_2(k_1x+k_2)^{\frac{3(m-1)}{2}}$	$-\frac{1}{3}$	arbitrary	0
V_i'''	IV.b	k_1x+k_2	$c(k_1x+k_2)^{-2}$	$-\frac{1}{3}$	$+\frac{1}{3}$	0
V_i	IV.b	k_1x+k_2	c	$-\frac{1}{3}$	arbitrary,1	0
V_i^1	IV.c	arbitrary	arbitrary	arbitrary	arbitrary	$n-1$
V_i^1	IV.c	$\frac{c}{p(x)}$	$\frac{c}{p(x)^2}$	arbitrary	$-\frac{1}{3}$	m
V_i^2	V	$c_1(k_3x+k_1)^\alpha$	$c_2(k_3x+k_1)^\beta$	1	arbitrary	$\neq 1, 0$
V_i^2	V	$c_1 \exp\left(-\frac{k_4 s}{k_1}\right)$	$\exp\left(-\frac{k_4(m-1)}{k_1}\right)$	1	arbitrary	$\neq 1, 0$
V_i^2	V	arbitrary	arbitrary	1	arbitrary	$\neq 1, 0$
V_i^3	VI	$\frac{b}{x+2a}$	$k-\frac{2}{(x+2a)^2}$	1	1	1
V_i^3	VI	$c(x+2a)^{2b+1}$	$\frac{d}{(x+2a)^2}$	1	1	1

Table A3. Each row shows the functions $E(x)$, $g(x)$ and the constants for which (18) can be reduced to an ODE.

V_i	$E(x)$	$g(x)$	n	m	s
V_i^5	ax^2+c	$\exp\left(\frac{1}{2}(m-1)\int f + \frac{k_1x^2}{4}\right)$	1	$\neq 1$	0
V_i^5	ax^2+c	$c\exp(\frac{1}{2}(m-1)\int f)(k_1x+2k_3)^{k_4/2k_1}$	1	$\neq 1$	0

Table A4. Commutator table for the Lie algebra $\{V_i\}$.

(a)	V_1	V_2	V_3	V_4	V_5
V_1	0	0	V_1	0	$2V_3+4V_4$
V_2	0	0	$2V_2$	$-V_2$	0
V_3	$-V_1$	$-2V_2$	0	0	V_5
V_4	0	V_2	0	0	0
V_5	$-2V_3-4V_4$	0	$-V_5$	0	0

Table A5. Adjoint table for the Lie algebra $\{V_i\}$.

Ad	V_1	V_2	V_3	V_4	V_5
V_1	V_1	V_2	$V_3-\epsilon V_1$	V_4	$V_5-2\epsilon V_3-4\epsilon V_4$
V_2	V_1	V_2	$V_3-2\epsilon V_2$	$V_4+\epsilon V_2$	V_5
V_3	$\mathrm{e}^{\epsilon}V_1$	$\mathrm{e}^{2\epsilon}V_2$	V_3	V_4	$\mathrm{e}^{\epsilon}V_5$
V_4	V_1	$\mathrm{e}^{-\epsilon}V_2$	V_3	V_4	V_5
V_5	$V_1+2\epsilon V_3+4\epsilon V_4$	V_2	$V_3+\epsilon V_5$	V_4	V_5

Table A6. Commutator table for the Lie algebra $\{V_i'\}$.

(a)	V_1	V_2	V_3'	V_4'	V_5'
V_1	0	0	V_1	0	$2V_3'$
V_2	0	0	0	$k(n-1)V_4'$	0
V_3'	$-V_1$	0	0	0	V_5'
V_4'	0	$-k(1-n)V_4'$	0	0	0
V_5'	$-2V_3'$	0	$-V_5'$	0	0

Table A7. Adjoint table for the Lie algebra $\{V_i'\}$.

Ad	V_1	V_2	V_3'	V_4'	V_5'
V_1	V_1	V_2	$V_3'-\epsilon V_1$	V_4'	$V_5'-2\epsilon V_3'$
V_2	V_1	V_2	V_3'	$V_4'\mathrm{e}^{-k(1-n)\epsilon}$	V_5'
V_3'	$\mathrm{e}^{\epsilon}V_1$	V_2	V_3'	V_4'	$\mathrm{e}^{-\epsilon}V_5'$
V_4'	V_1	$V_2+k(n-1)\mathrm{e}^{-\epsilon}V_4'$	V_3'	V_4'	V_5'
V_5'	$V_1+2\epsilon V_3'$	V_2	$V_3'+\epsilon V_5'$	V_4'	V_5'

Table A8. Commutator table for the Lie algebra $\{V_i^1\}$.

(a)	V_1^1	V_2^1	V_3^1
V_1^1	0	V_2^1	0
V_2^1	$-V_2^1$	0	0
V_3^1	0	0	0

Table A9. Adjoint table for the Lie algebra $\{V_i^1\}$.

Ad	V_1^1	V_2^1	V_3^1
V_1^1	V_1^1	$e^{\epsilon} V_2^1$	V_3^1
V_2^1	$V_1^1 + \epsilon V_2^1$	V_2^1	V_3^1
V_3^1	V_1^1	V_2^1	V_3^1

Table A10. Commutator table for the Lie algebra $\{V_i''\}$.

(a)	V_1	V_2	V_3''	V_4''	V_5''
V_1	0	0	$2V_4''$	V_1	0
V_2	0	0	$2kV_4''$	kV_1	$V_2 - kV_1$
V_3''	$-2V_4''$	$-2kV_4''$	0	$-V_3''$	0
V_4''	$-V_1$	$-kV_1$	V_3''	0	0
V_5''	0	$kV_1 - V_2$	0	0	0

Table A11. Adjoint table for the Lie algebra $\{V_i''\}$.

Ad	V_1	V_2	V_3''	V_4''	V_5''
V_1	V_1	V_2	$V_3'' - 2\epsilon V_4''$	$V_4'' - \epsilon V_1$	V_5''
V_2	V_1	V_2	$V_3'' - 2k\epsilon V_4''$	$V_4'' - k\epsilon V_1$	$V_5' - \epsilon(V_2 - kV_1)$
V_3''	$V_1 + 2\epsilon V_4''$	$V_2 + 2k\epsilon V_4''$	V_3''	$V_4'' + \epsilon V_3''$	V_5''
V_4''	$e^{\epsilon} V_1$	$V_2 + k\epsilon V_1$	$e^{-\epsilon} V_3''$	V_4''	V_5''
V_5''	V_1	$V_2 - \epsilon(kV_1 - V_2)$	V_3''	V_4''	V_5''

Table A12. Commutator table for the Lie algebra $\{V_i'''\}$.

(a)	V_1'''	V_2'''	V_3'''	V_4'''	V_5'''
V_1'''	0	0	0	$-V_1'''$	$2V_4'''$
V_2'''	0	0	$2k_1 V_2'''$	0	0
V_3'''	0	$-2k_1 V_2'''$	0	0	0
V_4'''	V_1'''	0	0	0	$-V_5'''$
V_5'''	$2V_4'''$	0	0	V_5'''	0

Table A13. Adjoint table for the Lie algebra $\{V_i'''\}$.

Ad	V_1'''	V_2'''	V_3'''	V_4'''	V_5'''
V_1'''	V_1'''	V_2'''	V_3'''	$V_4''' + \epsilon V_1'''$	$V_5''' + 2\epsilon V_4'''$
V_2'''	V_1'''	V_2'''	$V_3''' - 2k_1\epsilon V_2'''$	V_4'''	V_5'''
V_3'''	V_1'''	$e^{2k_1\epsilon} V_2'''$	V_3'''	V_4'''	$e^{k_1\epsilon} V_5'''$
V_4'''	$e^{-\epsilon} V_1'''$	V_2'''	V_3'''	V_4'''	$e^{-\epsilon} V_5'''$
V_5'''	$V_1''' - 2\epsilon V_4'''$	V_2'''	V_3'''	$V_4''' + \epsilon V_5'''$	V_5'''

Table A14. Commutator table for the Lie algebra $\{V_i^2\}$.

(a)	V_1^2	V_2^2	V_3^2	V_4^2
V_1^2	0	0	V_1^2	0
V_2^2	0	0	$2V_2^2$	0
V_3^2	$-V_1^2$	$-2V_2^2$	0	0
V_4^2	0	0	0	0

Table A15. Adjoint table for the Lie algebra $\{V_i^2\}$.

Ad	V_1^2	V_2^2	V_3^2	V_4^2
V_1^2	V_1^2	V_2^2	$V_3^2 - \epsilon V_1^2$	V_4^2
V_2^2	V_1^2	V_2^2	$V_3^2 - 2\epsilon V_2^2$	V_4^2
V_3^2	$e^{\epsilon} V_1^2$	$e^{2\epsilon} V_2^2$	V_3^2	V_4^2
V_4^2	V_1^2	V_2^2	V_3^2	V_4^2

Table A16. Commutator table for the Lie algebra $\{V_i^3\}$.

(a)	V_1^3	V_2^3
V_1^3	0	$k_1 V_2^3$
V_2^3	$-k_1 V_1^3$	0

Table A17. Adjoint table for the Lie algebra $\{V_i^3\}$.

Ad	V_1^3	V_2^3
V_1^3	V_1^3	$e^{k_1 \epsilon} V_2^3$
V_2^3	$V_1^3 + k_1 \epsilon V_2^3$	V_2^3

Table A18. Commutator table for the Lie algebra $\{V_i^4\}$.

(a)	V_1^4	V_2^4
V_1^4	0	$-V_2^4$
V_2^4	V_2^4	0

Table A19. Adjoint table for the Lie algebra $\{V_i^4\}$.

Ad	V_1^4	V_2^4
V_1^4	V_1^4	$e^{\epsilon} V_2^4$
V_2^4	$V_1^4 - \epsilon V_2^4$	V_2^4

Table A20. Commutator table for the Lie algebra $\{V_i^5\}$.

(a)	V_1^5	V_2^5	V_3^5
V_1^5	0	$-k_1 V_1^5$	0
V_2^5	$k_1 V_1^5$	0	$\frac{1}{2} k_1 V_3^5$
V_3^5	0	$-\frac{1}{2} k_1 V_3^5$	0

Table A21. Adjoint table for the Lie algebra $\{V_i^5\}$.

Ad	V_1^5	V_2^5	V_3^5
V_1^5	V_1^5	$V_2^5 - k_1 \epsilon V_1^5$	V_3^5
V_2^5	$e^{k_1 \epsilon} V_1^5$	V_2^5	$e^{k_2/2} V_3^5$
V_3^5	V_3^5	$V_2^5 - \frac{1}{2} k_1 \epsilon V_3^5$	V_3^5

Table A22. Commutator table for the Lie algebra $\{V_i^6\}$.

(a)	V_1^6	V_2^6	V_3^6	V_4^6
V_1^6	0	$-V_2^6$	$-\frac{1}{2} V_3^6$	0
V_2^6	V_2^6	0	0	0
V_3^6	$-\frac{1}{2} V_3^6$	0	0	0
V_4^6	0	0	0	0

Table A23. Adjoint table for the Lie algebra $\{V_i^6\}$.

Ad	V_1^6	V_2^6	V_3^6	V_4^6
V_1^6	V_1^6	$e^{\epsilon} V_2^6$	$e^{\epsilon/2} V_3^6$	V_4^6
V_2^6	$V_1^6 - \epsilon V_2^6$	V_2^6	V_3^6	V_4^6
V_3^6	$V_1^6 - \frac{1}{2} \epsilon V_3^6$	V_2^6	V_3^6	V_4^6
V_4^6	V_1^6	V_2^6	V_3^6	V_4^6

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