J. Phys. A: Math. Gen. **32** (1999) 3621–3635. Printed in the UK PII: S0305-4470(99)98123-4

Hidden integrable hierarchies of AKNS type*

Boris Konopelchenko†, Luis Martínez Alonso‡ and Elena Medina§

† Dipartimento di Fisica, Universitá de Lecce, 73100 Lecce, Italy
‡ Departamento de Física Teórica, Universidad Complutense, E28040 Madrid, Spain
§ Departamento de Matemáticas, Universidad de Cádiz, 11510 Cádiz, Spain

Received 5 October 1998, in final form 15 February 1999

Abstract. A method for constructing integrable hierarchies by restricting AKNS flows on manifolds of finite codimension in the space of independent parameters is provided. Two particular types of hierachies are characterized: one is given by nonlinear differential equations with coordinate-dependent coefficients and the other is related to Schrödinger spectral problems with energy-dependent potentials of even degree.

1. Introduction

We have recently introduced the notion of *hidden hierarchies* [1] to describe integrable models which arise by restricting the KP flows and their reductions to certain submanifolds $\mathcal{M}_m$ of finite codimension m

$$t_1 = b_1(t_{m+1}, t_{m+2}, \ldots) \quad t_2 = b_2(t_{m+1}, t_{m+2}, \ldots) \quad \ldots \quad t_m = b_m(t_{m+1}, t_{m+2}, \ldots) \tag{1}$$

in the space of independent parameters $\boldsymbol{t} = (t_1, t_2, \ldots) \in \mathbb{C}^{\infty}$. In [1–3] hidden hierarchies associated with KdV flows are analysed and they are found to provide integrable models related to the *energy-dependent* Schrödinger spectral problems

$$\partial_{xx} f = \left(z^{2m+1} + \sum_{n=0}^{2m} z^n u_n(x)\right) f. \tag{2}$$

Furthermore, it is proved that these hierarchies are connected to the zero manifolds of τ-functions and, consequently, their corresponding flows in the Grassmannian take place outside the big cell.

The methods used in [1–3] provide the starting point of a general technique for deriving integrable models which are based on the consideration of constrained flows on the Grassmannian. The input of this technique is a wavefunction $\Psi(z, \boldsymbol{t})$ of a KP hierarchy or of one of their reductions and the aim is to characterize submanifolds $\mathcal{M}_m$ such that the restriction Ψ_{res} of Ψ to $\mathcal{M}_m$ satisfies an infinite system of linear problems which determines the dependence of Ψ_{res} on the parameters $(t_{m+1}, t_{m+2}, \ldots)$. Under these conditions, the compatibility of the system of linear problems leads to a hidden hierarchy.

In the present paper these methods are applied to study hidden hierarchies arising from AKNS flows. In this case a fundamental difference arises with respect to our previous results about the hidden KP hierarchy; namely, *for arbitrary $\mathcal{M}_m$ the restrictions of AKNS*

* Partially supported by CICYT proyecto PB95-0401.

wavefunctions satisfy a system of linear problems (this feature is also considered from another point of view in recent works, see for instance [4, 5]). Consequently, these restricted wavefunctions determine hidden hierarchies which, generically, flow in the big cell of the Grassmannian. Furthermore, their integrable systems turn out to be non-autonomous evolution equations depending on the arbitrary functions $b_i(t_{m+1}, t_{m+2}, \ldots)$, $i = 1, \ldots, m$ which define $\mathcal{M}_m$. We also derive a different class of hidden AKNS hierarchies which flow outside the big cell of the Grassmannian. These hierarchies are determined by imposing appropriate conditions on $\mathcal{M}_m$ and, under certain transformations, turn out to describe the integrable systems associated with the class of *energy-dependent* Schrödinger operators

$$\partial_{xx} f = \left(z^{2m+2} + \sum_{n=0}^{2m+1} z^n u_n(x) \right) f. \tag{3}$$

Case (3) with $m = 0$, which does not correspond to any hidden AKNS hierarchy, it is associated with the Jaulent–Miodek hierarchy [6], which in turn becomes the standard AKNS hierarchy under an appropriate transformation.

The existence of the hidden AKNS flows not only outside the big cell but also on the big cell is a new feature with respect to the previously studied one-component KP hierarchy.

Observe that (2) and (3) represent the whole set of Schrödinger operators with a potential function which has a polynomial dependence on the spectral parameter. The corresponding hierarchies of integrable systems have already been described in [7], but no indications about methods of solution nor its group-theoretical interpretation from the point of view of Birkhoff factorization were provided. Our results in the present paper as well as those in [1–3] fill these gaps. In this sense we notice that there is a direct relationship between the Birkhoff factorization of a flow and the stratum in the Grassmannian on which it lies [8, 9, 12].

This paper is organized as follows. In section 2 we recall the main ideas about the Grassmannian and its stratified structure (subsection 2.1), we describe the AKNS flows in the Grassmannian (subsection 2.2) and the relation between the AKNS wavefunction and the corresponding tau-functions is established (subsection 2.3). Some details about this fact are provided in the appendix, at the end of the paper.

In section 3 we describe the hidden AKNS flows in the big cell of the Grassmannian, the hierarchies of integrable systems associated with them (subsection 3.1), the hidden AKNS flows outside the big cell and the relation with the hierarchies connected to energy-dependent Schrödinger operators (subsection 3.2). Finally, we consider reductions of the AKNS flows in the big cell (subsection 3.3).

2. AKNS flows on the Grassmannian

2.1. The stratification of the Grassmannian

It is well known that a wavefunction of the N-component KP hierarchy leads to a flow in the Grassmannian which can be formulated in several ways [8–11]. In what follows we will take advantage of the lexicographic isomorphism [8, 9] for applying the stratification of the standard one-component KP Grassmannian to study the hidden AKNS flows. To this end let us introduce the Hilbert space $H := L^2(S^1, \mathbb{C})$ of square-integrable complex-valued functions on S^1, with the scalar product being defined by

$$\langle w', w \rangle := \int_{S^1} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \overline{w}'(z)\, w(z).$$

We consider the decomposition of H as the direct sum of the closed subspaces $H_\pm$ generated by $z^n : n \geqslant 0$ and $z^{-n} : n \geqslant 1$, respectively. The Grassmannian $\mathrm{Gr}(H)$ is the set of all closed subspaces W of H such that

(a) The orthogonal projections $P_\pm\colon W \longrightarrow H_\pm$ are operators of Fredholm and compact types, respectively.
(b) The virtual dimension of W (i.e. the index of P_+) is zero.

It can be proved that $\mathrm{Gr}(H)$ is a Hilbert manifold with a stratified structure. The strata of $\mathrm{Gr}(H)$ can be described by introducing the set $\boldsymbol{S}_0$ of increasing sequences of integers

$$S = \{s_0, s_1, s_2, \ldots\}$$

such that $s_n = n$ for all sufficiently large n. Each $W \in \mathrm{Gr}(H)$ determines a sequence of this type. To see this point recall that an element $W \in H$ is said to be of finite order n if it can be expressed in the form $w = \sum_{m \leqslant n} a_m z^m$, with $a_n \neq 0$. Thus, due to the fact that the virtual dimension of W is zero, it can be shown that the sequence

$$S_W = \{n \in \mathbb{Z} : W \text{ contains an element of order } n\}$$

is an element of $\boldsymbol{S}_0$. Then, given $S \in \boldsymbol{S}_0$ we may define the subset of $\mathrm{Gr}(H)$

$$\Sigma_S = \{W \in \mathrm{Gr}(H) : S_W = S\}$$

which is called the stratum corresponding to S. In any $W \in \mathrm{Gr}(H)$ the elements of finite order form a dense open subspace denoted by W^{alg}. Therefore, W belongs to Σ_S when W^{alg} has a basis $\{w_n\}_{n\geqslant 0}$ such that

$$w_n(z) = z^{s_n}(1 + \mathcal{O}(z^{-1})) \qquad n \geqslant 0.$$

In particular, if S is the set of non-negative integers the corresponding stratum is a dense open subset of $\mathrm{Gr}(H)$ which is called the *big cell* of the Grassmannian.

In the analysis of the KdV and AKNS hierarchies one is led to consider the subset of $\mathrm{Gr}(H)$ given by

$$\mathrm{Gr}(H)^{(2)} = \{W \in \mathrm{Gr}(H) : z^2 W \subset W\}.$$

Here $z^2 W$ denotes the action of the multiplication operator by the function z^2 on W. It is obvious that $S_W + 2 \subset S_W$ for all $W \in \mathrm{Gr}(H)^{(2)}$, and consequently the stratification of $\mathrm{Gr}(H)^{(2)}$ turns out to be

$$\mathrm{Gr}(H)^{(2)} = \bigcup_{m\geqslant 0} \Sigma_m \qquad \Sigma_m := \Sigma_{S_m} \cap \mathrm{Gr}(H)^{(2)} \tag{4}$$

where

$$S_m = \{-m, -m+2, -m+4, \ldots, m, m+1, m+2, \ldots\}.$$

For describing the AKNS flows in the Grassmannian it is useful to introduce the Hilbert space $\mathcal{H} := L^2(S^1, \mathbb{C}^2)$ of square-integrable functions from S^1 into $\mathbb{C}^2$. The scalar product in $\mathcal{H}$ is defined by

$$\langle w', w\rangle := \int_{S^1} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \sum_{i=1}^{2} \overline{w_i}'(z)\, w_i(z).$$

There is a canonical isomorphism, also called lexicographic isomorphism, between H and $\mathcal{H}$,

$$H \longleftrightarrow \mathcal{H} \qquad w \longleftrightarrow \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

given by

$$w_1(z^2) = \frac{w(z) + w(-z)}{2}$$
$$w_2(z^2) = \frac{w(z) - w(-z)}{2z}$$
$$w(z) = w_1(z^2) + z w_2(z^2).$$

This isomorphism extends to the corresponding Grassmannians. In what follows, in order to avoid confusion, given a subspace W in H we will denote by $\mathcal{W}$ the corresponding subspace in $\mathcal{H}$. In particular, notice that the image of $\mathrm{Gr}(H)^{(2)}$ under the lexicographic isomorphism is

$$\mathrm{Gr}(\mathcal{H})^{(2)} = \{\mathcal{W} \in \mathrm{Gr}(\mathcal{H}) : z\mathcal{W} \subset \mathcal{W}\}.$$

2.2. AKNS flows on the big cell

Let us consider the AKNS linear system of equations for the wavefunction

$$\partial_n \Psi = P_n \Psi \qquad \partial_n = \frac{\partial}{\partial t_n} \quad n \geqslant 1. \tag{5}$$

Here $\Psi = \Psi(z, \boldsymbol{t})$ denotes a (2×2)-matrix function such that $\det \Psi = 1$, which depends on a complex parameter z and an infinite set of time parameters

$$\boldsymbol{t} := (x \equiv t_1, t_2, t_3, \ldots)$$

and $P_n = P_n(z, \boldsymbol{t})$ $(n \geqslant 1)$ are given (2×2)-matrix functions with polynomial dependence on z. We will henceforth assume that Ψ is an analytic function of z on some domain containing the unit circle $|z| = 1$, and that it admits a factorization of the form

$$\Psi = \chi \cdot \Psi_0 \tag{6}$$

where

$$\chi(z, \boldsymbol{t}) = 1 + \sum_{n \geqslant 1} \frac{A_n(\boldsymbol{t})}{z^n} \qquad \Psi_0 = \exp\left[\left(\sum_{n \geqslant 1} z^n t_n\right)\sigma_3\right] \qquad |z| = 1 \tag{7}$$

with $\sigma_3 = \mathrm{diag}(1, -1)$. The compatibility conditions between the flows $\partial_x \equiv \partial_1$ and ∂_n, $(n \geqslant 2)$ lead to the system of equations

$$\partial_n P_1 = \partial_x P_n + [P_n, P_1] \qquad n \geqslant 2 \tag{8}$$

which constitutes the AKNS hierarchy of the nonlinear integrable system for the functions $q(\boldsymbol{t})$ and $r(\boldsymbol{t})$ such that

$$P_1 = z\sigma_3 - \begin{pmatrix} 0 & q(\boldsymbol{t}) \\ r(\boldsymbol{t}) & 0 \end{pmatrix}.$$

Consider now the following two-component vector functions:

$$\psi_1 := \begin{pmatrix} \Psi_{11} \\ \Psi_{12} \end{pmatrix} \qquad \psi_2 := \begin{pmatrix} \Psi_{21} \\ \Psi_{22} \end{pmatrix}.$$

From (5) it follows that ψ_i verify a linear system of the form

$$\partial_n \psi_i = a_{ni}(z, \boldsymbol{t})\psi_1 + b_{ni}(z, \boldsymbol{t})\psi_2 \qquad i = 1, 2 \quad n \geqslant 1 \tag{9}$$

where $a_{ni}(z, \boldsymbol{t})$ and $b_{ni}(z, \boldsymbol{t})$ are complex-valued functions polynomially dependent on z. In particular,

$$\partial_x \psi_1 = z\psi_1 - q(\boldsymbol{t})\psi_2 \qquad \partial_x \psi_2 = -r(\boldsymbol{t})\psi_1 - z\psi_2. \tag{10}$$

If we define

$$\chi_i := \Psi_0^{-1} \cdot \psi_i \qquad i = 1, 2$$

then on S^1 we have

$$\chi_1(z, \boldsymbol{t}) = \begin{pmatrix} 1 + \mathcal{O}(1/z) \\ \mathcal{O}(1/z) \end{pmatrix} \qquad \chi_2(z, \boldsymbol{t}) = \begin{pmatrix} \mathcal{O}(1/z) \\ 1 + \mathcal{O}(1/z) \end{pmatrix}. \tag{11}$$

Each AKNS wavefunction determines an element $\mathcal{W}$ of $\mathrm{Gr}(\mathcal{H})$ defined by

$$\mathcal{W} := \mathrm{span}\{\psi_1(z, \boldsymbol{t}), \psi_2(z, \boldsymbol{t}), \text{all admissible } \boldsymbol{t}\}$$

where an *admissible* value of $\boldsymbol{t}$ means that $\psi_1(z, \boldsymbol{t})$ and $\psi_2(z, \boldsymbol{t})$ are non-singular at $\boldsymbol{t}$. Here span denotes the closure in $\mathcal{H}$ of the set of all linear combinations of the form

$$\sum_{n\geqslant 1}^{N} (a_n(\boldsymbol{t}_n)\psi_1(z, \boldsymbol{t}_n) + b_n(\boldsymbol{t}'_n)\psi_2(z, \boldsymbol{t}'_n)) \tag{12}$$

with a_n, b_n being arbitrary functions in $\boldsymbol{t}$ and $\boldsymbol{t}_n, \boldsymbol{t}'_n$ are arbitrary admissible points in $\mathbb{C}^\infty$. From (10) it is clear that $z^n\psi_i(z, \boldsymbol{t}) \in \mathcal{W}$, $i = 1, 2$, $n \geqslant 0$, then, taking into account (9) and by using a Taylor expansion around any value $\boldsymbol{t}$ it follows that

$$\mathcal{W} := \mathrm{span}_{\mathbb{C}[z]}\{\psi_1(z, \boldsymbol{t}), \psi_2(z, \boldsymbol{t}), \text{any fixed admissible } \boldsymbol{t}\} \tag{13}$$

where $\mathrm{span}_{\mathbb{C}[z]}$ is defined as span but now, functions a_n, b_n in (12) are arbitrary polynomials in z. As a consequence of (13) we have that each AKNS wavefunction determines a flow in $\mathrm{Gr}(\mathcal{H})$ given by

$$\mathcal{W}(\boldsymbol{t}) := \Psi_0(z, \boldsymbol{t})^{-1}\mathcal{W} = \mathrm{span}_{\mathbb{C}[z]}\{\chi_1(z, \boldsymbol{t}), \chi_2(z, \boldsymbol{t})\}.$$

2.3. *Tau-functions for AKNS flows*

There is a natural embedding of $\mathrm{Gr}(\mathcal{H})$ in the projective space $P(\wedge^\infty\mathcal{H})$ of the infinite wedge space $\wedge^\infty\mathcal{H}$. It assigns to each $\mathcal{W}$ the ray in $\wedge^\infty\mathcal{H}$ containing the vector

$$|\mathcal{W}\rangle := w_0 \wedge w_1 \wedge \cdots \wedge w_n \wedge \cdots$$

where $\{w_n\}_{n\geqslant 0}$ is any admissible basis of $\mathcal{W}$ [8].

Let $\mathcal{W}$ be the element in $\mathrm{Gr}(\mathcal{H})$ generated by a given AKNS wavefunction Ψ, we define the associated tau-functions

$$\tau_{\mathcal{W}}^{(l)}(\boldsymbol{t}) := \frac{\langle \mathcal{H}_+ | z^{-l\sigma_3}\mathcal{W}(\boldsymbol{t})\rangle}{\langle \mathcal{H}_+ | \mathcal{W}\rangle} \qquad l = 0, \pm 1 \tag{14}$$

where the scalar product $\langle \mathcal{W}' | \mathcal{W}\rangle$ denotes the determinant of the matrix whose (i, j)th element is $\langle w'_i, w_j\rangle$. (Here $\{w'_n\}_{n\geqslant 0}$ and $\{w_n\}_{n\geqslant 0}$ are given admissible basis for $\mathcal{W}'$ and $\mathcal{W}$, respectively.)

The AKNS wavefunction Ψ can be recovered from its associated tau-functions according to the following expressions:

$$\chi_1(z,\boldsymbol{t}) = \frac{1}{\tau_{\mathcal{W}}^{(0)}(\boldsymbol{t})}\begin{pmatrix} \tau_{\mathcal{W}}^{(0)}(\boldsymbol{t}-\frac{1}{2}[z]) \\ -\dfrac{1}{z}\tau_{\mathcal{W}}^{(1)}(\boldsymbol{t}+\frac{1}{2}[z]) \end{pmatrix} \tag{15}$$

$$\chi_2(z,\boldsymbol{t}) = \frac{1}{\tau_{\mathcal{W}}^{(0)}(\boldsymbol{t})}\begin{pmatrix} \dfrac{1}{z}\tau_{\mathcal{W}}^{(-1)}(\boldsymbol{t}-\frac{1}{2}[z]) \\ \tau_{\mathcal{W}}^{(0)}(\boldsymbol{t}+\frac{1}{2}[z]) \end{pmatrix} \tag{16}$$

where $[z] := (1/z, 1/2z^2, \ldots, 1/nz^n, \ldots)$.

To derive (15) and (16) we first need the following basic relation which is an immediate consequence of the definition (14):

$$\tau_{\mathcal{W}}^{(l)}(\boldsymbol{t}+\boldsymbol{s}) = \tau_{\mathcal{W}}^{(0)}(\boldsymbol{t})\cdot\tau_{\mathcal{W}(\boldsymbol{t})}^{(l)}(\boldsymbol{s}) \qquad l = 0, \pm 1. \tag{17}$$

On the other hand, one has (see the appendix)

$$\begin{aligned} &\tau_{\mathcal{W}}^{(0)}(-\tfrac{1}{2}[z]) = (\chi_1(z,\boldsymbol{0}))_1 \qquad &&\tau_{\mathcal{W}}^{(1)}(\tfrac{1}{2}[z]) = -z(\chi_1(z,\boldsymbol{0}))_2 \\ &\tau_{\mathcal{W}}^{(0)}(\tfrac{1}{2}[z]) = (\chi_2(z,\boldsymbol{0}))_2 \qquad &&\tau_{\mathcal{W}}^{(-1)}(-\tfrac{1}{2}[z]) = z(\chi_2(z,\boldsymbol{0}))_1. \end{aligned} \tag{18}$$

Hence, from (17) and (18) the expressions (15) and (16) follow at once.

There are two immediate consequences of (15) and (16). Firstly, the admissible values of $\boldsymbol{t}$ are obviously those such that $\tau_{\mathcal{W}}^{(0)}(\boldsymbol{t}) \neq 0$. Secondly, a subspace $z^{-l\sigma_3}\mathcal{W}(\boldsymbol{t})$ is in the big cell of $\mathrm{Gr}(\mathcal{H})$ if and only if $\tau_{\mathcal{W}}^{(l)}(\boldsymbol{t}) \neq 0$. In this way, by taking into account that

$$q(\boldsymbol{t}) = -2\frac{\tau_{\mathcal{W}}^{(1)}(\boldsymbol{t})}{\tau_{\mathcal{W}}^{(0)}(\boldsymbol{t})} \qquad r(\boldsymbol{t}) = -2\frac{\tau_{\mathcal{W}}^{(-1)}(\boldsymbol{t})}{\tau_{\mathcal{W}}^{(0)}(\boldsymbol{t})}$$

we conclude that provided $q(\boldsymbol{t}) \neq 0$ and $r(\boldsymbol{t}) \neq 0$, the AKNS flows $z^{-l\sigma_3}\mathcal{W}(\boldsymbol{t})$ take place in the big cell of $\mathrm{Gr}(\mathcal{H})$.

3. Hidden AKNS flows in the Grassmannian

3.1. Hidden AKNS flows in the big cell

Let us suppose we have an AKNS wavefunction $\Psi(z,\boldsymbol{t})$ and take an arbitrary submanifold $\mathcal{M}_m$ of finite codimension m in $\mathbb{C}^\infty$ of the form

$$t_1 = b_1(\boldsymbol{s}) \quad t_2 = b_2(\boldsymbol{s}) \quad \ldots \quad t_m = b_m(\boldsymbol{s}). \tag{19}$$

Here $\boldsymbol{s}$ denotes

$$\boldsymbol{s} = (t_{m+1}, t_{m+2}, \ldots)$$

and $b_j(\boldsymbol{s})$ $(j = 1, \ldots, m)$ are given functions. This submanifold can be expressed parametrically as

$$\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{s}) = (b_1(\boldsymbol{s}), b_2(\boldsymbol{s}), \ldots, b_m(\boldsymbol{s}), \boldsymbol{s}).$$

Consider the restriction of Ψ on $\mathcal{M}_m$

$$\Psi_{\mathrm{res}}(z,\boldsymbol{s}) := \Psi(z,\boldsymbol{t}(\boldsymbol{s})). \tag{20}$$

Then, if $\mathcal{W}$ is the element of $\mathrm{Gr}(\mathcal{H})$ generated by Ψ we have

$$\mathcal{W} = \mathrm{span}_{\mathbb{C}[z]}\{\psi_{\mathrm{res},1}(z, \boldsymbol{s}), \psi_{\mathrm{res},2}(z, \boldsymbol{s}), \text{any fixed admissible } \boldsymbol{s}\} \tag{21}$$

so that there are decompositions of the form

$$\partial_n \psi_{\mathrm{res},i} = a_{ni}(z, \boldsymbol{s})\psi_{\mathrm{res},1} + b_{ni}(z, \boldsymbol{s})\psi_{\mathrm{res},2} \qquad n \geqslant m+1 \quad i = 1, 2$$

with a_{ni}, b_{ni} being polynomials in z. Equivalently, in terms of $\Psi_{\mathrm{res}}(z, \boldsymbol{s})$, we have

$$\partial_n \Psi_{\mathrm{res}} = P_{\mathrm{res},n-m}(z, \boldsymbol{s})\Psi_{\mathrm{res}} \qquad n \geqslant m+1 \tag{22}$$

with $P_{\mathrm{res},n-m}$ being (2×2)-matrix functions with polynomial dependence on z given by

$$P_{\mathrm{res},n-m} = \left[\left(z^n + \sum_{j=1}^{m} z^j \partial_n b_j\right)\chi_{\mathrm{res}}\sigma_3\chi_{\mathrm{res}}^{-1}\right]_+ \qquad n \geqslant m+1.$$

Here $[\ ]_+$ denotes the Taylor part of a Laurent series at $z = 0$. Notice that $\mathrm{tr}\, P_{\mathrm{res},n-m} = 0$, so that, in particular, for $n = m+1$ we have

$$\partial_x \Psi_{\mathrm{res}} = P_{\mathrm{res},1}(z, \boldsymbol{s})\Psi_{\mathrm{res}} \qquad x \equiv t_{m+1} \tag{23}$$

where $P_{\mathrm{res},1}$ takes the form

$$P_{\mathrm{res},1} = \begin{pmatrix} p(z, \boldsymbol{s}) & q(z, \boldsymbol{s}) \\ r(z, \boldsymbol{s}) & -p(z, \boldsymbol{s}) \end{pmatrix} \tag{24}$$

with

$$p(z, \boldsymbol{s}) := z^{m+1} + \sum_{j=0}^{m} z^j p_j(\boldsymbol{s})$$

$$q(z, \boldsymbol{s}) := \sum_{j=0}^{m} z^j q_j(\boldsymbol{s})$$

$$r(z, \boldsymbol{s}) := \sum_{j=0}^{m} z^j r_j(\boldsymbol{s}).$$

We are going to prove that the compatibility conditions of (22) and (23) determine an integrable hierarchy of non-autonomous nonlinear partial differential equations in (1 + 1)-dimensions. To this end we introduce the matrix function

$$R(z, \boldsymbol{s}) := \chi_{\mathrm{res}}\sigma_3\chi_{\mathrm{res}}^{-1} = R_0 + \sum_{n\geqslant 1} \frac{R_n(\boldsymbol{s})}{z^n} \qquad R_0 = \sigma_3. \tag{25}$$

One immediately finds that

$$\mathrm{tr}\, R = 0 \qquad \det R = -1, \qquad \partial_x R = [P_{\mathrm{res},1}, R]$$

$$P_{\mathrm{res},n-m} = \left[\left(z^n + \sum_{j=1}^{m} z^j \partial_n b_j\right) \cdot R\right]_+ \qquad n \geqslant m+1. \tag{26}$$

In this way, if we write the coefficients of the expansion of R in the form

$$R_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & -\alpha_n \end{pmatrix}$$

we obtain the recursion relations

$$\begin{aligned}
\alpha_k &= -\tfrac{1}{2}\sum_{j=1}^{k-1}(\beta_j\gamma_{k-j} + \alpha_j\alpha_{k-j}) && k \geqslant 2 \quad \alpha_1 = 0 \\
\beta_{m+1-k} &= q_k + \sum_{j=k+1}^{m}(q_j\alpha_{j-k} - p_j\beta_{j-k}) && k = 0,\ldots,m \\
\beta_{m+1+k} &= \tfrac{1}{2}\beta_{k,x} - \sum_{j=0}^{m}(p_j\beta_{j+k} - q_j\alpha_{j+k}) && k \geqslant 0 \\
\gamma_{m+1-k} &= r_k + \sum_{j=k+1}^{m}(r_j\alpha_{j-k} - p_j\gamma_{j-k}) && k = 0,\ldots,m \\
\gamma_{m+1+k} &= -\tfrac{1}{2}\gamma_{k,x} + \sum_{j=0}^{m}(r_j\alpha_{j+k} - p_j\gamma_{j+k}) && k \geqslant 0.
\end{aligned} \tag{27}$$

Moreover, by using equation (26) for $P_{\mathrm{res},1}$ we find

$$\begin{aligned}
p_k &= b_{k,x} + \alpha_{m+1-k} + \sum_{j=k+1}^{m} b_{j,x}\alpha_{j-k} && k = 1,\ldots,m \\
p_0 &= \alpha_{m+1} + \sum_{j=1}^{m} b_{j,x}\alpha_j.
\end{aligned} \tag{28}$$

These recursion relations involve the matrix elements $\alpha_n, \beta_n, \gamma_n$ of R_n ($n \geqslant 0$) and the coefficients p_k, q_k, r_k ($k = 0,\ldots,m$) which determine the potential function $P_{\mathrm{res},1}$ of the *spectral problem* (23). It is straightforward to see that they allow us to express $\alpha_n, \beta_n, \gamma_n$ ($n \geqslant 0$) and p_k ($k = 0,\ldots,m$) in terms of q_k, r_k ($k = 0,\ldots,m$). Therefore, the compatibility conditions between (22) and (23)

$$\partial_n P_{\mathrm{res},1} = \partial_x P_{\mathrm{res},n-m} + [P_{\mathrm{res},n-m}, P_{\mathrm{res},1}] \qquad n \geqslant m+1 \tag{29}$$

become nonlinear evolution equations for q_k, r_k ($k = 0,\ldots,m$). We notice that due to the form of (27) and (28) the functions $\alpha_n, \beta_n, \gamma_n$ and p_k are differential polynomials in q_k, r_k ($k = 0,\ldots,m$) and the given fixed functions b_j ($j = 1,\ldots,m$), so that (29) are non-autonomous partial differential equations.

For example, let us find the explicit form of the simplest nonlinear integrable system for the case $m = 1$. We take a submanifold $\mathcal{M}_1$ of finite codimension 1 in $\mathbb{C}^\infty$ of the form

$$t_1 = b(\boldsymbol{s}) \qquad \boldsymbol{s} = (x := t_2, t := t_3, t_4, \ldots) \tag{30}$$

with $b(\boldsymbol{s})$ being a given function. The first member of the hierarchy derives from the equation

$$\partial_t P_{\mathrm{res},1} = \partial_x P_{\mathrm{res},2} + [P_{\mathrm{res},2}, P_{\mathrm{res},1}]. \tag{31}$$

From (26) and using the expansion (25) it follows that

$$\begin{aligned}
P_{\mathrm{res},1} &= [(z^2 + zb_x)R]_+ = z^2\sigma_3 + z(R_1 + b_x\sigma_3) + R_2 + b_x R_1 \\
P_{\mathrm{res},2} &= [(z^3 + zb_t)R]_+ = z^3\sigma_3 + z^2 R_1 + z(R_2 + b_t\sigma_3) + R_3 + b_t R_1.
\end{aligned}$$

In order to write (31) as a system of nonlinear differential equations for q_0, q_1, r_0 and r_1, we have to calculate the explicit form of the coefficients R_1, R_2 and R_3. By using (27) and (28) one finds at once

$$\begin{aligned}
&p_0 = -\tfrac{1}{2}q_1 r_1 \qquad p_1 = b_x \\
&\alpha_1 = 0 \qquad \alpha_2 = -\tfrac{1}{2}q_1 r_1 \qquad \alpha_3 = -\tfrac{1}{2}(q_1 r_0 + q_0 r_1) + b_x r_1 q_1 \\
&\beta_1 = q_1 \qquad \beta_2 = q_0 - q_1 b_x \qquad \beta_3 = \tfrac{1}{2}q_{1,x} + q_1 b_x^2 - q_0 b_x \\
&\gamma_1 = r_1 \qquad \gamma_2 = r_0 - r_1 b_x \qquad \gamma_3 = -\tfrac{1}{2}r_{1,x} + r_1 b_x^2 - r_0 b_x.
\end{aligned}$$

Thus, one finds that (31) reduces to the following system:

$$q_{1,t} = -2q_{1,x}b_x - q_1b_{xx} + q_{0,x} + q_1^2r_1b_x - 2q_1b_xb_t + 2q_0b_t \\ -q_1^2r_0 - q_0q_1r_1 - 2q_1b_x^3 + 2q_0b_x^2 \qquad (32)$$

$$r_{1,t} = -2r_{1,x}b_x - r_1b_{xx} + r_{0,x} - r_1^2q_1b_x + 2r_1b_xb_t - 2r_0b_t \\ +r_1^2q_0 + r_0q_1r_1 + 2r_1b_x^3 - 2r_0b_x^2 \qquad (33)$$

$$q_{0,t} = \tfrac{1}{2}q_{1,xx} + 2q_1b_{xx}b_x + q_{1,x}b_x^2 - b_{xx}q_0 - b_xq_{0,x} + b_{xt}q_1 + b_tq_{1,x} \\ +q_1q_0r_1b_x - q_1q_0r_0 - q_0^2r_1 + \tfrac{1}{2}q_1r_1q_{1,x} + q_1^2r_1b_x^2 + q_1^2r_1b_t \qquad (34)$$

$$r_{0,t} = -\tfrac{1}{2}r_{1,xx} + 2r_1b_{xx}b_x + r_{1,x}b_x^2 - b_{xx}r_0 - b_xr_{0,x} + b_{xt}r_1 + b_tr_{1,x} \\ -r_1r_0q_1b_x + q_0r_0r_1 + r_0^2q_1 + \tfrac{1}{2}q_1r_1r_{1,x} - r_1^2q_1b_x^2 - q_1r_1^2b_t. \qquad (35)$$

From (21), it can be seen that generically the hidden AKNS flows $z^{-l\sigma_3}\mathcal{W}(\boldsymbol{t}(\boldsymbol{s}))$ take place in the big cell of $\mathrm{Gr}(\mathcal{H})$. This follows from the fact that the image $W_l(\boldsymbol{t}(\boldsymbol{s}))$ of $z^{-l\sigma_3}\mathcal{W}(\boldsymbol{t}(\boldsymbol{s}))$ under the lexicographic isomorphism contains elements of arbitrary non-negative orders.

At this point it is important to consider the reductions of this flows. For the sake of illustration we will just consider the analogue of the standard AKNS reduction which leads to the nonlinear Schrödinger equation.

Let us first assume that the flows are restricted to the subspace of $\mathbb{C}^\infty$ of the form

$$\boldsymbol{t} := (\mathrm{i}t_1, \mathrm{i}t_2, \mathrm{i}t_3, \ldots) \qquad \text{with} \quad t_j \in \mathbb{R} \quad j = 1, 2, \ldots$$

and consider elements $\mathcal{W}$ in the Grassmannian satisfying the reduction condition:

$$\text{if } w = w(z) \text{ is in } \mathcal{W} \text{ then } \sigma_1\overline{w(\bar{z})} \text{ is in } \mathcal{W} \qquad (36)$$

where an overbar denotes complex conjugation and σ_1 is the first Pauli matrix. From (6), (7), (11) and (36), it is easy to see that the wavefunction associated with $\mathcal{W}$ satisfies

$$\sigma_1\overline{\Psi(\bar{z}, \boldsymbol{t})}\sigma_1 = \Psi(z, \boldsymbol{t}).$$

Thus, by taking an arbitrary submanifold $\mathcal{M}_m$ of finite codimension m in $\mathbb{R}^\infty$ given by (19), where now $b_j(\boldsymbol{s})$ $(j = 1, 2, \ldots, m)$ are real-valued functions, it is found

$$\sigma_1\overline{\Psi_{\mathrm{res}}(\bar{z}, \boldsymbol{s})}\sigma_1 = \Psi_{\mathrm{res}}(z, \boldsymbol{s}). \qquad (37)$$

In order to study how the equations in the hierarchies are reduced when the above condition is imposed, we notice that according to (37) and (25) we have

$$\sigma_1\overline{R(\bar{z}, \boldsymbol{s})}\sigma_1 = -R(z, \boldsymbol{s})$$

besides, taking into account that in terms of the new set of times $P_{\mathrm{res},1}(z, \boldsymbol{s}) = [\mathrm{i}(z^{m+1} + \sum_{j=1}^m z^j\partial_x b_j) \cdot R]_+$ it is clear that

$$\sigma_1\overline{P_{\mathrm{res},1}(\bar{z}, \boldsymbol{s})}\sigma_1 = P_{\mathrm{res},1}(z, \boldsymbol{s})$$

and consequently $\overline{r(\bar{z}, \boldsymbol{s})} = q(z, \boldsymbol{s})$, or equivalently

$$\overline{r_n(\boldsymbol{s})} = q_n(\boldsymbol{s}) \qquad n = 0, 1, \ldots, m.$$

Then, systems in the hierarchies reduce from systems of $2m + 2$ equations to systems of $m + 1$ equations. For example, for $m = 1$ and the first system in the hierarchy (32)–(35); equations (32) and (33) reduce to

$$\mathrm{i}q_{1,t} = -2\mathrm{i}q_{1,x}b_x - \mathrm{i}q_1b_{xx} + \mathrm{i}q_{0,x} - q_1|q_1|^2b_x + 2q_1b_xb_t - 2q_0b_t \\ +q_1^2\overline{q_0} + q_0|q_1|^2 + 2q_1b_x^3 - 2q_0b_x^2$$

while (34) and (35) transform into

$$\mathrm{i}q_{0,t} = \tfrac{1}{2}q_{1,xx} + 2\mathrm{i}q_1b_{xx}b_x + \mathrm{i}q_{1,x}b_x^2 - \mathrm{i}b_{xx}q_0 - \mathrm{i}b_xq_{0x} + \mathrm{i}b_{xt}q_1 + \mathrm{i}b_tq_{1x} - q_0|q_1|^2b_x + q_1|q_0|^2 \\ +\overline{q_1}q_0^2 + \tfrac{1}{2}\mathrm{i}q_{1,x}|q_1|^2 - q_1|q_1|^2b_x^2 - q_1|q_1|^2b_t.$$

3.2. Hidden AKNS flows outside the big cell

Let $\Psi(z, \boldsymbol{t})$ be a given AKNS wavefunction, from (10) we have

$$\psi_2 = -\frac{1}{q(\boldsymbol{t})}(\partial_x \psi_1 - z\psi_1).$$

Hence, provided $q(\boldsymbol{t}) \neq 0$, the element $\mathcal{W} \in \mathrm{Gr}(\mathcal{H})$ associated with Ψ can be generated in the form

$$\mathcal{W} := \mathrm{span}_{\mathbb{C}[z]}\{\psi_1(z, \boldsymbol{t}), \partial_x \psi_1(z, \boldsymbol{t}), \text{ any fixed } \boldsymbol{t} \text{ such that } q(\boldsymbol{t}) \neq 0\}.$$

Thus, the AKNS spectral problem (10) can be reduced to a second-order differential equation for ψ_1

$$\psi_{1,xx} - (\log q)_x \cdot \psi_{1,x} = (z^2 - z(\log q)_x + qr)\psi_1 \tag{38}$$

which in turn can be written as an *energy-dependent* Schrödinger spectral problem

$$\partial_{xx} f = (z^2 + zu_1 + u_0) f \tag{39}$$

with

$$u_1 = -(\log q)_x \qquad u_0 = -\frac{q_{xx}}{2q} + \frac{3}{4}(\partial_x \log q)^2 + qr \tag{40}$$

where the wavefunction f is obtained from ψ_1 by applying the Liouville transformation

$$f(z, \boldsymbol{t}) = \left(\frac{2}{q(\boldsymbol{t})}\right)^{1/2} \psi_1(z, \boldsymbol{t}) \tag{41}$$

to (38). It is also clear that

$$\mathcal{W} := \mathrm{span}_{\mathbb{C}[z]}\{f(z, \boldsymbol{t}), \partial_x f(z, \boldsymbol{t}), \text{ any fixed } \boldsymbol{t} \text{ such that } q(\boldsymbol{t}) \neq 0\}$$

and, consequently, there are decompositions of the form

$$\partial_n f = a_n(z, \boldsymbol{t}) f + b_n(z, \boldsymbol{t}) \partial_x f \qquad n \geqslant 2$$

with a_n and b_n being polynomials in z. The compatibility between these equations and (39) leads to the Jaulent–Miodek hierarchy of integrable equations for the potentials u_0, u_1 [6].

Our next goal is to characterize a special class of hidden AKNS hierarchies for which a generalization of the Jaulent–Miodek transformation (40) exists and connects them to integrable hierarchies associated with more general energy-dependent Schrödinger operators. We start again with a given AKNS wavefunction Ψ and look for those submanifolds $\mathcal{M}_m \in \mathbb{C}^\infty$

$$t_i = b_i(\boldsymbol{s}) \qquad i = 1, \ldots, m \quad \boldsymbol{s} := (x \equiv t_{m+1}, t_{m+2}, \ldots) \tag{42}$$

such that the following condition is satisfied:

$$\mathcal{W} := \mathrm{span}_{\mathbb{C}[z]}\{\psi_{\mathrm{res},1}(z, \boldsymbol{s}), \partial_x \psi_{\mathrm{res},1}(z, \boldsymbol{s}), \text{all admissible } \boldsymbol{s}\}. \tag{43}$$

According to (23) and (24) we have

$$\begin{aligned} \partial_x \psi_{\mathrm{res},1} &= p(z, \boldsymbol{s})\psi_{\mathrm{res},1} + q(z, \boldsymbol{s})\psi_{\mathrm{res},2} \\ \partial_x \psi_{\mathrm{res},2} &= r(z, \boldsymbol{s})\psi_{\mathrm{res},1} - p(z, \boldsymbol{s})\psi_{\mathrm{res},2}. \end{aligned} \tag{44}$$

Theorem 1. *The condition (43) is satisfied if and only if*

$$q_i(\boldsymbol{s}) \equiv 0 \qquad i = 1, \ldots, m. \tag{45}$$

Proof. If (45) holds, then from (23)

$$\psi_{\mathrm{res},2} = \frac{1}{q_0(\boldsymbol{s})}(\partial_x \psi_{\mathrm{res},1} - p(z, \boldsymbol{s})\psi_{\mathrm{res},1})$$

so that it is clear that (43) is satisfied.

Reciprocally, if we assume (43) then there is a decomposition of the form

$$\psi_{\mathrm{res},2} = u(z, \boldsymbol{s})\psi_{\mathrm{res},1} + v(z, \boldsymbol{s})\partial_x \psi_{\mathrm{res},1}$$

with u and v being polynomials in z. By substituting this expression in (23) and (24) we obtain

$$\partial_x \psi_{\mathrm{res},1} = (p + qu)\psi_{\mathrm{res},1} + qv\partial_x \psi_{\mathrm{res},1}.$$

This identity implies $qv = 1$ so that $q_i = 0$ $(i = 1, \ldots, m)$ and $v = q_0^{-1}$. □

We notice that according to this theorem the class of hidden AKNS hierarchies of nonlinear evolution equations for q_k, r_k $(k = 0, \ldots, m)$ satisfying (43) is characterized by a single hierarchy for b_i $(i = 1, \ldots, m), q_0, r_k$ $(k = 1, \ldots, m)$. Indeed, now the functions $b_l(\boldsymbol{s})$ characterizing the submanifold $\mathcal{M}_m$ are no longer fixed but they become dynamical variables with evolution equations which derive from the conditions $\partial_n q_i = 0, n \geqslant m+1, i = 1, \ldots, m$. For example, consider the hidden hierarchies with $m = 1$ which satisfy (43), the evolution equation corresponding to $t := t_3$ is given by (32)–(35), so that by setting $q_1 = 0$ it follows that

$$\begin{aligned}
b_t &= -\frac{1}{2}\frac{q_{0,x}}{q_0} - b_x^2 \\
q_{0,t} &= -b_{xx}q_0 - b_x q_{0,x} - q_0^2 r_1 \\
r_{0,t} &= -\frac{1}{2}r_{1,xx} - b_{xx}r_0 - b_x r_{0,x} + q_0 r_0 r_1 - \frac{1}{2}\frac{q_{0,x}}{q_0}r_{1,x} - \frac{1}{2}\left(\frac{q_{0,xx}}{q_0} - \frac{q_{0,x}^2}{q_0^2}\right)r_1 \\
r_{1,t} &= -2r_{1,x}b_x - r_1 b_{xx} + r_{0,x} + q_0 r_1^2 + \frac{q_{0,x}}{q_0}(r_0 - b_x r_1).
\end{aligned}$$

These special hidden hierarchies can be related to energy-dependent Schrödinger spectral problems as follows. Firstly, notice that as a consequence of the theorem, if (43) is satisfied then

$$\partial_{xx}\psi_{\mathrm{res},1} - \frac{q_{0,x}}{q_0}\partial_x \psi_{\mathrm{res},1} = \left(p^2 - \frac{q_{0,x}}{q_0}p + p_x + q_0 r\right)\psi_{\mathrm{res},1}. \tag{46}$$

Therefore, we can perform a Liouville transformation

$$f := \left(\frac{2}{q_0(\boldsymbol{s})}\right)^{1/2}\psi_{\mathrm{res},1} \tag{47}$$

which converts (46) into an energy-dependent Schrödinger spectral problem

$$\partial_{xx} f = \left(z^{2m+2} + \sum_{n=0}^{2m+1} u_n(\boldsymbol{s})z^n\right) f \tag{48}$$

where the potential coefficients u_n are obtained from p and q_0 by identifying powers of z in the identity

$$z^{2m+2} + \sum_{n=0}^{2m+1} u_n(\boldsymbol{s})z^n = p^2 + p_x + q_0 r - \frac{q_{0,x}}{q_0}p - \frac{q_{0,xx}}{2q_0} + \frac{3}{4}\left(\frac{q_{0,x}}{q_0}\right)^2.$$

The important point is that (47) implies

$$\mathcal{W} := \mathrm{span}_{\mathbb{C}[z]}\{f(z,\boldsymbol{s}), \partial_x f(z,\boldsymbol{s}), \text{any fixed } \boldsymbol{s} \text{ such that } q_0(\boldsymbol{s}) \neq 0\}$$

and consequently there are decompositions of the form

$$\partial_n f = a_n(z,\boldsymbol{s}) f + b_n(z,\boldsymbol{s})\partial_x f \qquad n \geqslant 2$$

with a_n and b_n being polynomials in z. The compatibility between these equations and (48) leads to a hierarchy of integrable equations for the potentials u_n $(n = 0, \ldots, 2m+1)$ [7].

The following theorem provides a method for constructing solutions for this hierarchy. Furthermore, it is useful to find the strata of the Grassmannian on which these hidden hierarchies flow.

Theorem 2. *The condition (43) is satisfied if and only if* $\psi_{\mathrm{res},1}$ *is of the form*

$$\psi_{\mathrm{res},1}(z,\boldsymbol{s}) = \Psi_0(z,\boldsymbol{t}(\boldsymbol{s}))\begin{pmatrix} 1+\mathcal{O}(1/z) \\ \mathcal{O}(1/z^{m+1}) \end{pmatrix}. \tag{49}$$

Proof. If (43) holds then as a consequence of theorem 1 the first equation in (44) reads

$$\partial_x \psi_{\mathrm{res},1} = \left(z^{m+1} + \sum_{n=0}^{m} z^n p_n(\boldsymbol{s})\right)\psi_{\mathrm{res},1} + q_0(\boldsymbol{s})\psi_{\mathrm{res},2}.$$

By using the expansions

$$\psi_{\mathrm{res},1} = \Psi_0(z,\boldsymbol{t}(\boldsymbol{s}))\begin{pmatrix} 1 + \sum_{n=1}^{\infty} a_n(\boldsymbol{s})/z^n \\ \sum_{n=1}^{\infty} c_n(\boldsymbol{s})/z^n \end{pmatrix}$$

$$\psi_{\mathrm{res},2} = \Psi_0(z,\boldsymbol{t}(\boldsymbol{s}))\begin{pmatrix} \sum_{n=1}^{\infty} d_n(\boldsymbol{s})/z^n \\ 1 + \sum_{n=1}^{\infty} e_n(\boldsymbol{s})/z^n \end{pmatrix}$$

one finds

$$\left(2z^{m+1} + \sum_{n=0}^{m} z^n (p_n + b_{nx})\right)\sum_{n=1}^{\infty}\frac{c_n(\boldsymbol{s})}{z^n} + q_0 = \mathcal{O}\left(\frac{1}{z}\right)$$

so that $c_1 = c_2 = \cdots = c_m = 0$ and (49) follows.

Reciprocally, assume that (49) is satisfied, then we may write

$$\Psi_{\mathrm{res}} = \chi_{\mathrm{res}}\Psi_0(z,\boldsymbol{t}(\boldsymbol{s})) \qquad \chi_{\mathrm{res}} := \begin{pmatrix} a(z,\boldsymbol{s}) & c(z,\boldsymbol{s}) \\ d(z,\boldsymbol{s}) & e(z,\boldsymbol{s}) \end{pmatrix}$$

where

$$a(z,\boldsymbol{s}) = 1 + \mathcal{O}\left(\frac{1}{z}\right) \qquad c(z,\boldsymbol{s}) = \mathcal{O}\left(\frac{1}{z^{m+1}}\right)$$

$$d(z,\boldsymbol{s}) = \mathcal{O}\left(\frac{1}{z}\right) \qquad e(z,\boldsymbol{s}) = 1 + \mathcal{O}\left(\frac{1}{z}\right).$$

On the other hand,

$$P_{\mathrm{res},1} = \begin{pmatrix} p(z,\boldsymbol{s}) & q(z,\boldsymbol{s}) \\ r(z,\boldsymbol{s}) & -p(z,\boldsymbol{s}) \end{pmatrix} = \left[\left(z^{m+1} + \sum_{j=1}^{m} z^j \partial_x b_j\right)\chi_{\mathrm{res}}\sigma_3\chi_{\mathrm{res}}^{-1}\right]_+. \qquad (50)$$

By direct computation one finds

$$\chi_{\mathrm{res}}\sigma_3\chi_{\mathrm{res}}^{-1} = \begin{pmatrix} 1+2\dfrac{cd}{ae-cd} & -2\dfrac{ac}{ae-cd} \\ 2\dfrac{de}{ae-cd} & -1-2\dfrac{cd}{ae-cd} \end{pmatrix} = \begin{pmatrix} 1+\mathcal{O}(1/z^{m+2}) & \mathcal{O}(1/z^{m+1}) \\ \mathcal{O}(1/z) & -1-\mathcal{O}(1/z^{m+2}) \end{pmatrix}$$

so that from (50) we find $q(z,\boldsymbol{s}) = q_0(\boldsymbol{s})$ and then, as a consequence of theorem 1, we conclude that (43) is satisfied. □

According to (49) the hidden hierarchies satisfying (43) are associated with submanifolds $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{s})$ such that

$$\tau_{\mathcal{W}}^{(1)}(\boldsymbol{t}(\boldsymbol{s})) = 0.$$

Therefore, the flow $z^{-\sigma_3}\mathcal{W}(\boldsymbol{t}(\boldsymbol{s}))$ lies outside the big cell. In order to analyse this feature let us observe that provided (43) holds then

$$\psi_{\mathrm{res},1} = \Psi_0(z,\boldsymbol{t}(\boldsymbol{s}))\begin{pmatrix} 1+\displaystyle\sum_{n=1}^{\infty} a_n(\boldsymbol{s})/z^n \\ \displaystyle\sum_{n=m+1}^{\infty} c_n(\boldsymbol{s})/z^n \end{pmatrix}$$

so that by introducing

$$\begin{aligned}\phi_{\mathrm{res},1} &:= \frac{1}{2c_{m+1}}\left(\left(z^{m+1} + \sum_{n=1}^{m} b_{nx} z^n\right)\psi_{\mathrm{res},1} - \partial_x\psi_{\mathrm{res},1}\right) \\ &= \Psi_0(z,\boldsymbol{t}(\boldsymbol{s}))\begin{pmatrix} \mathcal{O}(1/z) \\ 1+\dfrac{c_{m+2}+b_{mx}c_{m+1}}{c_{m+1}}\dfrac{1}{z}+\mathcal{O}(1/z^2) \end{pmatrix}\end{aligned}$$

we may describe $z^{-\sigma_3}\mathcal{W}$ as

$$\mathrm{span}_{\mathbb{C}[z]}\{z^{-\sigma_3}\psi_{\mathrm{res},1}(z,\boldsymbol{s}), z^{-\sigma_3}\phi_{\mathrm{res},1}(z,\boldsymbol{s}), \text{all admissible } \boldsymbol{s}\}.$$

Let us now consider the image $W_1(\boldsymbol{s})$ of $z^{-\sigma_3}\mathcal{W}(\boldsymbol{t}(\boldsymbol{s}))$ in $\mathrm{Gr}(H)$ under the lexicographic isomorphism, we have

$$W_1(\boldsymbol{s}) = \mathrm{span}_{\mathbb{C}[z^2]}\{g(z,\boldsymbol{s}), \tilde{g}(z,\boldsymbol{s})\}$$

where g and $\tilde{g}$ denote the corresponding images of $\Psi_0(z,\boldsymbol{t}(\boldsymbol{s}))^{-1}z^{-\sigma_3}\psi_{\mathrm{res},1}$ and $\Psi_0(z,\boldsymbol{t}(\boldsymbol{s}))^{-1}z^{-\sigma_3}\phi_{\mathrm{res},1}$, respectively. Two situations arise:

(a) $m = 1$. In this case

$$g = \frac{c_2}{z} + \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \qquad \tilde{g} = z^3 + \frac{c_3 + b_{1x}c_2}{c_2}z + \mathcal{O}\left(\frac{1}{z}\right)$$

so that

$$z^4 g - c_2\tilde{g} = z^2 + \mathcal{O}(z).$$

Hence, it follows easily that the set $S_{W_1(\boldsymbol{s})}$ of orders of elements in W_1 is given by

$$S_{W_1(\boldsymbol{s})} = \{-1, 1, 2, 3, \ldots\}.$$

This means that $W_1(\boldsymbol{s})$ is in the stratum Σ_1 of $\mathrm{Gr}(H)$.

(b) $m > 1$. Now we have

$$g = \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \qquad \tilde{g} = z^3 + \frac{c_3 + b_{1x}c_2}{c_2} z + \mathcal{O}\left(\frac{1}{z}\right)$$

then it follows at once that

$$S_{W_1(s)} = \{-2, 0, 2, 3, \ldots\}.$$

Therefore, $W_1(s)$ is in the stratum Σ_2 of $\mathrm{Gr}(H)$.

Appendix

We devote this appendix to check expressions (18), giving the relation between the AKNS tau-functions and the corresponding wavefunction. In order to compute the determinants involved in these expressions, we choose for the rays $|\mathcal{H}_+\rangle$ and $|\mathcal{W}\rangle = |\mathcal{W}(\mathbf{0})\rangle$:

$$\begin{aligned} |\mathcal{H}_+\rangle &= e_1 \wedge e_2 \wedge z e_1 \wedge z e_2 \wedge z^2 e_1 \wedge z^2 e_2 \wedge \cdots \\ |\mathcal{W}\rangle &= w_{0,1} \wedge w_{0,2} \wedge w_{1,1} \wedge w_{1,2} \wedge w_{2,1} \wedge w_{2,2} \wedge \cdots \end{aligned} \tag{A1}$$

where $\{e_1, e_2\}$ is the canonical basis in $\mathbb{C}^2$ and $w_{n,i}$, $i = 1, 2$, $n \geqslant 0$ denotes the element in $\mathcal{W}$ such that $(w_{n,i})_+ = z^n e_i$ (the existence and uniqueness of such an element derive from (11) and (13)). On the other hand, from (11) it is clear that $\chi_1(z, \mathbf{0}) = w_{0,1}(z)$ and $\chi_2(z, \mathbf{0}) = w_{0,2}(z)$.

By using the definition of $\mathcal{W}(\boldsymbol{t})$ one obtains at once

$$\begin{aligned} |\mathcal{W}\big(-\tfrac{1}{2}[z]\big)\rangle &= \left(\left(1 - \frac{z'}{z}\right)^{-1} E_{11} + E_{22}\right) |\mathcal{W}\rangle \\ |\mathcal{W}\big(\tfrac{1}{2}[z]\big)\rangle &= \left(E_{11} + \left(1 - \frac{z'}{z}\right)^{-1} E_{22}\right) |\mathcal{W}\rangle \\ |(z')^{-\sigma_3}\mathcal{W}\big(\tfrac{1}{2}[z]\big)\rangle &= \left(\frac{1}{z'} E_{11} + z'\left(1 - \frac{z'}{z}\right)^{-1} E_{22}\right) |\mathcal{W}\rangle \\ |(z')^{\sigma_3}\mathcal{W}\big(-\tfrac{1}{2}[z]\big)\rangle &= \left(z'\left(1 - \frac{z'}{z}\right)^{-1} E_{11} + \frac{1}{z'} E_{22}\right) |\mathcal{W}\rangle \end{aligned}$$

where as usual $(E_{11})_{ij} = \delta_{1i}\delta_{1j}$, $(E_{22})_{ij} = \delta_{2i}\delta_{2j}$ $(i, j = 1, 2)$. Now, for example, in order to obtain $\langle\mathcal{H}_+|\mathcal{W}(-\frac{1}{2}[z])\rangle$ we need to compute the scalar products determined by the basis chosen in (A1):

$$\langle z'^r e_i| \left(\left(1 - \frac{z'}{z}\right)^{-1} E_{11} + E_{22}\right) w_{s,j}(z')\rangle \qquad r, s \geqslant 0 \quad i, j = 1, 2$$

which are given by

$$\begin{aligned} &\langle z'^r e_1| \left(\left(1 - \frac{z'}{z}\right)^{-1} E_{11} + E_{22}\right) w_{s,1}(z')\rangle = z^{-r}(w_{s,1}(z))_1 - z^{s-r}\theta(s \geqslant r+1) \\ &\langle z'^r e_1| \left(\left(1 - \frac{z'}{z}\right)^{-1} E_{11} + E_{22}\right) w_{s,2}(z')\rangle = z^{-r}(w_{s,2}(z))_1 \\ &\langle z'^r e_2| \left(\left(1 - \frac{z'}{z}\right)^{-1} E_{11} + E_{22}\right) w_{s,1}(z')\rangle = 0 \\ &\langle z'^r e_2| \left(\left(1 - \frac{z'}{z}\right)^{-1} E_{11} + E_{22}\right) w_{s,2}(z')\rangle = \delta_{rs} \end{aligned} \tag{A2}$$

where

$$\delta_{rs} = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \theta(s \geqslant r) = \begin{cases} 1 & \text{if } s \geqslant r \\ 0 & \text{otherwise.} \end{cases}$$

Thus, from (A2), we have

$$\langle \mathcal{H}_+ | \mathcal{W}(-\tfrac{1}{2}[z]) \rangle$$

$$= \begin{vmatrix} (w_{0,1}(z))_1 & (w_{0,2}(z))_1 & (w_{1,1}(z))_1 - z & (w_{1,2}(z))_1 & (w_{2,1}(z))_1 - z^2 & (w_{2,2}(z))_1 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ z^{-1}(w_{0,1}(z))_1 & z^{-1}(w_{0,2}(z))_1 & z^{-1}(w_{1,1}(z))_1 & z^{-1}(w_{1,2}(z))_1 & z^{-1}(w_{2,1}(z))_1 - z & z^{-1}(w_{2,2}(z))_1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ z^{-2}(w_{0,1}(z))_1 & z^{-2}(w_{0,2}(z))_1 & z^{-2}(w_{1,1}(z))_1 & z^{-2}(w_{1,2}(z))_1 & z^{-2}(w_{2,1}(z))_1 & z^{-2}(w_{2,2}(z))_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

$$= (w_{0,1}(z))_1 = (\chi_1(z, \mathbf{0}))_1.$$

Then, by taking into account that $\langle \mathcal{H}_+ | \mathcal{W} \rangle = 1$, we obtain

$$\tau_{\mathcal{W}}^{(0)} = (\chi_1(z, \mathbf{0}))_1.$$

Proceeding in the same way one easily finds the other formulae in equation (18).

References

[1] Konopelchenko B and Martínez Alonso L 1997 *Phys. Lett.* A **236** 431

[2] Konopelchenko B, Martínez Alonso L and Medina E 1999 Singular sector of the KP hierarchy, $\bar{\partial}$-operators of non-zero index and associated integrable systems *J. Math. Phys.* to appear

[3] Mañas M, Martínez Alonso L and Medina E 1997 *J. Phys. A: Math. Gen.* **30** 4815

[4] Degasperis A, Manakov S V and Zenchuk A I 1998 *Phys. Lett.* A **249** 307

[5] Zenchuk A I 1997 *Pis. ZETF* **66** 206

[6] Jaulent M and Miodek Y 1976 *Lett. Math. Phys.* **1** 243

[7] Martínez Alonso L 1980 *J. Math. Phys.* **21** 2342

[8] Segal G and Wilson G 1985 *Loop Groups and Equations of KdV Type* (*Publ. Math. IHES* vol 61) p 5

[9] Pressley A and Segal G 1986 *Loop Groups* (Oxford: Oxford University Press)

[10] Wilson G 1985 Algebraic curves and soliton equations *Geometry Today (Rome, 1984)* (*Progress in Mathematics* vol 60) (Boston, MA: Birkhäuser) pp 303–29

[11] Bergvelt M J and ten Kroode A P E 1988 *J. Math. Phys.* **29** 1308

[12] Mañas M, Martínez Alonso L and Medina E 1999 *J. Geom. Phys.* **29** 1–2
Mañas M, Martínez Alonso L and Medina E 1999 *J. Geom. Phys.* **29** 13