Journal of Number Theory **74**, 298–306 (1999)
Article ID jnth.1998.2329, available online at http://www.idealibrary.com on IDEAL®

On the Expansion of a Product Formula Related with the Drinfeld Discriminant Function

Bartolomé López*

Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain

and

Bodo Wack

FB 9 Mathematik, Universität des Saarlandes, D-66041 Saarbrücken, Germany
E-mail: bodo@math.uni-sb.de

Communicated by D. Goss

Received March 5, 1998; revised July 27, 1998

In this work, we study the expansion of a product function U related to the Drinfeld discriminant $\Delta(z)$; U is the analogue of the classical η-function. The main result is the formula given in Theorem 3.1. From this formula, we derive the fact that the expansion of U is lacunary for $q>2$ (Theorem 3.3) and the expansion (up to a certain bound) of U valid for any q, as in [Gekeler, *Invent. Math.* **93** (1988), 667–700]. © 1999 Academic Press

INTRODUCTION

There is a classical formula for the discriminant function $\Delta(z)$,

$$(2\pi)^{-12}\,\Delta(z) = q \prod_{n\geqslant 1} (1-q^n)^{24},$$

where $q=e^{2\pi iz}$. The coefficients of the q-expansion of $(2\pi)^{-12}\,\Delta$ are given by the Ramanujan function $\tau(n)$. This function has been extensively studied, although no explicit formula is known for $\tau(n)$.

* Partially supported by DGICYT PB94-291 and a postdoctoral grant of the M.E.C. (Spain).

0022-314X/99 $30.00

In this context, it is natural to consider the expansion of the η-function, where $\eta = \prod_{n \geqslant 1} (1 - q^n)$. Here, the situation becomes much more simple, and an explicit formula can be given as

$$\prod_{n \geqslant 1} (1 - q^n) = \sum_{n \geqslant 0} (-1)^n \, q^{n(3n \pm 1)/2}. \tag{1}$$

This expansion was studied already by Euler in the frame of the theory of partitions.

In [1], Gekeler obtains the following formula for the Drinfeld discriminant function (see Section 1 for notations),

$$\bar{\pi}^{1-q^2} \Delta(z) = -t^{q-1} \prod_{\substack{a \in A \\ a \text{ monic}}} f_a(t)^{(q^2-1)(q-1)} = -t^{q-1} \cdot U(t)^{(q^2-1)(q-1)}. \tag{2}$$

As a first step in the study of the expansion of Δ, we consider the expansion of the product U (Eq. (4)).

In Section 1, we introduce the notations used along the work. In Section 2 we get recursive formulas related with U. In Section 3, we get a formula for U (Theorem 3.1). This formula does not give the explicit expansion of U, but we get from it the fact, that the expansion is lacunary, that is, with growing degree the rate of nonzero coefficients goes to zero (Theorem 3.3). This property is also satisfied by the expansion of the classical η-function (Eq. (1)). Finally, in Section 4, we give the expansion of the product U up to a certain bound.

We express our gratitude to Ernst-Ulrich Gekeler who proposed to prove the lacunarity of the expansion of U. We thank him also for his encouragement.

1. PRELIMINARIES

Let $A = \mathbb{F}_q[T]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ in the variable T. Consider a field k such that there exists a monomorphism $A \to k$. Let $k\{\tau\}$ be the ring of non-commutative polynomials over k, where τ is the Frobenius endomorphism. The ring $k\{\tau\}$ can be identified with the ring of q-additive polynomials $\sum_{i=0}^{l} \alpha_i X^{q^i}$, where the product is given by substitution.

Definition 1.1. A Drinfeld module of rank r over k is an $\mathbb{F}_q$-morphism ϕ: $A \to k\{\tau\}$ given by

$$\phi_T = T\tau^0 + \sum_{i=1}^{r} \alpha_i \tau^i,$$

where $\alpha_i \in k$ and $\alpha_r \neq 0$.

Let $K=\mathbb{F}_q(T)$. We consider the field $K_\infty=\mathbb{F}_q((1/T))$, its algebraic closure $\bar{K}_\infty$ and the completion C of $\bar{K}_\infty$.

An A-lattice in C of rank r is a discrete, finitely generated A-module $\Lambda\subset C$ such that $\dim_K K\Lambda=r$. We can associate with a lattice Λ a function

$$e_\Lambda(z)=z\prod_{\lambda\in\Lambda-\{0\}}(1-z/\lambda).$$

Through the function $e_\Lambda(z)$, we can construct a Drinfeld module ϕ_Λ of rank r. This construction establishes a bijection between lattices in C and Drinfeld modules over C.

In the rank one case, we have the *Carlitz* module given by

$$\rho_T=T\tau^0+\tau=TX+X^q.$$

Let $L=\bar{\pi}A$ be the lattice corresponding to ρ. The element $\bar{\pi}$ is well defined up to an element of $\mathbb{F}_q^*$. We consider the functions

$$t(z)=e_L(z)^{-1} \qquad \text{and} \qquad s(z)=t(z)^{q-1}. \tag{3}$$

The function $s(z)$ is an analogue to the classical function $e^{2\pi iz}$.

The A-lattices of rank two inside C are of the form $u(zA+A)$, where $u\in C^*$ and $z\in C-K_\infty$. Since homothetic lattices correspond to isomorphic Drinfeld modules, any Drinfeld module of rank two is isomorphic to one of the form

$$\phi_T=T\tau^0+g(z)\,\tau+\Delta(z)\,\tau^2,$$

where $z\in C-K_\infty$. The function $\Delta(z)$ is called the Drinfeld discriminant function.

In Eq. (2), $\Delta(z)$ is written as a product function with respect to the parameter $t(z)$. The polynomials f_a which appear in this formula are defined as follows. Let $a\in A$; consider the polynomial $\rho_a(X)$. Then we define

$$f_a(X)=\rho_a(X^{-1})\,X^{q^{\deg a}}.$$

Observe that we can consider f_a as polynomial in the variable $s=X^{q-1}$.

In the following sections, we will study the expansion of the product

$$U=\prod_{\substack{a\in A\\ a\text{ monic}}} f_a \tag{4}$$

as a formal power series in s.

2. RECURSIVE FORMULAS

Following [2], we consider the following factors of U,

$$U_d = \prod_{\substack{\deg a = d \\ a \text{ monic}}} f_a. \tag{5}$$

For practical purposes, we consider also the polynomials

$$V_d = \prod_{\substack{\deg a = d \\ a \text{ monic}}} \rho_a. \tag{6}$$

There exist recursive formulas for U_d which allow us to give some information about the expansion of U_d; especially Corollary 2.6 will be used several times in the paper. We prove the formulas for the polynomials V_d, and then we easily deduce the corresponding formulas for U_d. First, we define some intermediate products.

Notation 2.1. Let $i, n \in \mathbb{N}$, $i < n$. We denote by $G_{n,-1} = \rho_{T^n}$ and $H_{n,-1} = f_{T^n}$, and by $G_{n,i}$ and $H_{n,i}$ the products

$$G_{n,i} = \prod_{(c_0, ..., c_i) \in \mathbb{F}_q^{i+1}} \rho_{T^n + c_i T^i + \cdots + c_0},$$

$$H_{n,i} = \prod_{(c_0, ..., c_i) \in \mathbb{F}_q^{i+1}} f_{T^n + c_i T^i + \cdots + c_0}.$$

Observe that $V_d = G_{d,d-1}$ and $U_d = H_{d,d-1}$.

Consider the following operators β_k, defined recursively by

$$\gamma(Y, Z) = Y^q - YZ^{q-1},$$

$$\beta_1(X_1, X_0) = \gamma(X_1, X_0),$$

$$\beta_k(X_k, X_{k-1}, ..., X_0) = \gamma(\beta_{k-1}(X_k, X_{k-2}, ..., X_0), \beta_{k-1}(X_{k-1}, X_{k-2}, ..., X_0)).$$

The β_k satisfy the following property which is proved using an easy induction on k.

LEMMA 2.2. *Let $c \in \mathbb{F}_q$. Then*

$$\beta_k(Y + cZ, Y_{k-1}, ..., Y_0) = \beta_k(Y, Y_{k-1}, ..., Y_0) + c\beta_k(Z, Y_{k-1}, ..., Y_0).$$

LEMMA 2.3. *Let $b = T^n + c_{n-1}T^{n-1} + \cdots + c_{i+1}T^{i+1}$ be a polynomial in A. Then*

$$\prod_{(c_0, ..., c_i) \in \mathbb{F}_q^{i+1}} \rho_{b + c_i T^i + \cdots + c_0} = \beta_{i+1}(\rho_b, \rho_{T^i}, ..., \rho_1).$$

Proof. By induction on i. We have

$$\begin{aligned}\prod_{(c_0, ..., c_i) \in \mathbb{F}_q^{i+1}} \rho_{b + c_i T^i + \cdots + c_0} &= \prod_{c_i \in \mathbb{F}_q} \left(\prod_{(c_0, ..., c_{i-1}) \in \mathbb{F}_q^{i}} \rho_{b + c_i T^i + \cdots + c_0} \right) \\ &= \prod_{c_i \in \mathbb{F}_q} \beta_i(\rho_{b + c_i T^i}, \rho_{T^{i-1}}, ..., \rho_1) \\ &= \prod_{c_i \in \mathbb{F}_q} \beta_i(\rho_b + c_i \rho_{T^i}, \rho_{T^{i-1}}, ..., \rho_1).\end{aligned}$$

By Lemma 2.2, the last product equals

$$\prod_{c_i \in \mathbb{F}_q} (\beta_i(\rho_b, \rho_{T^{i-1}}, ..., \rho_1) + c_i \beta_i(\rho_{T^i}, \rho_{T^{i-1}}, ..., \rho_1)).$$

This finishes the proof. ∎

The following proposition gives a recursive formula for the products $G_{n,i}$, and in particular, for V_d. It is a consequence of Lemma 2.3.

PROPOSITION 2.4. *Let $G_{n,i}$ be as in Notation* 2.1. *We have that*

$$G_{n,i} = G_{n,i-1}^q - G_{n,i-1} G_{i,i-1}^{q-1}.$$

Now, let $H_{n,i}$ be as in Notation 2.1. Recall that f_a can be considered as a polynomial in the variable $s = X^{q-1}$.

COROLLARY 2.5. *We have that*

$$H_{n,i} = H_{n,i-1}^q - H_{n,i-1} H_{i,i-1}^{q-1} s^{q^i(q^n - q^i)}.$$

Proof. The statement follows from the identity $f_a(X) = \rho_a(X^{-1}) X^{q^{\deg a}}$ and Proposition 2.4. ∎

COROLLARY 2.6. *We have that*

$$U_d = 1 + us^{q^{2d-1} - q^{2d-2}},$$

where $u \in A[s]$.

Proof. This follows from Corollary 2.5. ∎

Remark 2.7. We have that

$$\deg_s U_1 U_2 \cdots U_d = \frac{q(q^d-1)(q^{d+1}-1)}{(q-1)(q^2-1)}.$$

By Corollary 2.6, we have that $U_{d+1} = 1 + us^{q^{2d+1}-q^{2d}}$, $u \in A[s]$. For $q > 2$, the number $q^{2d+1} - q^{2d}$ is larger than $q(q^d-1)(q^{d+1}-1)/(q-1)(q^2-1)$. This implies that the expansion of U in s has gaps between $q(q^d-1)(q^{d+1}-1)/(q-1)(q^2-1)$ and $q^{2d+1} - q^{2d}$.

3. A FORMULA FOR U

The product $V_0 V_1 \cdots V_d$ satisfies the following formula.

THEOREM 3.1. *Let V_d be as in Eq.* (6). *Let S_{d+1} be the group of permutations of the numbers* $\{0, 1, ..., d\}$. *Then*

$$V_0 V_1 \cdots V_d = \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma)\, \rho_1^{q^{\sigma(0)}} \rho_T^{q^{\sigma(1)}} \cdots \rho_{T^d}^{q^{\sigma(d)}} = \det(\rho_{T^i}^{q^j})_{i,\, j=0,\, 1,\, ...,\, d}.$$

Proof. More generally, one can prove

$$G_{i+1,\, i} G_{i+2,\, i+1} \cdots G_{n,\, n-1} = \det(G_{i+1+\nu}^{q^\mu})_{\nu,\, \mu=0,\, ...,\, n-(i+1)}$$

by induction using the formulas from Proposition 2.4. But, as Goss pointed out to us, the theorem is also an immediate consequence of the Moore determinant formula (cf. [3, Corollary 1.3.7])

$$\det \begin{pmatrix} w_0 & \cdots & w_d \\ w_0^q & \cdots & w_d^q \\ \vdots & \ddots & \vdots \\ w_0^{q^d} & \cdots & w_d^{q^d} \end{pmatrix} = \prod_{i=0}^{d} \prod_{k_{i-1} \in \mathbb{F}_q} \cdots \prod_{k_0 \in \mathbb{F}_q} (w_i + k_{i-1} w_{i-1} + \cdots + k_0 w_0)$$

taking $w_i = \rho_{T^i}$. ∎

COROLLARY 3.2. *Let U_d be as in Eq.* (5). *Then*

$$U_1 U_2 \cdots U_d = \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma)\, f_1^{q^{\sigma(0)}} f_T^{q^{\sigma(1)}} \cdots f_{T^d}^{q^{\sigma(d)}} s^{\delta(\sigma)},$$

where $\delta(\sigma) = (1/q-1) \sum_{i=0}^{d} (q^{2i} - q^{i+\sigma(i)})$.

An immediate consequence of this formula is that the expansion of U can be given independently of q (cf. Section 4).

Next, we estimate the number of non-zero terms of the expansion of U in s for $q>2$.

THEOREM 3.3. *Let the expansion of* $U=\prod f_a$ *be given as* $\sum_{i\geqslant 0} c_i s^i$ *and let* $C_m=\{i\leqslant m\colon c_i\neq 0\}$. *Suppose that* $q>2$. *If* $\alpha>\sqrt{3e}$ (*where* e *is Euler's* e) *and* $m\gg 0$, *then*

$$\# C_m \leqslant \alpha^2 m^{\log_q \alpha}.$$

In particular, because $q>\sqrt{3e}$, *we have that*

$$\lim_{m\to\infty} \frac{\# C_m}{m}=0.$$

Proof. The number of non-zero terms of the expansion of $U_1 U_2 \cdots U_d$ in s equals the number of non-zero terms of the expansion of $V_0 V_1 \cdots V_d$ in X because of $f_a(X)=\rho_a(X^{-1})\, X^{\deg a}$. Now, the polynomials ρ_{T^l} are of the form

$$\rho_{T^l}=T^l X+a_1(T)\, X^q+\cdots+a_l(T)\, X^{q^l}.$$

Thus, by Theorem 3.1, the set of exponents of non-zero terms in $V_0 V_1 \cdots V_d$ is a subset of

$$B_d=\{q^{\sigma(0)}+q^{\sigma(1)+k_1}+\cdots+q^{\sigma(d)+k_d}\colon \sigma\in S_{d+1},\, 0\leqslant k_j\leqslant j\}.$$

As $0\leqslant \sigma(j)+k_j\leqslant 2d$, we have that

$$\# B_d\leqslant \binom{(2d+1)+(d+1)-1}{d+1}=\binom{3d+1}{d+1}.$$

Consider the sequence $b_d=\binom{3d+1}{d+1}$. We have that

$$\frac{b_{d+1}}{b_d}=\frac{(3d+4)\cdots(2d+3)}{(d+2)!}\cdot\frac{(d+1)!}{(3d+1)\cdots(2d+1)}$$

$$=\frac{3d+4}{d+2}\cdot\frac{3d+3}{3d+1}\cdots\frac{2d+3}{2d+1}\leqslant 3\left(1+\frac{2}{2d+1}\right)^{d+1}.$$

Therefore,

$$\limsup_{d\to\infty}\frac{b_{d+1}}{b_d}\leqslant 3e.$$

Let $\beta(\in\mathbb{R})$ be such that $\beta>3e$. If $d\gg 0$, then

$$b_d \leqslant \beta^d.$$

Now, by Corollary 2.6, we have $U_{d+1}=1+us^{q^{2d+1}-q^{2d}}$, $u\in A[s]$, so it follows that $\# C_{(q-1)\,q^{2d}-1}\leqslant b_d$. Hence,

$$\# C_{q^{2d}}\leqslant \beta^d.$$

Thus, if $m\in\mathbb{N}$ and $m\gg 0$, then

$$\# C_m\leqslant \beta^{1+1/2\log_q m}. \quad \blacksquare$$

4. SOME TERMS OF THE EXPANSION OF U

In this section, we consider the expansion of U in s up to the term q^5+q^3. Our calculations extend the table given in [2, p. 691], and also correct some of the values given there. The calculations were made with a computer program based on the formula of Corollary 3.2; in fact, we obtained the product $U_1U_2U_3U_4$ with this program, but the calculations are too large to be included here.

The following table is valid for any $q>2$ and the coefficients of the expansion are expressed in terms of the polynomials $[n]=T^{q^n}-T$, for $n\in\mathbb{N}$.

i	c_i	i	c_i
0	1	q^3+q^2-1	$2[2]-[1]$
$q-1$	-1	q^3+q^2	$-[1][2]$
q	$[1]$	q^3+q^2+q-1	$-[2]^2+[1][2]$
q^3-q^2	-1	q^3+q^2+q	$[1][2]^2-[1]^2[2]$
q^3-1	2	q^5-q^4	-1
q^3	$[3]-[2]-[1]$	q^5-q^4+q-1	1
q^3+q-1	$-[3]-[2]+[1]$	q^5-q^4+q	$-[1]$
q^3+q	$[1][3]+[1][2]-[1]^2$	q^5-q^2	2
q^3+q^2-q-1	-1	q^5-1	-2

i	c_i
q^5	$[5]-[4]-[3]+[2]+[1]$
q^5+q-1	$-[5]+[4]+[3]+[2]-[1]$
q^5+q	$[1][5]-[1][4]-[1][3]-[1][2]+[1]^2$
$q^5+q^3-q^2-1$	-2
$q^5+q^3-q^2$	$-[5]-[4]+[3]+[2]+[1]$
q^5+q^3-q-1	2
q^5+q^3-1	$2[5]+2[4]-4[2]$
q^5+q^3	$[3][5]-[2][5]-[1][5]+[3][4]-[2][4]-[1][4]-[3]^2+[2]^2+2[1][2]+[1]^2$

Observe that different terms in the expansion may be equal for fixed q. For example, for $q=2$, we have that $q^3+q-1=q^3+q^2-q-1$. But of course, if one looks at a finite part of the expansion (as in the given table), this can only happen for finitely many q's.

REFERENCES

1. E.-U. Gekeler, A product expansion for the discriminant function of Drinfeld modules of rank two, *J. Number Theory* **21** (1985), 135–140.
2. E.-U. Gekeler, On the coefficients of Drinfeld modular forms, *Invent. Math.* **93** (1988), 667–700.
3. D. Goss, "Basic Structures of Function Field Arithmetic," Springer-Verlag, Berlin/New York, 1996.