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Self-similar blow-up for a reaction–diffusion system[1]

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Abstract

This work is concerned with the following system:

$$(\mathrm{S})\qquad \begin{cases} u_t = \Delta u - \chi\nabla(u\nabla v); & \chi>0,\\ \Delta v = -u, \end{cases}$$

which is a model to describe several phenomena in which aggregation plays a crucial role as, for instance, motion of bacteria by chemotaxis and equilibrium of self-attracting clusters. When the space dimension N is equal to three, we show here that (S) has radial solutions with finite mass that blow-up in finite time in a self-similar manner. When $N=2$, however, no radial solution with finite mass may give rise to self-similar blow-up.

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1. Introduction

Consider the equations:

$$u_t = \Delta u - \chi\nabla(u\nabla v); \quad \chi>0, \tag{1.1a}$$

$$\Delta v = -u. \tag{1.1b}$$

System (1.1) can be obtained from the pair of parabolic equations consisting of (1.1a) and:

$$v_t = D\nabla v + au - bv, \tag{1.2}$$

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where a, b and D are positive constants, when a suitable rescaling is performed and the assumption $D \gg 1$ is made (see [10]), and only terms relevant near blow-up are retained. System (1.1) (resp (1.1a), (1.2)) has been used as a model for several biological and physical problems, as for instance chemotaxis (cf. [11, 14]) and the evolution of self-attracting clusters (see [16, 17]). From a mathematical point of view, the most remarkable feature of these systems is the presence of the convective term $\chi \nabla(u \nabla v)$, that describes aggregation directed towards the origin with a velocity proportional to ∇v. In the biological model, where $u(x,t)$ represents the concentration of a species and $v(x,t)$ that of a chemical secreted by the organisms involved, this term induces a motion towards higher concentrations of the substance (chemoattractant) thus produced.

Due to the quadratic nature of the convective term in (1.1a), solutions of (1.1) may blow-up in finite time. By this we mean that sequences $\{x_n\}$ and $\{t_n\}$ exist, such that $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} t_n = T < +\infty$, and $\lim_{n\to\infty} u(x_n, t_n) = +\infty$. Blow-up for parabolic equations and systems is an important problem (both at a theoretical and a practical level), and as such it has deserved considerable attention. In general, blow-up is shown to occur either by applying the maximum principle to suitable auxiliary functions (subsolutions), or by deriving adequate differential inequalities for some integral norm(s) of the solution(s), and observing then that such quantities cease to be bounded after a finite time. This type of argument, however, does not provide insight about the manner of blow-up, i.e., about the precise asymptotic behaviour of solutions near the unfolding singularities.

Such kind of information is nevertheless crucial in many aspects. For instance, in combustion theory the form of the singularity is required to determine how the transition to a post-ignition regime takes place. Techniques suitable to describe the manner of blow-up have been developed during the last two decades, which in particular allow for rather complete analysis of some simple (but relevant) models as, for instance, semilinear parabolic equations of the type:

$$u_t = \Delta u + f(u),$$

where $f(u) = u^p$ $(p > 1)$ and $f(u) = e^u$ are typical choices for the reaction term in the equation above (see, for instance, [3–5, 12, 15] and references therein).

On the other hand, in biological problems it is important to know when does (1.1) (or (1.1a), (1.2)) have solutions exhibiting chemotactic collapse, i.e., such that they blow-up in a finite time by focusing into a multiple of Dirac's delta at the origin. These solutions actually concentrate a finite mass at the origin when blow-up happens. This last situation may be considered as a simplified model for the formation of sporae. The existence of that particular type of singularity formation has recently been shown to occur for radial solutions of (1.1) (see [6] for the bidimensional case $N = 2$, as well as [9] for $N = 3$). The corresponding analysis for the complete model (1.1a), (1.2) has been performed in [7, 8] for the case $N = 2$. We point out that these results show that, in the two-dimensional case $N = 2$, the mechanism of chemotactic collapse is substantially contained in the simplified model (1.1), which was obtained under the assumption that the diffusivity of the chemical is much faster than that of the species itself.

The results of [6, 9] point out an important feature of chemotactic collapse for (1.1) when $N = 2$ and when $N = 3$, respectively. These two cases actually correspond to different blow-up structures, but both of them are characterized by the fact that singularities unfold in a thin layer near the origin, whose width $R(t)$ is such that $R(t) \ll (T-t)^{1/2}$ for $t \sim T$. We say that this type of blow-up is not self-similar, since it does not occur over a region $y = O(1)$, where $y = x(T-t)^{-1/2}$ is the natural parabolic scaling associated to (1.1).

A question that naturally arises is that of ascertaining if (1.1) may have solutions that blow-up in a self-similar manner. In this article we shall show that the answer depends crucially on the space dimension. More precisely, our first result reads as follows:

Assume that $N=3$. *Then for any* $T>0$ *there exists a sequence* $\{\delta_n\}$ *with* $\lim_{n\to\infty}\delta_n=0$, *and a sequence of radial solutions of* (1.1), $\{u_n(r,t), v_n(r,t)\}$, *that blow up at* $r=0$ *and* $t=T$, *and are such that*

$$u_n(r,t) \text{ is self-similar,} \tag{1.3a}$$

and

$$u_n(r,T) \sim \left(\frac{8\pi}{\chi}+\delta_n\right)(4\pi r^2)^{-1} \quad \text{as } r\to 0. \tag{1.3b}$$

Notice that the solutions described in (1.3) are such that:

$$M(r,T)=\int_{|x|\leqslant r} u(x,T)\,\mathrm{d}x \to 0 \quad \text{as } r\to 0 \tag{1.4}$$

and, therefore, no concentration of mass at the origin occurs at the blow-up time. In the two-dimensional case, the situation is quite different. Namely, we have that:

$$\begin{array}{l}\textit{No radial, self-similar solutions of } (1.1) \textit{ exists}\\ \textit{such that } M(r,T)<\infty \quad \textit{as } r\to 0 \quad \textit{when } N=2.\end{array} \tag{1.5}$$

Concerning these results, a few remarks are in order. To begin with, the solutions described here are defined in the whole space $\mathbb{R}^N$ ($N=2$ or $N=3$). This is in contrast with most previous results, that are concerned with solutions defined in bounded domains. For instance, (1.1) has been often considered in bounded domains $\Omega\subset\mathbb{R}^N$ with homogeneous Neumann boundary conditions for all times for which solutions are defined. It is then known that blow-up never occurs in space dimension $N=1$ (cf. [2, 13]). The case $N=2$ is a critical one, since then a mass threshold appears that separates global existence and blow-up. More precisely, when $\Omega=B_R$ is a ball of radius $R>0$ centered at the origin, and the mass $M=(1/\pi R^2)\int_{B_R}u_0(x)\,\mathrm{d}x$ (which is preserved in time by the effect of the boundary conditions) satisfies the inequality $M<8/\chi$, radial solutions are global in time [10, 13]. However, if $M>8/\chi$, radial solutions must blow-up in a finite time [13]. In this case, a class of solutions exhibiting chemotactic collapse is known to exist (cf. [6]). Such solutions are not of a self-similar type. It should be mentioned that no complete classification of blow-up patterns has been obtained for (1.1) as yet, and therefore the possibility of other manners of singularity formation cannot be a priori discarded. We shall show herein that self-similar blow-up never occurs for radial solutions of (1.1) defined in the whole plane $\mathbb{R}^2$ and having finite mass (in the sense of (1.4)).

The case of three space dimensions exhibits a far richer structure. Not only do radial solutions exist that give raise to Dirac-delta-type blow-up, but moreover no restriction exists on the mass that these solutions focus at the blow-up point (cf. [9]).

Having already described the results provided by our analysis, and the way in which they add to the blow-up picture for system (1.1), it is perhaps appropiate to remark briefly on our approach. As it will be seen below, our technique consists in classical ODE theory and asymptotic analysis of the corresponding solutions. To simplify the presentation, as well as to highlight what we consider to be the main ingredients in our discussion, only the formal aspects of our arguments will be provided here.

We conclude this Introduction by describing the plan of the paper. The study of self-similar solutions in $N=3$ is undertaken in Section 2 below, where some basic results are established. Then in Section 3 we will study the asymptotics of the ODE that describes the self-similar solutions when a suitable bifurcation parameter approaches to zero. In Section 4 we will perform a shooting argument to show the existence of self-similar solutions with finite mass. In Section 5, we will describe a different blow-up mechanism that yields again solutions with finite mass, but not of a self-similar type. The nonexistence result (1.5) is then obtained in a short Section 6. Finally, a few concluding remarks are gathered in Section 7.

2. Self-similar solutions when $N=3$. Preliminary results

In this section we shall consider radial solutions of (1.1) in three space dimensions. Namely, we shall deal with functions $(u(r,t),\ v(r,t))$ such that:

$$u_t = u_{rr} + \frac{2u_r}{r} - \frac{\chi}{r^2}(r^2 u v_r)_r, \tag{2.1}$$

$$0 = v_{rr} + \frac{2v_r}{r} + u. \tag{2.2}$$

It is already known (cf. [13]) that system (2.1) and (2.2) possess solutions that blow-up in finite time. Our goal here consists in describing a class of such solutions which are of a self-similar type. To this end, we introduce an auxiliary mass function given by:

$$M(r,t) = \int_{|x|\leqslant r} u\,\mathrm{d}x = 4\pi \int_0^r u\rho^2\,\mathrm{d}\rho. \tag{2.3}$$

A routine check reveals that system (2.1) and (2.2) can be transformed into a single equation for $M(r,t)$, namely,

$$M_t = M_{rr} - \frac{2M_r}{r} + \chi M \frac{M_r}{4\pi r^2}. \tag{2.4}$$

We now proceed to introduce a number of auxiliary functions which will be useful in the sequel. To begin with, we consider self-similar variables given as follows:

$$\begin{aligned} &y = r(T-t)^{-1/2}, \quad \tau = -\log(T-t), \\ &M(r,t) = (T-t)^{1/2}\Phi(y,\tau). \end{aligned} \tag{2.5}$$

Here $T>0$ is fixed, but otherwise arbitrary. Actually, it will be shown that, for any choice of T, there exist solutions of (2.1) and (2.2) that blow up at $t=T$ in a manner to be described below. In terms of $\Phi(y,\tau)$, Eq. (2.4) is then transformed into:

$$\Phi_\tau = \Phi_{yy} - \left(\frac{2}{y} + \frac{y}{2}\right)\Phi_y + \frac{\Phi}{2} + \chi\Phi\frac{\Phi_y}{4\pi y^2}. \tag{2.6}$$

We shall look for solutions of (2.6) in the form:

$$\Phi(y,\tau)=yG(y,\tau), \tag{2.7}$$

so that $G(y,\tau)$ satisfies:

$$G_\tau=G_{yy}-\frac{yG_y}{2}-\frac{2G}{y^2}+\chi G\frac{G+yG_y}{4\pi y^2}. \tag{2.8}$$

Furthermore, we are interested in solutions of (2.8) which are self-similar, i.e., such that:

$$G(y,\tau)=G(y). \tag{2.9}$$

We remark on passing that we consider those limit solutions to be of a self-similar nature since they are invariant under the natural scaling corresponding to (2.1) and (2.2). One is thus led to the basic equation:

$$G''-\frac{yG'}{2}-\frac{2G}{y^2}+\frac{\chi}{4\pi y^2}(yGG'+G^2)=0 \quad \text{for } y>0. \tag{2.10}$$

Since our sought-for solutions will be locally bounded before blow-up, it is natural to impose:

$$G(0)=0. \tag{2.11}$$

We next observe that (2.10) has solutions such that:

$$G(y)\to C \quad \text{as } y\to\infty \quad \text{for any constant } C. \tag{2.12}$$

Indeed, on trying $G(y)=a+b/y^2+c/y^4+\cdots$ as $y\to\infty$, one recursively obtains from (2.10) that:

$$a \text{ arbitrary},\quad b=2a\left(1-\frac{\chi a}{8\pi}\right),\quad c=-4a\left(1-\frac{\chi a}{8\pi}\right),\ldots.$$

As a matter of fact, a standard fixed point argument can be implemented to show that (2.10) has solutions satisfying (2.12) (but not necessarily (2.11)). Notice that $C=8\pi/\chi$ is an explicit solution of (2.10).

We shall show the existence of solutions to (2.10)–(2.12) in a constructive way. To this end, we first observe that a straightforward local analysis reveals that solutions of (2.10) and (2.11) satisfy:

$$G(y)\sim Ky^2 \quad \text{as } y\to 0, \quad \text{where } K \text{ is arbitrary}. \tag{2.13a}$$

Since we are interested in nonnegative and nontrivial solutions, we may assume $K>0$. To highlight the relation between solutions of (2.10) and their corresponding behaviours near $y=0$, we now set:

$$K=\frac{1}{\varepsilon},\quad \eta=y\varepsilon^{-1/2},\quad V(\eta)=G(y). \tag{2.13b}$$

We then obtain the following equation for $V(\eta)$:

$$V''-\frac{2V}{\eta^2}+\frac{\chi}{4\pi\eta^2}(\eta VV'+V^2)-\frac{\varepsilon}{2}\eta V'=0, \tag{2.14}$$

with initial condition:

$$V(\eta)\sim\eta^2 \quad \text{as } \eta\to 0. \tag{2.15}$$

Let us denote by $V_0(\eta)$ the solutions of (2.14) and (2.15) corresponding to setting $\varepsilon=0$ in (2.14). On putting $\eta=e^s$ and $dV_0/ds \equiv \dot{V}_0 = H$, V_0 is shown to satisfy an autonomous system, namely:

$$\dot{V}_0 = H, \tag{2.16a}$$

$$\dot{H} = H + 2V_0 - \frac{\chi}{4\pi}(HV_0 + V_0^2). \tag{2.16b}$$

The corresponding phase portrait looks as indicated in Fig. 1 below:

It turns out that, when $\varepsilon=0$, (2.14) has a solution $V_0(\eta)$ such that $V_0(0)=0$, $V_0(\eta)\to 8\pi/\chi$ as $\eta\to\infty$, and $V_0(\eta)$ winds up around its limit value for $\eta\gg 1$. To obtain more information about the oscillating region of $V_0(\eta)$, we set:

$$V_0(\eta) = \frac{8\pi}{\chi} + \psi(\eta), \tag{2.17}$$

where $\psi(\eta)\to 0$ as $\eta\to\infty$. For large values of η, it is natural to retain only linear terms in the corresponding equation for ψ. We thus obtain that ψ should essentially satisfy:

$$\psi'' + \frac{2\psi'}{\eta} + \frac{2\psi}{\eta^2} = 0, \tag{2.18}$$

an equation of Euler type. Solutions of (2.18) are of the form:

$$\psi(\eta) = A_+\eta^{\alpha_+} + A_-\eta^{\alpha_-}, \tag{2.19a}$$

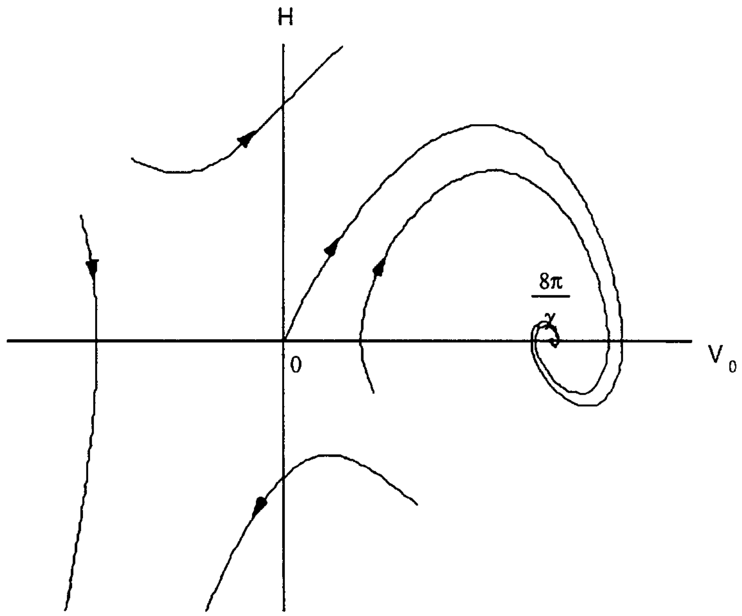

Fig. 1. Phase portrait of (2.16).

where:

$$\alpha_+ = -\frac{1}{2} + \frac{i\sqrt{7}}{2}, \quad \alpha_- = -\frac{1}{2} - \frac{i\sqrt{7}}{2}. \tag{2.19b}$$

Since $\psi(\eta)$ must be a real function, we necessarily have that $A_+ = \overline{A_-}$. We may thus write $A_+ = |A|e^{i\varphi}$, $A_- = |A|e^{-i\varphi}$, where $|A|$ and φ are uniquely determined by our choice of $V_0(\eta)$. In view of (2.19), we may then recast (2.17) in the form:

$$\begin{aligned} V_0(\eta) &= \frac{8\pi}{\chi} + |A|\eta^{-1/2}(\eta^{i\sqrt{7}/2}e^{i\varphi} + \eta^{-i\sqrt{7}/2}e^{-i\varphi}) + \cdots \\ &= \frac{8\pi}{\chi} + 2|A|\eta^{-1/2}\cos\left(\frac{\sqrt{7}}{2}\log\eta + \varphi\right) + \cdots \end{aligned} \tag{2.20}$$

when $\eta \gg 1$.

3. The case $0 < \varepsilon \ll 1$. Blow-up patterns when $N = 3$

Assume now that $\varepsilon > 0$ in (2.14) is small, and let us compare the solutions of (2.14), (2.15) with function $V_0(\eta)$ described above. A quick check reveals that the last term on the left in (2.14), while negligible for η small enough, becomes relevant when $V'' \sim -\varepsilon\eta V'/2$. This happens when $\eta \sim \varepsilon^{-1/2}$, i.e., at distances $y = O(1)$, as described in Fig. 2.

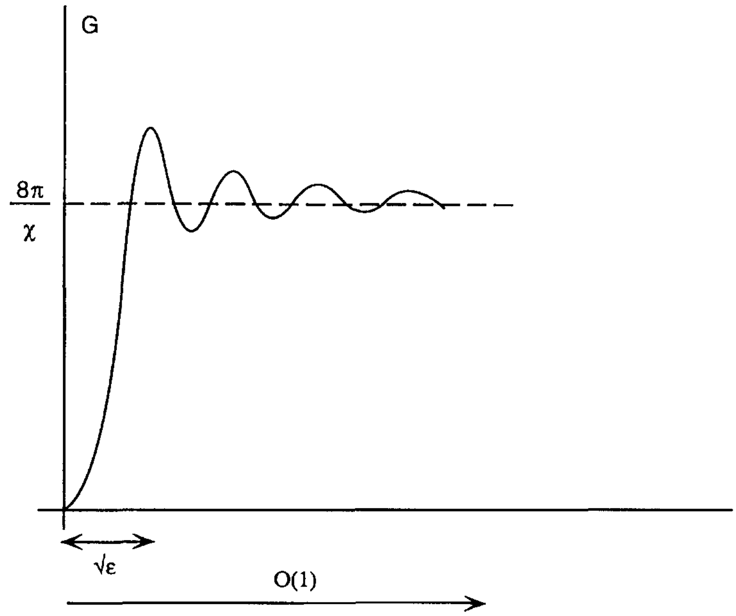

Fig. 2. Behaviour of solutions of (2.14), (2.15) in regions $y = O(1)$ when $0 < \varepsilon \ll 1$.

To unravel the subsequent asymptotics, it is then natural to set:

$$y=\eta\varepsilon^{-1/2}, \quad H(y)=\varepsilon^{-1/4}\left(V(\eta)-\frac{8\pi}{\chi}\right). \tag{3.1}$$

A straightforward computation gives then that $H(y)$ satisfies:

$$H''+\left(\frac{2}{y}-\frac{y}{2}\right)H'+\frac{2H}{y^2}+\frac{\chi\varepsilon^{1/4}}{4\pi y^2}(yHH'+H^2)=0. \tag{3.2}$$

Since we are assuming that $0<\varepsilon\ll 1$, one is led to consider first:

$$\mathscr{L}(H)\equiv H''+\left(\frac{2}{y}-\frac{y}{2}\right)H'+\frac{2H}{y^2}=0. \tag{3.3}$$

Actually, (3.3) can be recast as a confluent hypergeometric equation by setting $H(y)=s^{\alpha_+/2}F(s)$ with $s=y^2/4$ and $\alpha_+=-\frac{1}{2}+\mathrm{i}\sqrt{7}/2$. As it turns out, solutions of (3.3) may exhibit two possible behaviours as $y\to\infty$, namely,

$$H_1(y)=1+o(1) \tag{3.4}$$

or

$$H_2(y)\sim y^{-3}e^{y^2/4} \quad \text{(up to bounded, oscillatory terms)}. \tag{3.5}$$

We may therefore write the general solution of (3.3) in the form:

$$H(y)=A_1H_1(y)+A_2H_2(y), \tag{3.6}$$

for some arbitrary constants A_1 and A_2. On the other hand, when y is close to zero, Eq. (3.3) is equivalent to (2.18). It then turns out that:

$$H_1(y)\sim|B_1|y^{-1/2}\cos\left(\gamma_++\frac{\sqrt{7}}{2}\log y\right) \quad \text{as } y\to 0, \tag{3.7a}$$

$$H_2(y)\sim|B_2|y^{-1/2}\cos\left(\delta_++\frac{\sqrt{7}}{2}\log y\right) \quad \text{as } y\to 0, \tag{3.7b}$$

for some numbers $|B_1|,|B_2|$, γ_+ and δ_+ that are uniquely determined by our choice of H_1 and H_2 in (3.4), (3.5). Matching (3.7a) and (2.20) in a region where $0<y\ll 1$ and $\eta\gg 1$ gives:

$$|A_1|=\frac{2|A|}{|B_1|}; \quad \cos\left(\frac{\sqrt{7}}{2}\log y-\frac{\sqrt{7}}{4}\log\varepsilon+\varphi\right)=\pm\cos\left(\gamma_++\frac{\sqrt{7}}{2}\log y\right).$$

In particular, the second condition above allows the sought-for matching for a sequence of values $\{\varepsilon_n\}$, with $\lim_{n\to\infty}\varepsilon_n=0$, given by:

$$\varepsilon_n=\exp\left(\frac{4}{\sqrt{7}}(\varphi-\gamma_+-n\pi)\right); \quad n=1,2,3,\dots . \tag{3.8}$$

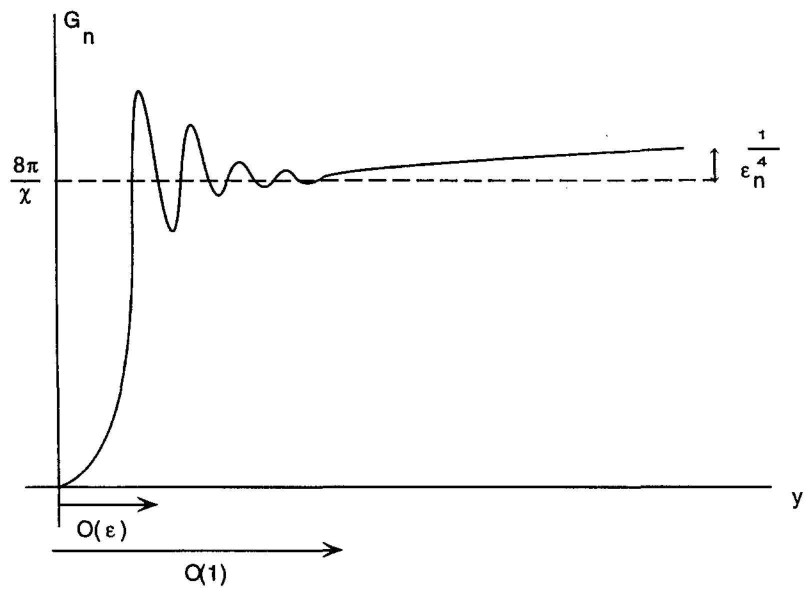

Fig. 3. The particular solutions $G_n(y)$ when $A_1>0$.

Back to the original variables, we have thus obtained a family of solutions $G_n(y)$ of (2.10)–(2.12) which are of the form:

$$G_n(y)\sim\frac{8\pi}{\chi}+A_1\varepsilon_n^{1/4}\quad\text{for } y\gg 1,\ \text{where the } \{\varepsilon_n\} \text{ are given in (3.8),}$$

$$\text{and } A_1 \text{ (positive or negative) is such that } |A_1|=2|A|/|B_1|. \tag{3.9}$$

The functions described in (3.9) and depicted in Fig. 3 above provide a countable family of blow-up patterns of the original system (2.1) and (2.2). Notice that any of these functions corresponds to a situation where several local maxima (in rescaled variables) exist right up to the blow-up time $t=T$, where all of them simultaneously collapse. Actually, the number of these maxima increases to infinity as $\varepsilon_n\to 0$.

We will show next that solutions of (3.3) with the asymptotic behaviour (3.5) do not provide suitable blow-up patterns. To check this we observe that if $H(y)$ satisfies (3.5), then, up to some algebraic factors:

$$N(H)\equiv\frac{\chi\varepsilon^{1/4}}{4\pi y^2}(yHH'+H^2)\sim\varepsilon^{1/4}e^{y^2/2}\quad\text{for } y\gg 1, \tag{3.10}$$

whence the nonlinear term in (3.2) ceases to be negligible at distances $y\sim|\log\varepsilon|^{1/2}$. We then introduce new variables by setting:

$$y=|\log\varepsilon|^{1/2}+\frac{5\log|\log\varepsilon|+u}{|\log\varepsilon|^{\frac{1}{2}}},\qquad H=\frac{|\log\varepsilon|}{\varepsilon^{1/4}}W. \tag{3.11}$$

Notice that, if (3.5) holds, then $N(H)$ in (3.10) actually satisfies $\mathrm{N}(H)\sim\varepsilon^{1/4}y^{-6}e^{y^2/2}$ when $y\gg 1$, whereas the terms on the left in (3.3) are such that $\mathscr{L}(H)\sim y^{-1}e^{y^2/4}$. On impossing $\mathrm{N}(H)\sim\mathscr{L}(H)$

one readily derives the space rescaling stated in (3.11). After a routine (but tedious) computation, we may now write the equation satisfied by $W(u)$ given in (3.11). The precise form of such equation is rather involved, but it suffices for our purposes to remark that it may be recast in the form:

$$W'' - \frac{W'}{2} + \frac{\chi W W'}{4\pi} + F(\varepsilon, u, W) = 0, \tag{3.12}$$

where

$$F(\varepsilon, u, W) = \mathrm{O}\left(\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\right) F_0(u, W)$$

and F_0 contains only algebraic terms in u, W. When $0<\varepsilon\ll 1$, we are thus led to analysing the problem:

$$W'' - \frac{W'}{2} + \frac{\chi W W'}{4\pi} = 0, \tag{3.13a}$$

$$W(u) \to 0 \quad \text{as } u \to -\infty, \tag{3.13b}$$

this last condition being a consequence of (2.15). Since (3.13a) is explicitly integrable, we may describe its solutions by means of Fig. 4.

Notice that solutions below the u-axis blow-up at finite values of u. As a matter of fact, they are of the form:

$$W(u) \sim \frac{1}{u - u_0} \quad \text{for some } u_0 < \infty.$$

Above the u-axis, one has that $W(u) \to 4\pi/\chi$ as $u \to \infty$. Consider any of these solutions. On taking, say, $u \sim |\log\varepsilon|^{1/2}$, we have that $y \sim |\log\varepsilon|^{1/2}$ and $W \sim 4\pi/\chi$. Bearing in mind (3.11), (3.1) and

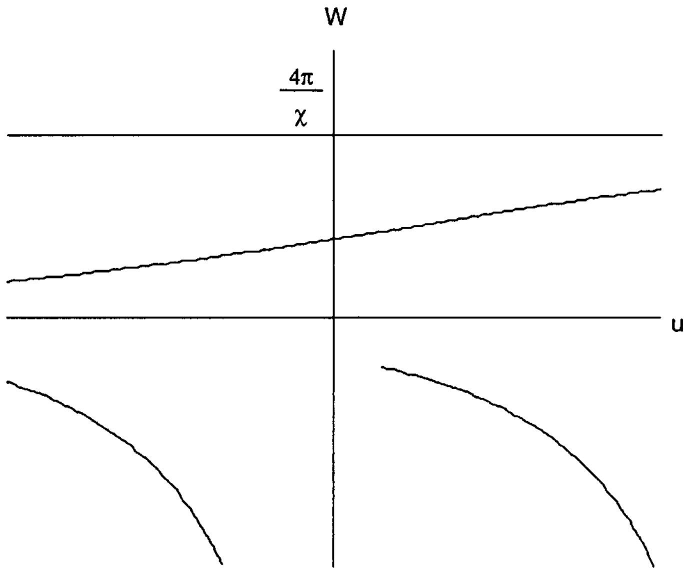

Fig. 4. The solutions of (3.3).

(2.13b), we then see that $G(y) \sim 4\pi/\chi|\log\varepsilon|^{1/2}$ for $y \sim |\log\varepsilon|^{1/2}$. On setting $y_0 = |\log\varepsilon|^{1/2}$, we deduce that the behaviour of $G(y)$ for $y > y_0$ and $(y - y_0)$ sufficiently small is encoded in the problem:

$$-\frac{yG'}{2} + \frac{\chi}{4\pi y^2}(yGG' + G^2) = 0 \quad \text{for } y > y_0, \tag{3.14a}$$

$$G(y_0) = \frac{4\pi}{\chi} y_0^2. \tag{3.14b}$$

Let us write now:

$$G = |\log\varepsilon| Q \equiv y_0^2 Q, \quad y = |\log\varepsilon|^{1/2}\xi. \tag{3.15}$$

Problem (3.14), (3.15) then transforms into:

$$-\frac{\xi\dot{Q}}{2} + \frac{\chi}{4\pi\xi^2}(\xi Q\dot{Q} + Q^2) = 0 \quad \text{for } \xi > 1, \tag{3.16a}$$

$$Q(1) = \frac{4\pi}{\chi}. \tag{3.16b}$$

Eq. (3.16a) becomes singular along the parabola $Q_1(\xi) = 2\pi\xi^2/\chi$. Analysis of (3.16) reveals that the corresponding solution decreases for $\xi > 1$ until passing through $Q_1(\xi)$ with infinite slope. Actually, if we denote by (ξ_0, Q_0) the point where $Q(\xi)$ crosses $Q_1(\xi)$, one easily sees that $Q(\xi) \sim Q_0 + k(\xi_0 - \xi)^{1/2}$ for some k as ξ approaches ξ_0. At this juncture, we therefore have to replace (3.16a) by an equation where second derivatives are retained. Plugging (3.15) into (2.10), we thus obtain

$$\frac{\ddot{Q}}{|\log\varepsilon|} - \frac{2Q}{|\log\varepsilon|\xi^2} - \frac{\xi\dot{Q}}{2} + \frac{\chi Q\dot{Q}}{4\pi\xi} + \frac{\xi Q^2}{4\pi\xi^2} = 0. \tag{3.17}$$

To analyse the evolution of $Q(\xi)$ near ξ_0, we set:

$$\xi = \xi_0 + \sigma|\log\varepsilon|^{-2/3}, \quad Q = Q_0 + \psi(\sigma)|\log\varepsilon|^{-1/3}.$$

This gives, up to first-order terms:

$$\psi'' + \frac{\chi}{4\pi\xi_0}\psi\psi' = -\frac{\chi Q_0^2}{4\pi\xi_0^2}, \tag{3.18a}$$

$$\psi \sim k(-\sigma)^{1/2} \quad \text{as } \sigma \to -\infty. \tag{3.18b}$$

Problem (3.18) has a first integral of the form:

$$\psi' + \frac{\chi}{8\pi\xi_0}\psi^2 + \frac{\chi Q_0^2\sigma}{4\pi\xi_0^2} = C,$$

whereupon one readily sees that $\psi(\sigma) \to -\infty$ as $\sigma \to \sigma_0 = \sigma_0(C)$. Summing up, we have obtained that solutions satisfying (3.5) become infinite at finite points, and therefore do not provide admissible blow-up profiles.

4. A shooting argument

In our previous section we have derived a class of self-similar blow-up patterns (cf. (3.9)). As a matter of fact, any of them can be characterized by the number of its oscillations around the value $8\pi/\chi$, a number that decreases as the parameter ε increases in the expansion for its profile near $y=0$ (cf. (2.13)). An important limit case is provided by the following explicit solution of (2.10):

$$\overline{G}(y)=\Gamma y^2 \quad \text{with } \Gamma=\frac{4\pi}{3\chi}, \tag{4.1}$$

which has a single crossing with the line $G=8\pi/\chi$. In view of (2.5), such solution has an infinite mass, as opposed to those described in (3.9). We shall next examine the behaviour of the solutions of (2.10) which start near $\overline{G}(y)$, i.e., such that:

$$G(y)\sim(\Gamma+\varepsilon)y^2 \quad \text{as } y\to 0 \text{ with } 0<|\varepsilon|\ll 1. \tag{4.2}$$

To this end, we try on (2.10) an expansion of the form:

$$G(y)=\Gamma y^2+\varepsilon\psi(y)+\cdots . \tag{4.3}$$

To the first order, we obtain:

$$\psi''-\frac{y\psi'}{6}+\left(\frac{4}{3}-\frac{2}{y^2}\right)\psi=0 \quad \text{for } y>0, \tag{4.4a}$$

$$\psi(y)\sim y^2 \quad \text{as } y\to 0. \tag{4.4b}$$

A standard asymptotic analysis reveals that solutions to (4.4a) may have two different asymptotic behaviours as $y\to\infty$, namely,

$$\psi(y)\sim y^{-9}e^{y^2/12} \quad \text{as } y\to\infty, \tag{4.5a}$$

and:

$$\psi(y)\sim y^8 \quad \text{as } y\to\infty. \tag{4.5b}$$

Actually, (4.4a) admits explicit polynomial solutions with the behaviour indicated in (4.5b). These are of the form:

$$\psi(y)=-\frac{1}{|C|}(y^8+Ay^6+By^4+Cy^2),$$

where

$$A=-162, \quad B=-42A, \quad C=-10B. \tag{4.6}$$

Let us remark on the region of validity of expansion (4.3). To this end, we observe that the second term on the right there becomes of the same order as Γy^2 when $y^6\sim 1/\varepsilon$. To analyse that region, we change variables as follows:

$$\xi=y\varepsilon^{1/6}, \quad G(y)=\varepsilon^{-1/3}Q(\xi). \tag{4.7}$$

To the first order, we then obtain the following equation for $Q(\xi)$:

$$\left(\frac{\chi Q}{4\pi\xi}-\frac{\xi}{2}\right)\dot{Q}=-\frac{\chi Q^2}{4\pi\xi^2} \quad \text{for } \xi>0, \tag{4.8a}$$

with initial condition

$$Q(\xi)\sim\left(\frac{4\pi}{3\chi}-\frac{\xi^6}{|C|}\right)\xi^2 \quad \text{as } \xi\to 0, \tag{4.8b}$$

where C is given in (4.6). Problem (4.8) can be simplified a bit by setting $Q=\xi^2 W$ there. We then obtain:

$$\xi\dot{W}=-W^2\left(W-\frac{2\pi}{\chi}\right)^{-1}-2W \quad \text{for } \xi>0, \tag{4.9a}$$

$$W(\xi)\sim\Gamma-\frac{\xi^6}{|C|} \quad \text{as } \xi\to 0. \tag{4.9b}$$

It is readily seen that the solution of (4.9) should decrease (and eventually go to zero) as $\xi\to\infty$. It is then necessary to look for a higher-order expansion in (4.3) in order to describe the asymptotics of our solutions for large y. Instead of (4.3), we therefore try:

$$G(y)=\Gamma y^2+\varepsilon\psi(y)+\varepsilon^2 f(y)+\cdots, \tag{4.10}$$

where $\psi(y)$ is given by (4.6). A routine computation then yields:

$$f''-\frac{yf'}{6}+\left(\frac{4}{3}-\frac{2}{y^2}\right)f=-\frac{\chi}{4\pi y^2}(y\psi\psi'+\psi^2) \quad \text{for } y>0, \tag{4.11a}$$

$$f(y)=o(y^2) \quad \text{as } y\to 0. \tag{4.11b}$$

Set now $f(y)=\psi(y)R(y)$. It then follows from (4.11) that:

$$e^{-y^2/12}\psi(y)^2R'(y)=-\frac{\chi}{4\pi}\int_0^y x^{-2}(x\psi'(x)+\psi(x))\psi(x)^2e^{-x^2/12}\,dx. \tag{4.12}$$

A crucial role in our argument is played by the fact that:

$$\int_0^\infty x^{-2}(x\psi'(x)+\psi(x))\psi(x)^2e^{-x^2/12}\,dx=-\frac{566784}{245}\sqrt{\pi}<0, \tag{4.13}$$

as can be readily checked by means of a computation with MAPLE V. Let us denote by M a positive generic constant, possibly changing from line to line. From (4.12) and (4.13), it turns out that:

$$R'(y)\sim\frac{Me^{y^2/12}}{y^{16}} \quad \text{as } y\to\infty \qquad R(y)\sim\frac{Me^{y^2/12}}{y^{17}} \quad \text{as } y\to\infty,$$

and recalling the fact that $f(y)=\psi(y)R(y)$, we finally obtain:

$$f(y)\sim-\frac{Me^{y^2/12}}{y^9} \quad \text{as } y\to\infty. \tag{4.14}$$

From (4.9) and (4.14), it follows that $G(y)$ admits an expansion of the form:

$$G(y) \sim \frac{4\pi}{3\chi}y^2 + \varepsilon y^8 - \frac{\varepsilon^2 M e^{y^2/12}}{y^9} + \cdots \quad \text{as } y \to \infty. \tag{4.15}$$

A new transition region thus appears when:

$$\frac{\varepsilon^2 M e^{y^2/12}}{y^9} \sim \frac{4\pi}{3\chi}y^2,$$

where the third term on the right of (4.15) overcomes the second one there, and balances $4\pi/3\chi y^2$. In order to describe that region, we set:

$$y = \lambda_\varepsilon + \frac{\xi}{|\log \varepsilon|^{1/2}}, \quad G(y) = |\log \varepsilon| W(\xi), \tag{4.16a}$$

where:

$$\lambda_\varepsilon^2 = 24|\log \varepsilon| + 66 \log|\log \varepsilon|, \tag{4.16b}$$

so that:

$$\lambda_\varepsilon \sim (24|\log \varepsilon|)^{1/2} + \frac{33}{\sqrt{24}} \cdot \frac{\log|\log \varepsilon|}{|\log \varepsilon|^{1/2}} \quad \text{when } \varepsilon \to 0.$$

Plugging (4.16) into (2.10), we obtain to the first order that $W(\xi)$ satisfies:

$$W'' - \sqrt{6}W' + \frac{\chi}{8\pi\sqrt{6}}WW' = 0 \quad \text{for } -\infty < \xi < \infty, \tag{4.17a}$$

with the following boundary condition:

$$W(\xi) \sim \frac{32\pi}{\chi} - M e^{\sqrt{2/3}\xi} \quad \text{for } \xi \to -\infty. \tag{4.17b}$$

Since (4.17) admits a first integral given by:

$$W'(\xi) - \sqrt{6}W + \frac{\chi W^2}{16\pi\sqrt{6}} = 0,$$

we readily see that any solution of (4.17) must go to $-\infty$ at a finite point. Summing up, we have obtained that any solution of (2.10) that behaves as indicated in (4.2) must become negative at a finite point y_0.

We shall now derive the following result:

There exists a sequence of values K_N such that $K_N \geqslant 4\pi/3\chi$, $\lim_{N\to\infty} K_N = \infty$, and the solution $G_N(y)$ of (2.10) satisfying $G_N(y) \sim K_N y^2$ as $y \to 0$ converges to a constant A_N as $y \to \infty$. Moreover, $G_N(y)$ crosses exactly N times the line $G^* = 8\pi/\chi$ (4.18)

To show (4.18), we shall make use of the following result concerning the asymptotic behaviour of solutions of (2.10).

Let $G(y)$ be a solution of (2.10) satisfying (2.13a). Then, if $G(y)\neq \bar{G}(y)=(4\pi/3\chi)y^2$, either $G(y)$ approaches towards a constant as $y\to\infty$, or $G(y^*)=0$ and $G'(y^*)<0$ for some finite y^*. (4.19)

To derive (4.19), we set $G=y^2W$ in (2.10) to obtain that W satisfies:

$$W''+\left(\frac{4}{y}-\frac{y}{2}\right)W'-W+\frac{\chi}{4\pi}(3W^2+yWW')=0. \tag{4.20}$$

We now claim that if $W(y)$ is a solution of (4.20) such that:

$$W(y_0)>\frac{4\pi}{3\chi},\quad W'(y_0)=0\quad \text{for some } y_0 \text{ large enough,} \tag{4.21}$$

then $W(y)$ becomes negative at some finite $y_1>y_0$. To see this, we rewrite (4.20) and (4.21) by rescaling the space variable in the form $y=y_0\xi$. We thus obtain:

$$\frac{1}{y_0^2}\left(\ddot{W}+\frac{4\dot{W}}{\xi}\right)-W-\frac{\xi}{2}\dot{W}+\frac{\chi}{4\pi}(3W^2+\xi W\dot{W})=0, \tag{4.22a}$$

$$W(1)=\alpha>\frac{4\pi}{3\chi},\quad \dot{W}(1)=0. \tag{4.22b}$$

Suppose first that $\alpha>2\pi/\chi$. Near $\xi=1$, $|\frac{4\dot{W}}{\xi y_0^2}|\ll 1$, since we are assuming $y_0\gg 1$, and therefore the dominant terms in (4.22a) are:

$$\frac{\ddot{W}}{y_0^2}=\left(W-\frac{3\chi}{4\pi}W^2\right)+\xi\left(\frac{1}{2}-\frac{\chi W}{4\pi}\right)\dot{W}\equiv I_1+I_2\dot{W}. \tag{4.23}$$

For $i=1,2$, let us write A_i to denote the value of I_i at $\xi=1$. Since $A_1<0$ and $A_2<0$ (by our choice of α), solutions of (4.23) are in the form:

$$W(\xi)=-\tfrac{A_1}{A_2}\xi+C_1+C_2e^{A_2y_0^2\xi}\quad \text{for } \xi>1 \text{ and } (\xi-1) \text{ small enough.} \tag{4.24}$$

In view of (4.24), we deduce that $W(\xi)$ quickly approaches towards the straight line $W(\xi)=-(A_1/A_2)\xi+C_1$. Once this happens, the second derivative in (4.23) becomes negligible, and that equation reduces to:

$$W-\frac{3\chi}{4\pi}W^2+\xi\left(\frac{1}{2}-\frac{\chi W}{4\pi}\right)\dot{W}=0. \tag{4.25}$$

Eq. (4.25) drives then the solution under consideration down to the line $\overline{W}=2\pi/\chi$. That line is hit upon at some point ξ_0, and there second derivatives become important again. Near $\xi=\xi_0$, (4.22a) reduces asymptotically to:

$$\frac{\ddot{W}}{y_0^2}=-A+\xi_0\left(\frac{1}{2}-\frac{\chi W}{4\pi}\right)\dot{W}, \tag{4.26}$$

where A is the value of $(-I_1)$ at $\xi=\xi_0$. An integration of (4.26) gives:

$$\frac{\dot{W}}{y_0^2}=-A\xi+C-\frac{2\pi\xi_0}{\chi}\left(\frac{1}{2}-\frac{\chi W}{4\pi}\right)^2, \tag{4.27}$$

for some constant C, which shows that $W(\xi)$ will continue its way down until crossing $\overline{W}=4\pi/3\chi$ at a point $\xi=\xi_1>\xi_0$. Near $\xi=\xi_1$, (4.23) will have the form:

$$\frac{\ddot{W}}{y_0^2}=\xi_1\left(\frac{1}{2}-\frac{\chi W}{4\pi}\right)\dot{W}, \tag{4.28}$$

whose integration gives $\dot{W}/y_0^2=-(2\pi\xi_1/\chi)(\frac{1}{2}-(\chi W/4\pi))^2$, which shows that $W(\xi)$ will then decrease until eventually becoming zero at some $\xi_2>\xi_1$. If $4\pi/3\chi<\alpha<2\pi/\chi$, then the last term in (4.23) is positive at $\xi=1$, and (4.24) is to be replaced by:

$$W(\xi)=\frac{A}{B}(\xi-1)+C_1(1-e^{By_0^2(\xi-1)})+\alpha,$$

for some positive constants A, B and C_1, and this implies that $W(\xi)$ will go down to the line $W=0$, which is reached at some value $\xi^*<\infty$. As a next step, we observe that:

If $W(\xi)$ is bounded, and $W(\xi)\neq 4\pi/3\chi$, then either $W(\xi)\to 0$ as $\xi\to\infty$, or otherwise there exists y_0 such that (4.21) holds. (4.29)

To check (4.29), we may reduce ourselves to the case where $W(\xi)<4\pi/3\chi$ for all ξ. Such solutions need necessarily be monotone. Indeed, by (4.20) any point ξ where $W'(\xi)=0$ should be a minimum, and solutions would be monotonically increasing afterwards. It then turns out that $W(\xi)\to C$ as $\xi\to\infty$, and an inspection of the equation reveals then that either $C=4\pi/3\chi$ or $C=0$, whence (4.29). Let us conclude now the argument leading to (4.19). If $G(y)$ is bounded and global, then (2.10) has the form:

$$G''-\frac{yG'}{2}=O\left(\frac{1}{y^2}\right)\quad\text{for } y\gg 1,$$

which shows that $G(y)\to C$ as $y\to\infty$ for some constant C. Suppose now that $W(y)$ is not bounded. Then, for $y\gg 1$, $W(y)>2\pi/\chi$, and Eq. (4.20) has asymptotically the form:

$$W''+\gamma yW'+\delta W=0,$$

for some positive γ and δ. Then a point $y_1>0$ will exist where $W'(y_1)=0$ and $W(y_1)>2\pi/\chi$, and our previous argument implies that $W(y_2)=0$ at some $y_2>y_1$. Suppose finally that $W(y)\to 0$ as $y\to\infty$. Then Eq. (4.20) reduces when $y\gg 1$ to:

$$W''-\frac{y}{2}W'-W=0,$$

and a local analysis shows then that $W(y)\sim C/y^2$ when $y\to\infty$. This concludes the derivation of (4.19). Once the possible asymptotic behaviours of $G(y)$ for large y have been obtained, our basic result in this section is derived as follows. Let us start from the explicit solution $\overline{G}(y)=(4\pi/3\chi)y^2$.

We have seen in Section 4 that if we consider solutions $G(y)$ such that $G(y)\sim(4\pi/3\chi+\varepsilon)y^2$ when $y\to 0$ for $0<|\varepsilon|\ll 1$, then these solutions cross twice the line $G^*=8\pi/\chi$ and then hit the line $G=0$ with negative slope. Let ε_2 denote the supremum of those $\varepsilon>0$ such that the solution $G(y)$ of (2.10) satisfies $G(y)\sim(4\pi/3\chi+\varepsilon)y^2$ as $y\to 0$, and G crosses twice the line $G^*=8\pi/\chi$. We claim that the function $G_2(y)$ such that $G_2(y)\sim(4\pi/3\chi+\varepsilon_2)y^2$ as $y\to 0$ cannot vanish at any finite value of y, since that would contradict the definition of ε_2. Then we necessarily have that $G_2(y)\to C_2$ as $y\to\infty$, for some constant C_2. Since solutions of (2.10) such that $G(y)\sim Ky^2$ as $y\to 0$ will cross the line $G^*=8\pi/\chi$ as many times as wished when $K\to\infty$, a repetition of our previous argument yields that, any time that the number of intersections changes, a new solution $G(y)$ approaching towards a constant when $y\to\infty$ will appear, and (4.18) follows.

5. Non-self-similar blow-up

In this section, we shall describe a family of blow-up profiles with finite mass that are obtained by linearizing around the explicit solution (4.1) which has infinite mass itself. This mechanism is closely related to that already observed in some semilinear parabolic equations (cf., for instance, [15]). To this end, we start from Eq. (2.8), namely,

$$G_\tau = G_{yy} - \frac{yG_y}{2} - \frac{2G}{y^2} + \frac{\chi G}{4\pi y^2}(G + yG_y). \tag{5.1}$$

Let us try now in (5.1) an expansion of the type:

$$G(y,\tau) = \frac{4\pi}{3\chi}y^2 - M\mathrm{e}^{\lambda\tau}\psi(y) + \cdots . \tag{5.2}$$

Retaining only first-order terms we then arrive at:

$$\psi'' - \frac{y\psi'}{6} + \left(\frac{4}{3} - \lambda - \frac{2}{y^2}\right)\psi = 0. \tag{5.3}$$

Standard asymptotic techniques reveal that solutions of (5.3) behave as $y\to\infty$ in one of the following forms. Either:

$$\psi(y) \sim y^{-9+6\lambda}\mathrm{e}^{y^2/12}, \tag{5.4}$$

or

$$\psi(y) \sim y^{8-6\lambda}. \tag{5.5}$$

We shall henceforth focus on solutions satisfying (5.5). To begin with, since (5.3) is invariant when changing y by $(-y)$, only even powers will appear in (5.5). It then turns out that the eigenvalue λ in (5.2), which has to be selected nonpositive, has the form:

$$\lambda = \tfrac{1}{3}(4-n), \quad n = 4,5,6,\ldots \tag{5.6}$$

Let us consider first the case where $n \geqslant 5$. Then (5.2) reads:

$$G(y,\tau) = \frac{4\pi}{3\chi}y^2 - M\mathrm{e}^{(4-n/3)\tau}y^{2n} + \cdots \quad \text{as } y\to\infty. \tag{5.7}$$

Both terms on the right of (5.7) become of same order at distances $y \sim e^{(n-4/6(n-1))\tau}$. This motivates the change of variables:

$$y = e^{(n-4/6(n-1))\tau}\xi, \quad G = e^{(n-4/3(n-1))\tau}Q(\xi,\tau). \tag{5.8}$$

We now substitute (5.8) into (5.1) to obtain that to the first order, when $\tau \gg 1$, Q satisfies:

$$\left(\frac{n-4}{3(n-1)}\right)Q + \left(\frac{1}{2} - \frac{n-4}{6(n-1)}\right)\xi\dot{Q} = \frac{\chi Q(Q+\xi\dot{Q})}{4\pi\xi^2} \tag{5.9}$$

with initial condition:

$$Q(\xi) \sim \frac{4\pi}{3\chi}\xi^2 - M\xi^{2n} \quad \text{as } \xi \to 0. \tag{5.10}$$

Analysis of (5.9), (5.10) can be done in a similar way as that corresponding to (4.8). As in that case, we may scale out the factor ξ^2 by setting $Q = \xi^2 W$. Eq. (5.9) transforms then into:

$$\left(\frac{2n+1}{6(n-1)} - \frac{\chi}{4\pi}W\right)(2W + \xi\dot{W}) = \frac{\chi W^2}{4\pi} - \frac{n-4}{3(n-1)}W.$$

As it happened in (4.9), solutions to the equation above decrease to zero as $\xi \to \infty$. When $\xi \gg 1$, the relevant terms there are the following:

$$\left(\frac{2n+1}{6(n-1)}\right)\xi\dot{W} = -W,$$

which determines a behaviour $W(\xi) \sim C\xi^{(-6(n-1)/2n+1)}$ for some $C > 0$ and for large ξ. In terms of the original variables (cf.(2.3), (2.5)) such profiles correspond to a mass concentration at $r = 0$ given by

$$M(r,T) \sim Cr^{9/(2n+1)} \quad \text{as } r \to 0. \tag{5.11}$$

It remains yet to discuss the case $n = 4$ in (5.6). We now expect (5.7) to be replaced by

$$G(y,\tau) = \frac{4\pi}{3\chi}y^2 + a(\tau)\psi(y) + \cdots \tag{5.12}$$

where $a(\tau)$ is an amplitude coefficient to be determined, and $\psi(y)$ is the polynomial given in (4.6). Trying (5.12) on (5.1), we obtain to the first order that, when $y \gg 1$:

$$\dot{a}(\tau)\psi(y) = \frac{\chi(a(\tau))^2\psi(y)}{4\pi y^2}(\psi(y) + y\psi'(y)). \tag{5.13}$$

Let us denote by $\langle f,g\rangle = \int_0^\infty f(y)g(y)e^{-y^2/12}\,dy$. We then deduce from (5.13) that

$$\dot{a}(\tau) = \frac{\chi(a(\tau))^2}{4\pi\langle\psi,\psi\rangle}\langle y^{-2}\psi(\psi + y\psi'),\psi\rangle.$$

Bearing in mind (4.13), as well as the fact that

$$\langle\psi,\psi\rangle = \frac{576}{35}\sqrt{3\pi},$$

which follows easily from computation with MAPLE V, we readily obtain that

$$a(\tau) \sim \frac{K}{\tau} \quad \text{for} \quad \tau \gg 1,$$

where

$$K = \frac{7\pi\sqrt{3}}{246\chi}.$$

On the other hand, condition (5.7) is now replaced by

$$G(y,\tau) = \frac{4\pi}{3\chi}y^2 - \frac{K_1 y^8}{\tau} + \cdots \quad \text{for some constant } K_1 \text{ as } y \to \infty.$$

This suggests that, as y increases, a transition region should appear which may be adequately analysed in the new scales:

$$y = \xi\tau^{1/6}, \quad G = \tau^{1/3}Q.$$

We then obtain that the asymptotics of W is described by:

$$-\frac{\xi\dot{Q}}{2} + \frac{\chi Q}{4\pi\xi^2}(Q + \xi\dot{Q}) = 0$$

with initial condition:

$$Q(\xi) \sim \frac{4\pi}{3\chi}\xi^2 - K\xi^8 \quad \text{as } \xi \to 0.$$

We may again scale out the factor ξ^2 by setting $Q(\xi) = \xi^2 W(\xi)$. Arguing as in the previous case $n \geqslant 5$, we then obtain that:

$$W(\xi) \sim \frac{C}{\xi^2} \quad \text{as } \xi \to \infty$$

for some $C > 0$. In terms of the variable $G(y,\tau)$, this means:

$$G(y,\tau) \sim C\tau^{1/3}.$$

Take now τ_0 large but fixed. Then, by (2.5), $\Phi(y,\tau_0) \sim C|y|\tau_0^{1/3}$. If we select y such that $|y| = A = O(1)$, then $r = Ae^{-\tau_0/2}$, whence $\tau_0 \sim |\log r|$. If we now take $\Phi(y,\tau_0)$ as an initial value at $\tau = \tau_0$, Eq. (2.6) reads:

$$\Phi_\tau = \frac{\Phi}{2} + (\text{lower-order terms}) \quad \text{for } \tau > \tau_0 \quad \text{when } |y| \geqslant A.$$

Therefore,

$$\Phi(y,\tau) \sim C|y|(2|\log r|)^{1/3} e^{(\tau-\tau_0)/2},$$

whence

$$M(r,T) \sim (2^{1/3}C)r(|\log r|)^{1/3} \quad \text{for } 0 < r \ll 1.$$

6. Nonexistence of self-similar blow-up patterns when $N=2$

In this section we shall consider radial solutions of (1.1) in two space dimensions. Eqs. (2.1) and (2.2) are to be replaced now by:

$$u_t = u_{rr} + \frac{u_r}{r} - \frac{\chi}{r}(ruv_r)_r, \tag{6.1}$$

$$v_{rr} + \frac{v_r}{r} + u = 0. \tag{6.2}$$

As in the case $N=3$, we introduce a mass function $M(r,t)$ given by:

$$M(r,t) = \int_{|x|\leqslant r} u\,\mathrm{d}x = 2\pi\int_0^r u\rho\,\mathrm{d}\rho. \tag{6.3}$$

We next change variables as follows:

$$y = r(T-t)^{-1/2}, \quad \tau = -\log(T-t); \quad M(r,t) = \Phi(y,\tau), \tag{6.4}$$

so that Φ satisfies:

$$\Phi_\tau = \Phi_{yy} - \frac{y\Phi_y}{2} + \left(\frac{\chi\Phi}{2\pi} - 1\right)\frac{\Phi_y}{y}. \tag{6.5}$$

Any self-similar behaviour near a blow-up time $t=T$ would then correspond to a nontrivial solution of the equation:

$$\Phi'' - \frac{y\Phi'}{2} + \left(\frac{\chi\Phi}{2\pi} - 1\right)\frac{\Phi'}{y} = 0. \tag{6.6}$$

A quick check reveals that:

$$\overline{\Phi}(y) = \frac{\pi}{\chi}y^2 \tag{6.7}$$

is an explicit solution of (6.6). Since $\overline{\Phi}$ has infinite mass, it may be considered as the analogue of (4.1) when $N=2$. Concerning finite-mass solutions, the following negative result holds:

There exists no nontrivial, nonnegative and bounded solution of (6.6) that is defined for all $y>0$. (6.8)

To check (6.8) we first observe that global solutions of (6.6) must be monotone. Indeed, if $\Phi'(y_0)=0$ at some $y_0>0$, then necessarily $\phi''(y_0)=0$ there, and that solution should necessarily be constant for $y>y_0$. However, in that case, only the first two terms in (6.6) would be relevant for $y\gg 1$, and these would imply an exponential growth as $y\to\infty$, which is a contradiction. This shows (6.8), whereupon the nonexistence of finite-mass patterns follows.

7. Concluding remarks

It follows from our analysis here that self-similar blow-up for radial solutions of (1.1), which is a simple but representative model for diffusion and aggregation, depends crucially on the space dimension N. Restricting our attention to finite-mass solutions, it turns out that when $N=2$ no self-similar blow-up may occur. In view of previously known results, finite-time focusing into a Dirac mass is the only type of blow-up behaviour that is known so far in this case.

In three space dimensions, however, self-similar blow-up may actually take place, although no mass concentration at the blow-up point occurs in such situation. Since chemotactic collapse has been shown to appear in that situation (cf. [9]), we see that the case $N=3$ allows for a richer variety of blow-up patterns than the bidimensional situation $N=2$.

Besides the cases referred to above, no other manners of blow-up for (1.1) are known as far as we know. It would be therefore interesting to derive a complete classification of blow-up patterns, similar to that obtained in [15] for the case of a single scalar reaction–diffusion equation.

On the other hand, only radial solutions have been considered here, as in most of the references listed below. Any information concerning non-radial blow-up profiles would be certainly of interest. A possibly related question worth to be examined would be the derivation of blow-up results at the boundary of a given domain for suitable initial values. All these questions are currently under intense study, and will hopefully yield interesting developments in the future.

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