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Symmetry reductions for a nonlinear diffusion-absorption equation in two spatial dimensions

M. L. GANDARIAS and L. DEL AGUILA

Departamento de Matemáticas, Universidad de Cádiz
P.O.Box 40, 11510 Puerto Real, Cádiz, Spain

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Abstract. – The complete Lie algebra of classical infinitesimal symmetries of the nonlinear two-dimensional (2D) diffusion-absorption equation is presented. The functional forms of absorption for which the two-dimensional diffusion-absorption equation can be fully reduced to an ordinary differential equation by classical Lie simmetries are derived. The two-dimensional optimal system is used to generate some new reductions of the 2D partial differential equation to ordinary differential equations. Some of these ordinary differential equations can be interpreted in terms of finite-time blow-up processes for the radial and the one-dimensional problem.

Introduction. – Nonlinear reaction-diffusion equations in their diverse forms serve as mathematical models to a number of interesting physical phenomena occurring in various research fields, as neurodynamics, developmental biology, chemical reactions, ecology, plasma physics etc. The generalised diffusion equation

$$T_t = (D_1(T)T_x)_x + a(D_2(T))_x + b(x,t)D_3(T), \tag{1}$$

where $T(x,t)$ denotes the temperature at a point, a is an arbitrary constant, D_1, D_2 and D_3 are arbitrary functions of temperature T and $b(x,t)$ is another arbitrary function of x and t, has been analysed via isovector approach, and some new exact solutions have been obtained by Bhutani [1]. For the case where $D_1(T) = r_0$, a constant parameter, $b(x,t) = k$, a constant and $D_3(T) = T(1 - T^{\alpha_0})$, $0 < \alpha_0 < \infty$, (1) reduces to the generalized Fisher's equation $T_t = r_0 T_{xx} + kT(1 - T^{\alpha_0})$; this equation describes the nonlinear evolution of population T in one-dimensional habitat. On taking $D_1(T) = 1$, $b(x,t) = 1$ and $a = 0$ in (1), we arrive at the reaction-diffusion equation

$$T_t = T_{xx} + D_3(T), \tag{2}$$

that has many applications in the area of biomathematics. A complete group classification for (2) was derived by Dorodnitsyn [2]. Classical and nonclassical symmetries of (2) are considered by Clarkson and Mansfield [3] by using the method of differential Grobner bases, and by Arrigo *et al.* [4] constructing several new exact solutions.

In [5] a group classification problem for the nonlinear diffusion equation with absorption and convection

$$u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x \tag{3}$$

was solved, by studying those spatial forms which admit the classical symmetry group. When $m = s + 1$ and $mg(x) = f'(x)$, eq. (3) can be written in a conserved form and potential (nonlocal) symmetries were derived in [6]. Nonclassical symmetry reductions, as well as exact solutions have been obtained in [7] for (3) when $f = 0$.

Although classical point symmetries as well as nonclassical symmetries of reaction-diffusion equations with two spatial dimensions have been obtained [8], [9], very few of these have been used to reduce the equation to ODEs, as well as to obtain exact solutions. These symmetries have only led to PDEs among a reduced number of variables. The model equation to be considered here is

$$u_t = u_{xx} + u_{yy} + f(u), \tag{4}$$

which is a diffusion-absorption equation. The third term on the right side is a nonlinear reaction rate $f(u)$ which represents volumetric absorption that in the case of plasma is caused by radiation to which plasma is trasparent. The machinery of Lie group theory provides the systematic method to search for group-invariant solutions. For PDEs with three independent variables, like eq. (4), a single group reduction transforms the original PDE into another PDE with two independent variables. However, if the two-dimensional nonlinear absorption-diffusion equation has two nontrivial symmetries X and Y, which are compatible because they obey the simple commutation property $[X,Y] = Y$, then this equation can be fully reduced to an ODE by classical Lie symmetry reductions. These ODEs are generally easier to solve than the original PDE. Besides, some important properties of a certain class of solutions can be derived from those equations and the similarity variables that reduce the PDE in a particular ODE. Most of the required theory and description of the method can be found in [10], [11].

The structure of the work is as follows: In the second section we study the Lie symmetries of eq. (4) and we list the functions $f(u)$ for which we obtain the Lie group of point transformations admitted by the corresponding equation, its Lie algebra as well as the corresponding one-dimensional and two-dimensional optimal systems. In the third section we report the reduction to ODEs obtained from the two-dimensional optimal system of subalgebras. Some of these ODEs can be related with finite-time blow-up for the radial and the one-dimensional problems; we derive for these cases solutions describing blow-up processes with the property that, if we interpret u as a concentration, the associated mass is finite in a bounded region near a blow-up point.

Lie symmetries and optimal systems. – We consider the classical Lie group symmetry analysis of the class of the (2D) equation (4). Invariance of this equation under a Lie group of point transformations with infinitesimal generator

$$V = p(x,y,t,u)\partial_x + q(x,y,t,u)\partial_y + s(x,y,t,u)\partial_t + r(x,y,t,u)\partial_u \tag{5}$$

leads to a set of determining equations which are linear partial differential equations in p, q, s and r. For totally arbitrary $f(u)$, the only symmetries are the group of the space and time traslations and the group of rotations which are defined by the infinitesimal generators

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = \frac{\partial}{\partial t}, \quad V_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{6}$$

The only functionals forms of $f(u)$ with $f(u) \neq$ constant and $f(u)$ nonlinear which have extra symmetries are the following listed in table I. We must remark that cases $i = 1$ and $i = 3$ with $b \neq 0$ are missing in [2], [9]. Unlike the two-dimensional diffusion-convection equation, that cannot be reduced to an ODE by classical Lie symmetries except for some cases involving constant diffusivity, the nonlinear two-dimensional diffusion-absorption equation can be fully

TABLE I. – *Symmetries for the 2D diffusion-absorption equation.*

$f(u)$	V_j
$c(au+b)^n$	$V_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + \frac{2}{1-n}\left(u+\frac{b}{a}\right)\frac{\partial}{\partial u}$
$ce^{(au+b)}$	$V_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - \frac{2}{a}\frac{\partial}{\partial u}$
$cu(\log(au)+b)$	$V_5 = e^{ct}\frac{\partial}{\partial x} - \frac{ce^{ct}}{2}xu\frac{\partial}{\partial u}$
$cu(\log(au)+b)$	$V_6 = e^{ct}\frac{\partial}{\partial y} - \frac{ce^{ct}}{2}yu\frac{\partial}{\partial u}$
$cu(\log(au)+b)$	$V_7 = e^{ct}u\frac{\partial}{\partial u}$

reduced to an ODE by classical Lie symmetry reductions for all the functionals forms of $f(u)$ listed in table I.

We desire to minimize the search for group-invariant solutions to that of finding non-equivalent branches of solutions, which leads to the concept of *optimal systems* of group-invariant solutions, from which every other solution can be derived. For further details and proofs see [12], [13], [10]. The one-dimensional optimal system for $f(u) = c(au+b)^n$ and $f(u) = ce^{au+b}$ is

$$\{\alpha V_1 + \beta V_3, \quad \alpha V_3 + \beta V_4, \quad \alpha V_4 + \beta V_5\} \tag{7}$$

with α and β arbitrary constants.

In order to construct the two-dimensional optimal system, we form a list of two-dimensional algebras $\mathcal{G}$ $\{w_1, w_2\}$. For each element w in (7), we set $w_1 = w$ and choose w_2 as a linear combination of all elements of $Nor(w_1)/w_1$, where Nor denotes the normalizer. Each pair of elements of the list of two-dimensional algebras will be simplified as much as possible using the adjoint transformation $Ad_{\exp(\epsilon v)}\langle w\rangle \equiv e^{-\epsilon v}we^{\epsilon v}$, which contains information about how group-invariant solutions transform under the action of other groups. In the multidimensional case the new elements can be, in general, linear combinations of the transformed elements. For the two-dimensional case we construct the adjoint transformation matrix, and we separate the

TABLE II. – *Each row shows the infinitesimal generators of the optimal system, the corresponding similarity variables and similarity solutions corresponding to* $f(u) = -(au+b)^n$. *Here* $n-1 = s$ *and* $\gamma = \alpha^2 + \beta^2$.

i	U_i	z_i	u_i	ODE
1	$\{V_4, V_5 - 2TV_3\}$	$\sqrt{\frac{x^2+y^2}{T-t}}$	$(T-t)^{-\frac{1}{s}}\phi(z) - \frac{b}{a}$	$\phi'' + \left(\frac{1}{z} - \frac{z}{2}\right)\phi' -$
				$-\frac{1}{s}\phi - (a\phi)^{s+1} = 0$
2	$\{V_2, V_5 - 2TV_3\}$	$\frac{x-x_0}{\sqrt{T-t}}$	$(T-t)^{-\frac{1}{s}}\phi(z) - \frac{b}{a}$	$2\phi'' - z\phi' - \frac{2}{s}\phi - 2(a\phi)^{s+1} = 0$
3	$\{V_1, \alpha V_2 + \beta V_3\}$	$\beta y - \alpha t$	$\phi(z) - \frac{b}{a}$	$\alpha\phi' + \beta^2\phi'' - (a\phi)^{s+1} = 0$
4	$\{V_3, \alpha V_4 + \beta V_5\}$	$\alpha\log(x^2+y^2)$	$\phi(z)(x^2+y^2)^{-\frac{1}{s}} - \frac{b}{a}$	$4\gamma s^2\phi'' - 8\alpha s\phi' + 4\phi -$
		$-2\beta a\tan\left(\frac{y}{x}\right)$		$-s^2(a\phi)^{s+1} = 0$

TABLE III. – *Each row shows the infinitesimal generators of the optimal system, the corresponding similarity variables and similarity solutions corresponding to* $f(u) = ce^{au}$. *Here* $\gamma = \alpha^2 + \beta^2$.

i	U_i	z_i	u_i	ODE
1	$\{V_4, V_5\}$	$\sqrt{\frac{x^2+y^2}{T-t}}$	$-\frac{1}{a}(\log(T-t)+\phi(z))$	$\phi'' + \left(\frac{1}{z} - \frac{z}{2}\right)\phi' -$ $-ace^{-\phi} + 1 = 0$
2	$\{V_2, V_5\}$	$\frac{x-x_0}{\sqrt{T-t}}$	$-\frac{1}{a}(\log(T-t)+\phi(z))$	$2\phi'' - z\phi' - 2ace^{-\phi} + 2 = 0$
3	$\{V_1, \alpha V_2 + \beta V_3\}$	$\beta y - \alpha t$	$\phi(z)$	$(\alpha+\beta)\phi' + \beta^2\phi'' + ce^{a\phi} = 0$
4	$\{V_3, \alpha V_4 + \beta V_5\}$	$\alpha \log(x^2+y^2) -$ $-2\beta a \tan\left(\frac{x}{y}\right)$	$\phi(z) - \frac{\log(x^2+y^2)}{a}$	$4\gamma\phi'' + ce^{a\phi} = 0$

list of two-dimensional algebras into equivalence classes under the adjoint action. The two-dimensional optimal systems for $f(u) = c(au+b)^n$, $f(u) = ce^{au+b}$ and $f(u) = cu(\log\,(au)+b)$ are, respectively, listed in tables II, III and IV.

Reductions to ODEs *and exact solutions.* – In order to obtain similarity solutions of eq. (4) we are interested in symmetry reductions to ODEs. This can be done with the members of the two-dimensional optimal system. Let $\mathcal{H}(w_1, w_2) \in \mathcal{G}$ denote the Lie algebra that is spanned by the vector fields $w_1, w_2 \in \mathcal{G}$. To perform this reduction we need to construct the invariants of the two groups which will then become the new variables. We calculate the invariants of the first of the two groups by solving the characteristic equations for that group. Next, the second group is written in terms of these invariants, which must be possible since the reduced equation is invariant under this second group. The integration constants from this second set of characteristic equations are then invariants of both groups and are the similarity variables. Tables II, III and IV list the similarity variables as well as the ODEs to which eq. (4) is reduced for each of the nontrivial symmetries from the optimal systems. As far as we know all these reductions are new.

ODE 1 and its corresponding self-similar variables are found when one is looking for solutions of (4) with $f(u) = -(au+b)^n$, describing finite-time blow-up at the origin in problems with radial symmetry. In this case using the variables z_1 and $\tau = -\log\,(T-t)$, solutions with blow-up at $t = T$ behave when $t \to T$ $(\tau \to \infty)$ as solutions of a stationary equation which is precisely ODE 1.

A particular solution of this equation is $\phi(z) = \frac{1}{a}(\frac{4}{as^2})^{\frac{1}{s}} z^{-\frac{2}{s}}$ or, equivalently,

$$u(r,t) = \frac{1}{a}\left(\frac{4}{as^2}\right)^{\frac{1}{s}} r^{\frac{-2}{s}} - \frac{b}{a}, \tag{8}$$

where r is the radial coordinate $r = \sqrt{x^2+y^2}$. If we define $M(R)$ as the mass in the disk $D(0,R)$, *i.e.*

$$M(R) = 2\pi \int_0^R u(r,t) r \mathrm{d}r,$$

TABLE IV. – *Each row shows the infinitesimal generators of the optimal system, the corresponding similarity variables and similarity solutions corresponding to* $f(u) = cu(\log(au)+b)$, $g(z) = 4(\alpha + e^{ct})$ *and* $\Gamma = -1 + 2b + 2\log a$.

i	U_i	z_i	u_i	ODE
1	$\{\alpha V_1 + V_3, V_2\}$	$\alpha x - t$	$\phi(z)$	$\alpha^2\phi'' + \phi' + c\phi(\log(a\phi) + b) = 0$
2	$\{V_3, V_5\}$	y	$e^{-\frac{cx^2}{4} - \phi(z)}$	$2\phi'' - 2(\phi')^2 + 2c\phi - c\Gamma = 0$
3	$\{V_3, V_4\}$	$\sqrt{x^2+y^2}$	$\phi(z)$	$\phi' + z\phi'' - cz\phi(\log\ (a\phi) + b) = 0$
4	$\{V_1, V_3\}$	y	$\phi(z)$	$\phi'' + c\phi(\log(a\phi) + b) = 0$
5	$\{\alpha V_1 + V_5, V_6\}$	t	$e^{-\frac{cy^2}{4} - \frac{ce^{ct}x^2}{g(z)} - \phi(z)}$	$2g\phi'(z) + cg(\Gamma - 2\phi) - 4ce^{cz} = 0$
6	$\{V_1 + \alpha V_3, V_6\}$	$\alpha x - t$	$e^{-\frac{cy^2}{4} - \phi}$	$2\alpha^2(\phi'' - \phi'^2) + 2\phi' + 2c\phi - c\Gamma = 0$

it is clear that solution (8) has a finite mass if and only if $n > 2$; in this case

$$M(R) = \frac{1}{a}\left(\frac{4}{as^2}\right)^{\frac{1}{s}} \frac{\pi s}{s-1} R^{2-\frac{2}{s}} - \frac{b}{a}\pi R^2.$$

We realize that, multiplying by ϕ and making $\phi = v^{\frac{2}{1-n}}$, ODE 1 can be transformed into

$$(\delta - 1)v_z^2 + vv_{zz} + \left(\frac{1}{z} - \frac{z}{2}\right) vv_z + \frac{1}{2}v^2 - \frac{a^{s+1}}{\delta} = 0, \tag{9}$$

where $\delta = -\frac{2}{s}$, which is a particular case of the Euler-Painlevé equation

$$yy'' - a(y')^2 + f(z)yy' + g(z)y^2 + by' + c = 0. \tag{10}$$

Equation (10) is an extension of the class of nonlinear ODEs which have been studied by Euler and Painlevé [14] and was introduced in [15] for describing the self-similar solutions of generalized Burgers equations. Performing a Painlevé analysis [16], it can be deduced that (9) is not of the Painlevé type and it is not to be expected that it can be solved in terms of elementary trascendents or elliptic functions. The particular solution $v = -\frac{1}{\delta}z$ of (9) corresponds to solution (8) of (4).

In the same way as above, ODE 2 describes the stationary solutions of the one-dimensional problem with finite-time blow-up at an arbitrary point x_0. In this case a particular solution is

$$\phi(z) = \frac{1}{a}\left[\frac{2}{as}\left(\frac{2}{s} + 1\right)\right]^{\frac{1}{s}} z^{-\frac{2}{s}},$$

or equivalently

$$u(x,t) = \frac{1}{a}\left[\frac{1}{a}\left(\frac{4}{s^2} + \frac{2}{s}\right)\right]^{\frac{1}{s}} (x - x_0)^{-\frac{2}{s}} - \frac{b}{a}, \tag{11}$$

solution with finite mass if and only if $n > 3$ as we can easily see:

$$M(\delta) = \int_{x_0-\delta}^{x_0+\delta} u(x,t)\mathrm{d}x = \frac{1}{a}\left[\frac{1}{a}\left(\frac{4}{s^2} + \frac{2}{s}\right)\right]^{\frac{1}{s}} \frac{2s}{s-2}\delta^{1-\frac{2}{s}} - \frac{2b}{a}\delta.$$

Making the same transformations that in the previous case ODE 2 takes the form of the Euler-Painlevé equation. A particular solution is now $v = \frac{n-1}{\sqrt{2(n+1)}} z$ which corresponds to solution (11) of (4). For the case $n = 3$, we have been able to find another particular solution: $v = \lambda z - \frac{6\lambda}{z}$ with $\lambda = \frac{1}{2\sqrt{2}}$. The corresponding solution of (4) is then

$$u = \frac{2\sqrt{2}(x - x_0)}{(x - x_0)^2 - 6(T - t)}.$$

Analogously to the case studied in table II, ODEs 1 and 2 follow the behaviour of stationary solutions describing finite-time blow-up in a problem with radial symmetry and a in one-dimensional problem, respectively.

Concluding remarks. – As far as we know, this is the first symmetry analysis of the nonlinear diffusion-absorption equation in two spatial dimensions in which (4) have been fully reduced to ordinary differential equations for several functional forms of absorption. These functions have been listed, some of them were missing in previous works. The two-dimensional optimal system has been used to obtain these reductions. We must point out that the two-step procedure did not yield new results, however we found classical reductions for the original eq. (4) that are nonclassical reductions for the intermediary equation. We finally remark that choosing the two-dimensional optimal systems in an appropiate form, some of our results can be interpreted in terms of finite-time blow-up processes for both the radial and the one-dimensional problem.

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