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AN ESTIMATE ON THE DISTORTION OF THE LOGARITHMIC CAPACITY

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Abstract. Let Γ be a Fuchsian group. We show that the existence of a set on $\partial\mathbb{D}$ with no Γ-equivalent points and positive logarithmic capacity does not imply that Γ is of convergence type.

Let Γ be a Fuchsian group, that is, a discontinuous group of Möbius transformation of $\mathbb{D} = \{z \in \mathbb{R} : |z| < 1\}$ onto itself.

In [P] Pommerenke asks the following question: If there exists a Borel set on $\partial\mathbb{D}$ of positive capacity that contains no Γ-equivalent points, does it follow that Γ is of convergence type?

The following theorem shows that the answer is NO.

Theorem 1. *There exist a Denjoy domain Ω and a normal fundamental domain $\mathcal{F}$ associated to Ω such that*

(a) $\operatorname{cap}(\partial\mathcal{F} \cap \partial\mathbb{D}) > 0$*, and*

(b) $\operatorname{cap}\partial\Omega = 0$ *(*cap *denotes the logarithmic capacity).*

The rest of this paper will be devoted to proving the theorem, but first I would like to thank C. Bishop and P. Jones for many helpful conversations.

Proof. We will construct a Denjoy domain satisfying (a) and (b), but before doing that we need an estimate on harmonic measure.

Suppose $\Omega \subset \mathbb{R}^+_2 = \{y > 0\}$ is a domain bounded by two orthogonal circles of radius 1 at distance δ. Denote by I this interval on $\mathbb{R}$ of length δ, which can be assumed to be centered at 0, i.e., $I = (-\delta, \delta)$. Let $z_0 = 2i$ (Figure 1). Then

$$\omega(z_0, I, \Omega) \le c_1 e^{-c_2 \frac{\pi}{\sqrt{\delta}}}, \tag{1}$$

where c_1, c_2 are universal constants.

We prove this statement by using an extremal length argument.

Let F be the family of curves in Ω separating z_0 from I, and let ρ be admissible for $F(\rho \in \mathcal{A}(F))$. For each t, $0 \le t \le 1$, consider the line $\gamma_t = it$ joining both orthogonal circles. Then

$$\int_{\gamma_t} \rho\, ds \ge 1.$$

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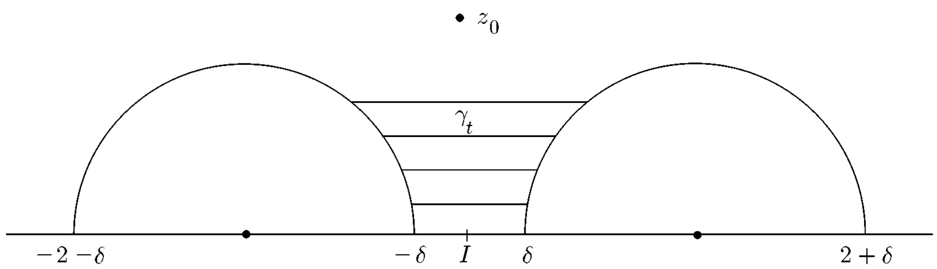

FIGURE 1

By Hölder's inequality

$$1 \leq \left(\int_{\gamma_t} \rho^2 \, ds \right) l(\gamma_t),$$

where $l(\gamma_t)$ denotes the length of γ_t. Integrating on t we get

$$\iint \rho^2 \, dx \, dy \geq \int_0^1 \left(\int_{\gamma_t} \rho^2 \, ds \right) dt \geq \int_0^1 \frac{dt}{l(\gamma_t)}.$$

Since $l(\gamma_t) \simeq \delta + 2t^2$, we obtain

$$\int_0^1 \frac{dt}{l(\gamma_t)} \simeq \int_0^1 \frac{dt}{\delta + 2t^2} \simeq \int_0^{\sqrt{\delta}/2} \frac{dt}{\delta} + \int_{\sqrt{\delta}/2}^1 \frac{dt}{2t^2} \simeq \frac{1}{\sqrt{\delta}}.$$

Hence,

$$M(F) = \inf_{\rho \in \mathcal{A}(F)} \iint \rho^2 \, dx \, dy \geq \frac{c}{\sqrt{\delta}}.$$

Therefore, by Beurling's Theorem

$$\omega(z_0, I, \Omega) \leq c_1 e^{-\pi M(F)} \leq c_1 e^{-c_2 \frac{1}{\sqrt{\delta}}},$$

which ends the proof of (1).

Next we consider a Cantor set $E = \bigcap E_n \subset \mathbb{R}$, where its n-th approximation E_n consists of 2^n intervals $\{I_n\}$ of length $l_n = e^{-n^2}$ (Figure 2). Then E has positive capacity if and only if $\sum 2^{-n} \log \frac{1}{l_n} < \infty$. (See [C, pg. 29].) In this case

$$\sum 2^{-n} \log e^{n^2} = \sum n^2/2^n < \infty,$$

and therefore $\operatorname{cap}(E) > 0$.

We are now ready to construct "half" of the fundamental domain, $\widetilde{\mathcal{F}}$. To do so we draw orthogonal circles supported on the intervals $[-1,1] \backslash E$ (Figure 2).

We obtain a normal fundamental domain $\mathcal{F}$ by reflecting $\tilde{\mathcal{F}}$ accross the orthogonal circle that contains the points 1 and -1. Note that $\operatorname{cap}(\partial \mathcal{F} \cap \mathbb{R}) > 0$.

To get the correspondent Denjoy domain Ω we send $\tilde{\mathcal{F}}$ conformally onto $\mathbb{R}^+_2$. Denote by Φ the conformal map so that $\Phi(i) = \infty$. Then $\partial\Omega = \Phi(E)$. Besides, $\partial\Omega$ looks like a "Cantor set" where the length of the intervals at the n-th stage are less than $c_1 e^{-c_2 e^n}$ (this is just a consequence of (1)). Therefore $\operatorname{cap}(\partial\Omega) = 0$. $\square$

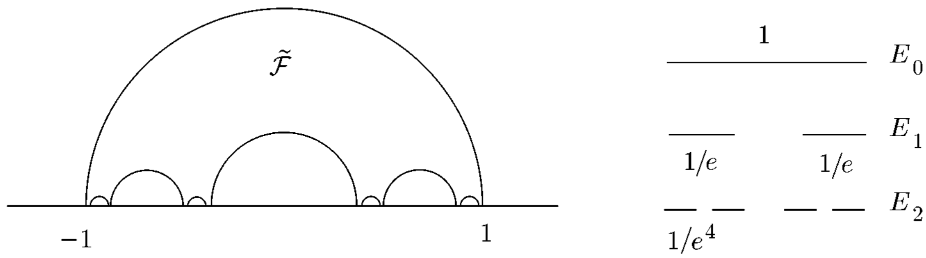

Figure 2

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