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Finite-time aggregation into a single point in a reaction–diffusion system

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Abstract. We consider the following system:

$$(\mathrm{S})\begin{cases} u_t = \Delta u - \chi\nabla(u\nabla v) & \chi > 0 \\ \Delta v = 1 - u \end{cases}$$

which has been used as a model for various phenomena, including motion of species by chemotaxis and equilibrium of self-attracting clusters. We show that, in space dimension $N = 3$, (S) possess radial solutions that blow-up in a finite time. The asymptotic behaviour of such solutions is analysed in detail. In particular, we obtain that the profile of any such solution consists of an imploding, smoothed-out shock wave that collapses into a Dirac mass when the singularity is formed. The differences between this type of behaviour and that known to occur for blowing-up solutions of (S) in the case $N = 2$ are also discussed.

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1. Introduction and description of results

This paper is concerned with the following reaction-diffusion system:

$$u_t = \Delta u - \chi\nabla(u\nabla v) \qquad \chi > 0 \tag{1.1a}$$

$$\Delta v = 1 - u. \tag{1.1b}$$

Equations (1.1) are obtained from the more general system consisting of (1.1*a*) and

$$v_t = D\Delta v + au - bv, \tag{1.2}$$

where a, b and D are positive constants, when a suitable rescaling is performed and the assumption $D \gg 1$ is made (cf [6] for details). It is easy to see that the second term on the right in (1.1*a*) induces solutions $u(x, t)$ to move towards the origin with a velocity that is proportional to ∇v. As a matter of fact, system (1.1) in two space dimensions has been used as a simplified model to describe chemotaxis (cf for instance [3, 7, 11]). This last term refers to the motion of organisms towards higher concentrations of a chemical that they themselves secrete, under the assumption that the motion velocity of the species is proportional to the gradient of some function (in our case, linear) of the chemical concentration $v(x, t)$. When the space dimension is $N = 3$, (1.1) has also been considered as a model of stellar dynamics in the limit of dominant friction terms (see [16, 17]).

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A question that naturally arises is that of ascertaining to what extent do solutions of (1.1) (resp of (1.1*a*), (1.2)) exhibit the behaviours to be expected in the situations they are supposed to model. For instance, in the biology-motivated literature, a feature that has deserved considerable interest is chemotactic collapse. This term often refers to the spatial shrinking of the total population so as to concentrate in a single point. In mathematical terms, this can be described by the fact that $u(x,t)$ converges to a Dirac mass in a finite time. This in turn may be considered as a particular case of blow-up. Actually, it is said that $u(x,t)$ blows up at, say $x = x_0$ and $t = T < +\infty$, if there exist sequences $\{x_n\}$ and $\{t_n\}$ such that $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} t_n = T$ and $\lim_{n\to\infty} u(x_n, t_n) = +\infty$. The main result in this paper is as follows.

Consider system (1.1) in space dimension $N = 3$. Then, for any $T > 0$ and any constant $C > 0$, there exists a radial solution $(u(r,t), v(r,t))$ of (1,1) that is smooth for all times $0 < t < T$, blows up at $r = 0$, $t = T$, and is such that:

$$\int_{|x|\leqslant r} u(s,T)\,\mathrm{d}s \to C \qquad \text{as } r \to 0. \tag{1.3}$$

Concerning the significance of our result, a few remarks are in order.

(1) Blow-up is an interesting fact in the theory of reaction-diffusion systems, and as such it has spawned a large literature. It has to be pointed out, however, that describing in detail the nature of the unfolding singularities is in general much more delicate that merely showing that singularities can actually occur. As a matter of fact, an acccurate description of blow-up mechanisms is so far available only for a limited number of simple scalar equations (cf [2, 5, 13–15]) and systems (see for instance [1, 8–10]).

(2) We shall provide here a detailed description of the structure of the blowing-up solutions referred to above by means of asymptotic methods. To begin with, they are not of a self-similar nature. Actually, we shall describe a singularity mechanism that yields concentration of mass in a manner which is completely unrelated to the self-similar scales. Roughly speaking, our solution $u(r,t)$ will be shown to consist of an imploding, smoothed-out shock wave which moves towards the origin. As $t \to T$, the bulk of such wave is concentrated at distances $0((T-t)^{\frac{1}{3}})$ from the origin, has a width $0((T-t)^{\frac{2}{3}})$, and at its peak it reaches a height of order $0((T-t)^{-\frac{4}{3}})$ (cf figure 1.1).

As a matter of fact, the internal structure of $u(r,t)$ will be described in detail in section 3. In particular we shall show that, when written in suitable rescaled variables, $u(r,t)$ can be

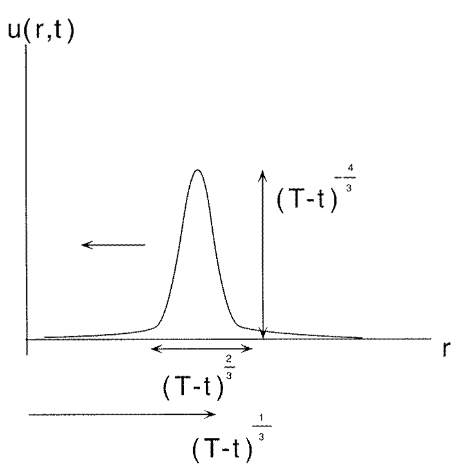

Figure 1.1. The profile of $u(r,t)$.

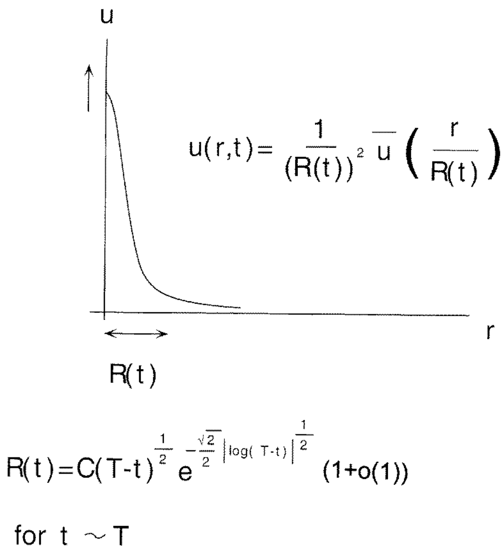

Figure 1.2. Chemotactic collapse in two space dimensions.

represented near the unfolding peak by means of a slowly moving travelling wave to a Burgers-type equation.

(3) A question that naturally arises is that of the influence of the space dimension in the manner of blow-up for (1.1). In [8] it was shown that when $N = 2$, (1.1) has radial solutions such that $u(r, t)$ develops a Dirac-delta-type singularity at the origin in a finite time. However, the cases $N = 2$ and $N = 3$ display major differences. For instance, when $N = 2$, the solutions obtained in [8] are such that $u(r, t)$ approaches (in rescaled variables) to the function $\bar{u}(r) = 8(\chi(1 + r^2)^2)^{-1}$ in a narrow layer around the origin as $t \to T$ (see figure 1.2).

It is to be noticed that, together with $\bar{v}(r) = -\frac{2}{\chi}\log(1 + r^2)$, $\bar{u}$ is an explicit solution of the elliptic system:

$$\Delta u - \chi\nabla(u\nabla v) = 0,$$
$$\Delta v + u = 0,$$

which corresponds to the stationary version of (1.1), once the constant term -1 (that is irrelevant near blow-up) is discarded there. We point out that the emergence of a stationary solution as a rescaled blow-up profile is reminiscent of similar situations already observed in singularity formation in geometrical problems, as for instance minimal surface theory (cf [15]) and motion of surfaces by mean curvature (cf for instance [2]). Incidentally, the blow-up pattern described in [8], while not self-similar, actually takes places in scales that are not far from the self-similar ones (see for instance the expression for the inner layer $R(t)$ in figure 1.2). No such consideration applies to the solutions discussed here, where blow-up scales are quite different from the self-similar ones.

There is yet another difference between the cases $N = 2$ and $N = 3$ that is worth to be stressed. When $N = 2$, there is a mass threshold for chemotactic collapse that seems to be absent when $N = 3$. More precisely, if (1.1) is considered in a ball $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$ with homogeneous Neumann conditions, it is shown in [8] that solutions concentrating into a Dirac mass at the origin exist if $\frac{1}{\pi R^2}\int_{B_R} u(x, 0)\,dx = \frac{1}{\pi R^2}\int_{B_R} u(x, t)\,dx > \frac{8}{\chi}$. Such a condition is known to be necessary for blow-up to occur in two dimensions (cf [11]). No such restriction appears when $N = 3$, since constant C in (1.3) is arbitrary.

This paper is organized as follows. A heuristic motivation of the blow-up mechanism to be obtained here is provided in section 2. The detailed structure of our solutions will then be given in section 3. Section 4 contains a discussion of the stability of such solutions. Finally, section 5 gives some concluding remarks.

2. A heuristic motivation

There is a simple, intuitive explanation of the way in which the scales in our collapse mechanism develop, that we describe here for convenience of the reader. Let us consider radial solutions of (1.1). On dispensing with the constant term (-1) there (which will be negligible with respect to u near blow-up), we may write the corresponding system in the form:

$$u_t = u_{rr} + \frac{2u_r}{r} - \frac{\chi}{r^2}(r^2 u v_r)_r, \tag{2.1a}$$

$$v_{rr} + \frac{2v_r}{r} = -u. \tag{2.1b}$$

The solutions to be constructed here will blow up at $t = T < \infty$ in such a way that the mass M of $u(r,t)$ will be concentrated, as $t \to T$, in a small layer of width $\delta(t)$, located at a distance $R(t)$ from the origin, where $\delta(t)$ and $R(t)$ are such that:

$$R(t) \to 0 \qquad \text{as } t \to T \qquad \delta(t) \ll R(t) \qquad \text{when } t \to T. \tag{2.2}$$

Let us denote by $h(t)$ the maximum height of $u(r,t)$ (see figure 2.1). The condition of mass conservation for $u(r,t)$ then gives:

$$4\pi R^2(t)\delta(t)h(t) \sim M = \mathrm{O}(1). \tag{2.3}$$

In view of (2.2), we readily see that, for $t \sim T$, system (2.1) becomes asymptotically equivalent to:

$$u_t = u_{rr} - \chi(u v_r)_r, \tag{2.4a}$$

$$v_{rr} = -u. \tag{2.4b}$$

Indeed, one has for instance that $\frac{u_r}{r} = \mathrm{O}(\frac{u}{\delta R(t)})$ and $u_{rr} = \mathrm{O}(\frac{u}{\delta^2})$, so that $\frac{u_r}{r} \ll u_{rr}$ as $t \to T$. It then turns out that the structure of $u(r,t)$ will be locally one-dimensional near $r = R(t)$.

As a next step, we shall consider travelling wave solutions of (2.4) of the form:

$$u(r,t) = \phi(\xi) \qquad v(r,t) = \psi(\xi) \qquad \text{where } \xi = r - ct.$$

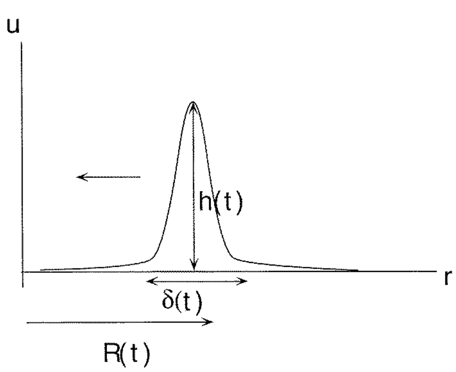

Figure 2.1. The form of the sought-for solutions.

Assuming that $\phi(\xi)$ and $\psi'(\xi)$ decrease to zero as $\xi \to \pm\infty$, we readily see that, if the wavefront is to be located at $r = R(t)$, then the wave speed c should satisfy the Rankine–Hugoniot-type condition:

$$c[\psi']_{R(t)} = [(\psi')^2]_{R(t)}, \tag{2.5}$$

where $[\psi']_{R(t)} = \lim_{\substack{x\to R(t)\\ x>R(t)}} (\psi'(x)) - \lim_{\substack{x\to R(t)\\ x<R(t)}} (\psi'(x))$. Our sought-for solutions will have locally a structure described by such a type of waves with variable speed $c = R'(t)$ satisfying condition (2.5), where the jump of $\psi' = v_r$ at $r = R(t)$ will be obtained from conditions:

$$v_r(R(t)^+, t) = -\frac{M}{4\pi R^2(t)}, \tag{2.6a}$$

$$v_r(R(t)^-, t) = 0, \tag{2.6b}$$

that follow from (2.4*b*) by classical potential theory. Putting together (2.5) and (2.6) we therefore obtain:

$$\dot{R}(t) = -\frac{M}{4\pi R^2(t)} \qquad \text{for } t \sim T,$$

whence:

$$R(t) \sim C(T-t)^{\frac{1}{3}} \qquad \text{for some } C = C(M) > 0 \text{ as } t \to T. \tag{2.7}$$

On rescaling u with $h(t)$ and r with $\delta(t)$, in order to make all terms in (2.4) of same order of magnitude, we also derive that:

$$h(t)\delta^2(t) \sim 1, \tag{2.8}$$

and from (2.8) and (2.3) we finally obtain that:

$$h(t) = (T-t)^{-\frac{4}{3}} \qquad \delta(t) = (T-t)^{\frac{2}{3}}. \tag{2.9}$$

3. Blowing-up solutions when $N = 3$

3.1. Preliminaries

In this section we shall consider radial solutions of (1.1) in three space dimensions. Namely, we shall deal with functions $(u(r,t), v(r,t))$ such that:

$$u_t = u_{rr} + \frac{2u_r}{r} - \frac{\chi}{r^2}(r^2 u v_r)_r, \tag{3.1}$$

$$0 = v_{rr} + \frac{2v_r}{r} + u - 1. \tag{3.2}$$

We next define an auxiliary mass function given by:

$$M(r,t) = \int_{|x|\leqslant r} (u-1)\,\mathrm{d}x = 4\pi \int_0^r (u-1)\rho^2\,\mathrm{d}\rho. \tag{3.3}$$

A routine check reveals that system (3.1), (3.2) can be transformed into a single equation for $M(r,t)$, namely:

$$M_t = M_{rr} - \frac{2M_r}{r} + \chi M\left(\frac{M_r}{4\pi r^2} + 1\right). \tag{3.4}$$

We next introduce rescaled variables as follows

$$\begin{aligned} &y = r(T-t)^{-\frac{1}{2}} \qquad \tau = -\log(T-t),\\ &M(r,t) = (T-t)^{\frac{1}{2}}\phi(y,\tau). \end{aligned} \tag{3.5}$$

Here $T > 0$ is fixed, but otherwise arbitrary. In terms of $\phi(y, \tau)$, equation (3.4) reads

$$\Phi_\tau = \Phi_{yy} - \left(\frac{2}{y} + \frac{y}{2}\right)\Phi_y + \frac{\Phi}{2} + \chi\Phi\left(\frac{\Phi_y}{4\pi y^2} + \mathrm{e}^{-\tau}\right). \tag{3.6}$$

As a further simplification, we shall drop the last term on the right in (3.6), which later will be shown to be negligible near blow-up. We are thus led to analysing the following equation:

$$\Phi_\tau = \Phi_{yy} - \left(\frac{2}{y} + \frac{y}{2}\right)\Phi_y + \frac{\Phi}{2} + \frac{\chi\Phi\Phi_y}{4\pi y^2}. \tag{3.7}$$

Let us now set:

$$\Phi(y, \tau) = y^3 G(y, \tau). \tag{3.8}$$

Then $G(y, \tau)$ solves:

$$G_\tau = G_{yy} + \left(\frac{4}{y} - \frac{y}{2}\right)G_y - G + \frac{\chi G}{4\pi}(3G + yG_y). \tag{3.9}$$

We shall look for solutions of (3.9) whose hyperbolic part has a shock at $y = R(\tau)$, which is smoothed out by the effect of the diffusive term there. This motivates our rescaling of the space coordinate by setting $\xi = \frac{y}{R(\tau)}$. Equation (3.9) then transforms into:

$$G_\tau = \frac{1}{R(\tau)^2}\left(G_{\xi\xi} + \frac{4}{\xi}G_\xi\right) + \left(\frac{\dot{R}(\tau)}{R(\tau)} - \frac{1}{2} + \frac{\chi G}{4\pi}\right)\xi G_\xi + \left(\frac{3\chi G^2}{4\pi} - G\right). \tag{3.10}$$

In (3.10) we want to have $R(\tau) \to \infty$ as $\tau \to \infty$, so that for $\tau \gg 1$ (i.e. near the blow-up time T) the dominant terms will correspond to the hyperbolic equation:

$$G_\tau = \left(\frac{\dot{R}(\tau)}{R(\tau)} - \frac{1}{2} + \frac{\chi G}{4\pi}\right)\xi G_\xi + \left(\frac{3\chi G^2}{4\pi} - G\right). \tag{3.11}$$

The actual forms of equations (3.10) and (3.11) clearly depend on the choice of function $R(\tau)$. However, equations corresponding to different values of $R(\tau)$ are all equivalent under suitable rescaling. For instance, for given $R_1(\tau)$ and $R_2(\tau)$, setting $\xi_1 = \frac{y}{R_1(\tau)}$ and $\xi_2 = \frac{y}{R_2(\tau)}$ allows us to define $\xi_1 = \xi_1(\xi_2)$ or $\xi_2 = \xi_2(\xi_1)$, so that one can readily pass from one form of equations (3.10), (3.11) to another. A choice which is particularly convenient corresponds to setting:

$$R(\tau) = \mathrm{e}^{\frac{\tau}{6}}, \tag{3.12}$$

in which case (3.10) and (3.11) read respectively as follows

$$G_\tau = \mathrm{e}^{-\frac{\tau}{3}}\left(G_{\xi\xi} + \frac{4}{\xi}G_\xi\right) + + \left(\frac{\chi G}{4\pi} - \frac{1}{3}\right)(\xi G_\xi + 3G), \tag{3.13}$$

$$G_\tau = \left(\frac{\chi G}{4\pi} - \frac{1}{3}\right)(\xi G_\xi + 3G) = \left(\xi\left(\frac{\chi G^2}{8\pi} - \frac{G}{3}\right)\right)_\xi + \left(\frac{5\chi G}{8\pi} - \frac{2}{3}\right)G. \tag{3.14}$$

Note that, if a solution of (3.14) has a discontinuity at $\xi = s(\tau)$, the classical Rankine–Hugoniot condition will read as follows

$$-\dot{s}(\tau)[G]_{s(\tau)} = \left[\left(\frac{\chi G^2}{8\pi} - \frac{G}{3}\right)\right]_{s(\tau)}, \tag{3.15}$$

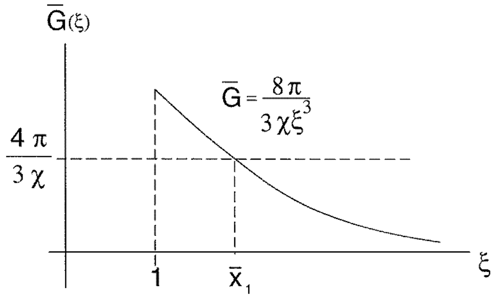

Figure 3.1. The rescaled asymptotic blow-up pattern.

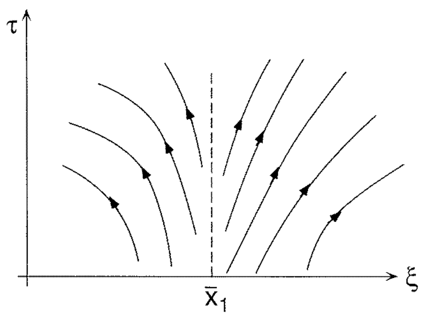

Figure 3.2. Drawing the characteristic lines of (3.14).

where, as usual, $[f(\xi,\tau)]_{s(\tau)} = \lim_{\substack{\xi\to s(\tau)\\ \xi>s(\tau)}}(f(\xi,\tau)) - \lim_{\substack{\xi\to s(\tau)\\ \xi<s(\tau)}}(f(\xi,\tau))$. If we now consider stationary solutions of (3.14) having a (stationary) shock at $\xi = \xi_0$, it follows from (3.14) and (3.15) that such solutions are of the form:

$$\begin{cases} \bar{G}(\xi) = \dfrac{K}{\xi^3} & \text{for } \xi > \xi_0, \\ \bar{G}(\xi) = 0 & \text{for } \xi < \xi_0, \\ \text{with } K = \dfrac{8\pi}{3\chi}\xi_0^3. \end{cases} \tag{3.16}$$

The functions given in (3.16) and displayed in figure 3.1 will play a crucial role in our approach, since they will correspond to the asymptotic blow-up patterns that the solutions under consideration will approach to as $\tau \to \infty$. For definiteness, we shall assume in the sequel $\xi_0 = 1$ in (3.16).

3.2. Asymptotics away from the shocks

A first step towards describing the sought-for asymptotics consists of examining the behaviour of our solutions around the point $\bar{x}_1$ where $\bar{G}(\bar{x}_1) = \frac{4\pi}{3\chi}$. This region is indeed important, since the hyperbolic part in (3.13) vanishes at $\xi = \bar{x}_1$, and diffusive terms become important therein. Notice that the characteristic curves of equation (3.14) have the form indicated in figure 3.2.

It is then readily seen that the values of the solutions of (3.14) are propagated along characteristic lines that emanate from $\xi = \bar{x}_1$. Analysing the behaviour of solutions of (3.13) near $\xi = \bar{x}_1$ is then crucial to describe the asymptotics of $G(\xi, \tau)$ away from the shocks. To this end, we introduce a new space variable λ given by:

$$\lambda = (\xi - \bar{x}_1)e^{\frac{\tau}{6}}, \tag{3.17}$$

and then linearize around $\bar{G}$ by setting:

$$\psi(\lambda,\tau) = G(\xi,\tau) - \frac{8\pi}{3\chi\xi^3} = G(\xi,\tau) - \frac{4\pi}{3\chi}\left(\frac{\bar{x}_1}{\xi}\right)^3$$
$$\text{where } \psi(\lambda,\tau)\to 0 \qquad \text{as } \tau\to\infty \qquad \text{when } |\lambda| = O(1). \tag{3.18}$$

Plugging (3.18) into (3.13), we obtain the following equation for ψ valid in regions where $\lambda = \mathrm{O}(1)$:

$$\psi_\tau = \psi_{\lambda\lambda} - \tfrac{7}{6}\lambda\psi_\lambda + (\text{lower-order terms}). \tag{3.19}$$

As $\tau\to\infty$, the dominant part in (3.19) is the linear equation:

$$\psi_\tau = \psi_{\lambda\lambda} - \tfrac{7}{6}\lambda\psi_\lambda, \tag{3.20}$$

whose general solution can be obtained by separation of variables method to give:

$$\psi(\lambda,\tau) = \sum_0^\infty \alpha_k \mathrm{e}^{-\frac{7k\tau}{6}} L_k(\lambda), \tag{3.21}$$

where the $\{\alpha_k\}$ are suitable real coefficients, and $L_k(r) = H_k(\sqrt{\tfrac{6}{7}}r)$, where for $k = 1, 2, \ldots$ $H_k(r)$ is the standard kth-Hermite polynomial. Since we are assuming that $\psi(\lambda,\tau)\to 0$ as $\tau\to\infty$ when $|\lambda| = \mathrm{O}(1)$, we generically have that, to the lower order:

$$\psi(\lambda,\tau) \sim \alpha_1\lambda\mathrm{e}^{-\frac{7}{6}\tau} \qquad \text{as } \tau\to\infty,$$

that in the original variables reads:

$$G(\xi,\tau) = \bar{G}(\xi) + \alpha_1\mathrm{e}^{-\tau}(\xi-\bar{x}_1) + \cdots \qquad \text{as } \xi\to\bar{x}_1. \tag{3.22}$$

On the other hand, as we move away from $\bar{x}_1$, we expect the following expansion to hold for solutions of (3.13):

$$G(\xi,\tau) = \bar{G}(\xi) + \mathrm{e}^{-\tau}G_1(\xi) + \cdots. \tag{3.23}$$

Substituting (3.23) into (3.13) shows that $G_1(\xi)$ has to solve the ODE:

$$G_1 + \left(\frac{\chi\bar{G}}{4\pi} - \frac{1}{3}\right)(\xi G_1' + 3G_1) = 0, \tag{3.24a}$$

whereas matching with (3.22) requires:

$$G_1(\xi) \sim \alpha_1(\xi-\bar{x}_1) \qquad \text{as } \xi\to\bar{x}_1. \tag{3.24b}$$

It is readily seen that the solution of (3.24) is given by:

$$G_1(\xi) = \frac{\alpha_1}{6}\left(1 - \frac{2}{\xi^3}\right),$$

whence the expansion away from the shock that matches with (3.22) for $\xi\sim\bar{x}_1$ is given by:

$$G(\xi,\tau) = \frac{8\pi}{3\chi\xi^3} + \frac{\alpha_1}{6}\left(1-\frac{2}{\xi^3}\right)\mathrm{e}^{-\tau} + \cdots, \tag{3.25}$$

where constant α_1 is not fixed by this analysis.

3.3. Matching with the region around the shock

We continue our study by describing the nature of the diffusive layers arising from the term $e^{-\frac{\tau}{3}} G_{\xi\xi}$ in (3.13). To this end, we define an inner variable given by:

$$\eta = (\xi - 1)e^{\frac{\tau}{3}}, \tag{3.26}$$

so that (3.13) is now rewritten in the form:

$$e^{-\frac{\tau}{3}} G_\tau = G_{\eta\eta} + \left(\frac{\chi G}{4\pi} - \frac{1}{3}\right) G_\eta + e^{-\frac{\tau}{3}} \Bigg(4G_\eta (1 + \eta e^{-\frac{\tau}{3}})^{-1} + \left(\frac{\chi G}{4\pi} - \frac{1}{3}\right)(\eta G_\eta + 3G) - \eta \frac{G_\eta}{3}\Bigg). \tag{3.27}$$

It is then natural to expect that the dynamics near $\xi = 1$ will be asymptotically of a Burgers-type nature. More precisely, for $|\eta| = \mathrm{O}(1)$ and $\tau \gg 1$, we expect $G(\xi, \tau)$ to behave as a solution of:

$$G'' + \left(\frac{\chi G}{4\pi} - \frac{1}{3}\right) G' = 0 \qquad \text{for } -\infty < \eta < \infty, \qquad \tau > 0, \tag{3.28a}$$

with boundary conditions:

$$G(\eta) \to L_l \equiv \bar{G}(1^-) \qquad \text{as } \eta \to -\infty, \tag{3.28b}$$

$$G(\eta) \to L_r \equiv \bar{G}(1^+) \qquad \text{as } \eta \to \infty. \tag{3.28c}$$

A quick check shows that the solution of (3.28) which has a jump of height $\frac{8\pi}{3\chi}$ at $\eta = 0$ is given by:

$$G^*(\eta) = \frac{L_r + L_l e^{-\frac{\chi}{8\pi}(L_r - L_l)(\eta - \eta_0)}}{1 + e^{-\frac{\chi}{8\pi}(L_r - L_l)(\eta - \eta_0)}}, \tag{3.29a}$$

where:

$$L_r + L_l = \frac{8\pi}{3\chi}. \tag{3.29b}$$

and η_0 is a free parameter. We shall denote by $G_0(\eta)$ the particular case of (3.29) obtained by setting $L_l = 0$ there, i.e.

$$G_0(\eta) = \frac{8\pi}{3\chi}(1 + e^{-\frac{1}{3}(\eta - \eta_0)})^{-1}. \tag{3.30}$$

To proceed further, we shall try on (3.27) an expansion of the type:

$$G(\eta, \tau) = G_0(\eta) + e^{-\frac{\tau}{3}} H_1(\eta) + \cdots. \tag{3.31}$$

A routine computation reveals that H satisfies:

$$\begin{aligned} H_1'' + \left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right) H_1' + \frac{\chi}{4\pi} G_0' H_1 &= -\left(4G_0' + 3\left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right) G_0 + \eta G_0' \left(\frac{\chi G_0}{4\pi} - \frac{2}{3}\right)\right) \\ &= -\left(4G_0' + 3\left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right) G_0 + (\eta - \eta_0) G_0' \left(\frac{\chi G_0}{4\pi} - \frac{2}{3}\right)\right) \\ &\quad -\eta_0 G_0' \left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right) \equiv F_1(\eta - \eta_0) + \eta_0 F_2(\eta - \eta_0). \end{aligned} \tag{3.32}$$

Equation (3.32) is a non-homogeneous, second-order linear ODE. The corresponding homogeneous equation:

$$H_1'' + \left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right) H_1' + \frac{\chi G_0'}{4\pi} H_1 = 0, \tag{3.33}$$

has two independent solutions, $H_{11}(\eta)$ and $H_{12}(\eta)$, such that:

$$H_{11}(\eta) \to 1 \qquad \text{when } \eta \to \infty \qquad H_{11}(\eta) \to -1 \qquad \text{when } \eta \to -\infty, \tag{3.34a}$$

and:

$$\begin{aligned} H_{12}(\eta) &\to \mathrm{e}^{-\frac{(\eta-\eta_0)}{3}} \qquad \text{when } \eta \to \infty, \\ H_{12}(\eta) &\to \mathrm{e}^{\frac{(\eta-\eta_0)}{3}} \qquad \text{when } \eta \to -\infty. \end{aligned} \tag{3.34b}$$

To check (3.34), we just remark that (3.33) has been obtained by linearization around $G_0(\eta)$ in equation (3.28*a*). Since:

$$\begin{aligned} G^*(\eta, a, b) = G_0(\eta) &+ \frac{\partial G^*}{\partial a}\left(\eta; \frac{8\pi}{3\chi}, \eta_0\right)\left(a - \frac{8\pi}{3\chi}\right) \\ &+ \frac{\partial G^*}{\partial b}\left(\eta; \frac{8\pi}{3\chi}\eta_0\right)(b - \eta_0) + \cdots \end{aligned}$$

it turns out that $H_{11} \equiv \frac{\partial G^*}{\partial a}(\eta; \frac{8\pi}{3\chi}, \eta_0)$ and $H_{12} \equiv -\frac{9\chi}{8\pi}\frac{\partial G^*}{\partial b}(\eta; \frac{8\pi}{3\chi}, \eta_0)$ are two independent solutions of (3.33), whereupon (3.34) follows by a direct computation.

Once the homogeneous equation (3.33) has been discussed, we proceed to analyse (3.32) by means of the classical variation of constant formula. Let us set:

$$U_1(\eta) \equiv U_1(\eta; \eta_0) = U_{11}(\eta - \eta_0) + \eta_0 U_{12}(\eta - \eta_0) \tag{3.35}$$

where U_{11} (resp. U_{12}) is a particular solution of (3.32) with $F_2 \equiv 0$ (resp. $F_1 \equiv 0$), satisfying the initial conditions $U_1(0) = U_1'(0) = 0$. One then has that:

$$U_{11}(\eta - \eta_0) \sim \begin{cases} -\dfrac{8\pi}{\chi}(\eta - \eta_0) + \Gamma_1 & \text{when } \eta \to \infty, \\ \gamma_1 & \text{when } \eta \to -\infty, \end{cases} \tag{3.36a}$$

and:

$$U_{12}(\eta - \eta_0) \sim \begin{cases} \Gamma_2 & \text{when } \eta \to \infty, \\ \gamma_2 & \text{when } \eta \to -\infty, \end{cases} \tag{3.36b}$$

where constants Γ_1, γ_1, Γ_2 and γ_2 are fixed in terms of $G_0(\eta)$. Therefore, the general solution of (3.32) is such that:

$$H_1(\eta; \eta_0) \sim -\frac{8\pi}{\chi}(\eta - \eta_0) + \Gamma_1 + \Gamma_2\eta_0 + \mathrm{O}(\mathrm{e}^{-\frac{\eta}{3}}) \qquad \text{as } \eta \to \infty \tag{3.37a}$$

$$H_1(\eta; \eta_0) \sim \gamma_1 + \gamma_2\eta_0 + \mathrm{O}(\mathrm{e}^{\frac{\eta}{3}}) \qquad \text{as } \eta \to -\infty. \tag{3.37b}$$

Putting together (3.31) and (3.36), we have obtained the following inner expansion for $G(\eta, \tau)$:

$$G(\eta, \tau) \sim \frac{8\pi}{3\chi} + \mathrm{e}^{-\frac{\tau}{3}}\left(\Gamma_1 + \Gamma_2\eta_0 - \frac{8\pi}{\chi}(\eta - \eta_0) + \mathrm{O}(\mathrm{e}^{-\frac{\eta}{3}})\right) \qquad \text{as } \eta \to \infty, \tag{3.38a}$$

$$G(\eta, \tau) \sim \mathrm{e}^{-\frac{\tau}{3}}(\gamma_1 + \gamma_2\eta_0 + \mathrm{O}(\mathrm{e}^{\frac{\eta}{3}})) \qquad \text{as } \eta \to -\infty. \tag{3.38b}$$

As an outer expansion, we now use (3.25), which when rewritten in terms of $\eta = (\xi - 1)\mathrm{e}^{\frac{\tau}{3}}$ gives:

$$\begin{aligned} G(\eta, \tau) &\sim \frac{8\pi}{3\chi}(1 + \eta\mathrm{e}^{-\frac{\tau}{3}})^{-3} + \frac{\alpha_+}{6}\left(1 - 2(1 + \eta\mathrm{e}^{-\frac{\tau}{3}})^{-3}\right)\mathrm{e}^{-\tau} + \cdots \\ &\sim \frac{8\pi}{3\chi} - \frac{8\pi}{\chi}\eta\mathrm{e}^{-\frac{\tau}{3}} + C_0\mathrm{e}^{-\frac{\tau}{3}} + \cdots \qquad \text{when } \eta \to \infty \end{aligned} \tag{3.39a}$$

$$G(\eta, \tau) = \mathrm{O}(\mathrm{e}^{\frac{\eta}{3}}) \qquad \text{when } \eta \to -\infty. \tag{3.39b}$$

Matching (3.38) and (3.39) yields:

$$\Gamma_1 + \eta_0 \left(\Gamma_2 + \frac{8\pi}{\chi} \right) = 0, \tag{3.40a}$$

$$\gamma_1 + \eta_0 \gamma_2 = 0. \tag{3.40b}$$

4. Discussing the stability of blow-up patterns

Once our blowing-up solutions have been obtained in section 3 by means of matched asymptotic expansion methods, a question that naturally arises is that of discussing the stability of these asymptotic behaviours. We shall now describe an argument which strongly suggests that these profiles are stable, although a fictitious instability appears which is due to the change of variables (3.5), and is related to the fact that small changes in the initial data produce generically changes in the corresponding blow-up time T.

4.1. The hyperbolic problem

To highlight the main points in our reasoning, we first consider the hyperbolic equation (3.14). We have already seen that function $\bar{G}(\xi)$ defined in (3.16) is a solution of (3.14). Let us take now as initial value for (3.14), at some time $\tau = \tau_0 \gg 1$, the following function $G(\xi, \tau_0) \equiv \bar{G}_0(\xi)$:

$$\bar{G}_0(\xi) = \begin{cases} \dfrac{8\pi}{3\chi\xi^3} & \text{if } \xi \geqslant s_0, \\ 0 & \text{if } 0 < \xi < s_0, \end{cases} \tag{4.1}$$

for some constant $s_0 \neq 1$. Equation (3.14) has then a solution $G(\xi, \tau)$ given by:

$$G(\xi, \tau) = \begin{cases} \dfrac{8\pi}{3\chi\xi^3} & \text{if } \xi \geqslant s(\tau), \\ 0 & \text{if } 0 < \xi < s(\tau), \end{cases}$$

where $s(\tau)$ satisfies the ODE:

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{1}{3}\left(1 - \frac{1}{s^3}\right)s, \tag{4.2}$$

with initial value:

$$s(\tau_0) = s_0. \tag{4.3}$$

Note that the point $s = 1$ is unstable for equation (4.2). Actually, if $s_0 > 1$, then $s(\tau) \sim C\mathrm{e}^{\frac{\tau}{3}}$ as $\tau \to \infty$ for some $C > 0$, whereas if $s_0 < 1$ the corresponding solution of (4.1), (4.2) blows up in finite time. Recalling (3.5) and the definition of ξ, we see that in the first case the shock would remain at a distance x of order one of the origin, which means that the corresponding solution will not blow up for any $t \leqslant T$. Conversely, in the second case we would have blow-up at some time $T^* < T$. It is worth noticing that when $s_0 \neq 1$ the shock moves away from $s = 1$ at an exponential rate. Indeed on setting $s(\tau) = 1 + \delta(\tau)$, and assuming $\delta(\tau) \ll 1$, we may formally linearize in (4.2) to obtain that $\dot{\delta}(\tau) = \delta(\tau)$, whereupon exponential growth follows.

Actually, the instability just described is a consequence of the change of variables (3.5) rather than an inherent feature of the problem, and it can be eliminated by a suitable change in the scaling in (3.5) as explained below. Let us denote by $\tilde{T}$ the new blow-up time corresponding to replacing an initial value $\bar{G}(\xi)$ (which blows up exactly at $t = T$) by a small perturbation of it, given for instance by $\bar{G}_0(\xi)$ in (4.1). Bearing in mind (3.5), we now write:

$$\tilde{y} = r(\tilde{T} - t)^{-\frac{1}{2}} \qquad \tilde{\tau} = -\log(\tilde{T} - t)$$
$$M(r, t) = (\tilde{T} - t)^{\frac{1}{2}}\tilde{\Phi}(\tilde{y}, \tilde{\tau}). \tag{4.4}$$

By (3.5) and (4.4), we have that:

$$\tilde{\Phi}(\tilde{y}, \tilde{\tau}) = \left(\frac{T - t}{\tilde{T} - t}\right)^{\frac{1}{2}} \Phi(y, \tau).$$

Set now:

$$\tilde{\Phi}(\tilde{y}, \tilde{\tau}) = \tilde{y}^3\tilde{G}(\tilde{y}, \tilde{\tau}) \qquad \tilde{y} = \mathrm{e}^{\frac{\tilde{\tau}}{6}}\tilde{\xi}.$$

By assumption, $G(\xi, \tau_0) = \bar{G}_0(\xi)$ at $\tau = \tau_0 \gg 1$. This implies that:

$$\tilde{G}(\tilde{\xi}, \tilde{\tau}_0) = \frac{8\pi}{3\chi\tilde{\xi}^3} \qquad \text{if } \tilde{\xi} \geqslant s_0(1 + \tfrac{1}{3}(\tilde{T} - T)\mathrm{e}^{\tau_0}), \tag{4.5a}$$

$$\tilde{G}(\tilde{\xi}, \tilde{\tau}_0) = 0 \qquad \text{if } \tilde{\xi} < s_0(1 + \tfrac{1}{3}(\tilde{T} - T)\mathrm{e}^{\tau_0}), \tag{4.5b}$$

where we have used the approximation:

$$\frac{\tilde{\xi}}{\xi} = (\mathrm{e}^{\tau - \tilde{\tau}})^{\frac{1}{3}} \sim 1 + \tfrac{1}{3}(\tilde{T} - T)\mathrm{e}^{\tau}. \tag{4.6}$$

In particular, the choice $s_0(1 + \frac{1}{3}(\tilde{T} - T)\mathrm{e}^{\tau_0}) = 1$ (i.e. $\tilde{T} = T + 3\mathrm{e}^{-\tau_0}(s_0^{-1} - 1)$) yields a stationary solution starting from the data in (4.5). We have thus absorbed the shift in the shock location by means of a suitable change in the blow-up time. Notice that a stationary shock at $\tilde{\xi} = 1$ corresponds, in the original variables (3.5), to a shock which initially moves according to:

$$\xi = s(\tau) \sim 1 - \tfrac{1}{3}(\tilde{T} - T)\mathrm{e}^{\tau} \tag{4.7}$$

at least while the nonlinear terms in (4.2) can be assumed to be negligible with respect to the linear ones there. The behaviour described in (4.7) corresponds to the exponential instability already mentioned at the beginning of this section.

4.2. Analysis of the complete equation (3.14)

We now proceed to discuss the instability phenomenon just described in the case where diffusive terms are retained. This will require of an additional analysis, since diffusive terms are particularly important near the unfolding shocks, and they could therefore have an effect on the asymptotics in such regions.

More precisely, we want to ascertain what would be the influence of a small perturbation of the initial profile (and the consequent shift in the shock location) in the matching procedure explained in section 3.3. To address this question, we shall replace the inner expansion (3.31) by:

$$G(\eta, \tau) = G_0(\eta + g(\tau)) + \mathrm{e}^{-\frac{\tau}{3}}G_1(\eta + g(\tau)) + \cdots \tag{4.8}$$

where $G_0(\eta)$ is given in (3.30), and the phase translation $g(\tau)$ will be determined presently. Substituting (4.8) into (3.27) gives:

$$\dot{g}G_0' = G_1'' + \left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right)G_1' + \frac{\chi G_0'}{4\pi}G_1 4G_0' + \eta\left(\frac{\chi G_0}{4\pi} - \frac{2}{3}\right)G_0' + 3\left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right)G_0.$$

Set now $\mathcal{L}(G_1) = G_1'' + \left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right) G_1' + \frac{\chi G_0'}{4\pi} G_1$. The previous equation can then be recast in the form:

$$\begin{aligned}\mathcal{L}G_1 &= \dot{g}G_0' - \eta\left(\frac{\chi G_0}{4\pi} - \frac{2}{3}\right)G_0' - 4G_0' - 3\left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right)G_0 \\ &= \dot{g}G_0' + g\left(\frac{\chi G_0}{4\pi} - \frac{2}{3}\right)G_0' - (\eta + g)\left(\frac{\chi G_0}{4\pi} - \frac{2}{3}\right)G_0' \\ &\quad -3\left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right)G_0 - 4G_0' \equiv F_1 + F_2 + F_3 + F_4 + F_5. \end{aligned} \tag{4.9}$$

We shall look for a solution of (4.9) such that:

$$G_1(g(\tau), \tau) = \frac{4\pi}{3\chi} \qquad G_1(\eta, \tau) \to 0 \qquad \text{as } \eta \to -\infty. \tag{4.10}$$

For $i = 1, 2, 3, 4, 5$, let G_{1i} be a solution of (4.9) with right-hand side F_i and satisfying (4.10). We now claim that, as $\eta \to \infty$:

$$\begin{aligned} &G_{11} \sim \frac{8\pi}{\chi}\dot{g} \qquad G_{12} \sim -g\left(\frac{8\pi}{3\chi}\right) \qquad G_{14} \sim -\frac{8\pi}{\chi}(\eta + g(\tau)) \\ &G_{13}, G_{15} \sim C \qquad \text{for some positive constant } C. \end{aligned} \tag{4.11}$$

To check (4.11), we argue as follows. Consider a function $G_c(\eta)$ that satisfies:

$$cG_c' = G_c'' + \left(\frac{\chi G_c}{4\pi} - \frac{1}{3}\right)G_c' \qquad -\infty < \eta < \infty \tag{4.12a}$$

$$G_c(-\infty) = 0 \qquad G_c(0) = \frac{4\pi}{3\chi} \tag{4.12b}$$

where c is a given real number. Clearly, for $c = 0$ we recover function $G_0(\eta)$ in (3.30). In general, $G_c(+\infty)$ will depend on c. An integration of (4.12a) gives at once that:

$$G_c(+\infty) \equiv h(c) = \frac{8\pi}{\chi}\left(c + \frac{1}{3}\right).$$

Suppose now that c is small, and let us try on (4.12a) an expansion of the type $G_c = G_0 + c\tilde{G}_1 + \cdots$. To the first order, G_1 will then satisfy

$$G_0' = G_1'' + \left(\frac{\chi G_0}{4\pi} - \frac{1}{3}\right)G_1' + \frac{\chi G_0'}{4\pi}G_1 \qquad -\infty < \eta < \infty \tag{4.13a}$$

$$G_1(-\infty) = G_1(0) = 0 \tag{4.13b}$$

and one easily sees that $\tilde{G}_1(+\infty) = \frac{\partial h}{\partial c} = \frac{8\pi}{\chi}$. Replacing c by $\dot{g}$, (4.11) readily follows by recalling that $G_0(\eta) \sim \frac{8\pi}{3\chi}$ as $\eta \to \infty$.

From (4.8) and (4.11), the following modified inner expansion follows when $\eta \gg 1$:

$$\begin{aligned} G(\eta, \tau) &\sim \frac{8\pi}{3\chi} + \mathrm{e}^{-\frac{\tau}{3}}\left(\frac{8\pi}{\chi}\dot{g} - g\left(\frac{8\pi}{3\chi}\right) - \frac{8\pi}{\chi}(\eta + g) + C\right) \\ &= \frac{8\pi}{3\chi} + \mathrm{e}^{-\frac{\tau}{3}}\left(\frac{8\pi}{\chi}\dot{g} - \frac{4}{3}\frac{8\pi}{\chi}g - \frac{8\pi}{\chi}\eta + C\right). \end{aligned} \tag{4.14}$$

On the other hand, if we now match (4.14) with the outer expansion (3.39a) we obtain:

$$\dot{g} = \tfrac{4}{3}g + \mathrm{O}(\mathrm{e}^{-\frac{\tau}{3}}),$$

so that $\delta(\tau) = g(\tau)\mathrm{e}^{-\frac{\tau}{3}}$ satisfies:

$$\dot{\delta} = \delta + \mathrm{O}(\mathrm{e}^{-\frac{2\tau}{3}}). \tag{4.15}$$

Integrating (4.15) between τ_0 and τ_1 ($\tau_0 < \tau_1$) now yields:

$$\delta(\tau_1) = \delta(\tau_0)e^{\tau_1-\tau_0} + e^{\tau_1}\int_{\tau_0}^{\tau_1} O(e^{-\frac{5s}{3}})\,ds.$$

Thus, in order to have $\delta(\tau_1) \to 0$ for $\tau_1 \to \infty$, it suffices to select $\delta(\tau_0) = O(e^{-\frac{2\tau_0}{3}})$ (compare with (4.7)), which in turn implies:

$$\delta(\tau) \sim Ce^{-\frac{2\tau}{3}} \qquad \text{for } \tau \to \infty.$$

Note that in (4.15) we recover, for the case of the complete equation (3.13), the instability previously described in section 4.1 for the simplified hyperbolic equation (3.14).

4.3. Stability of blow-up patterns

We conclude this section by remarking that the previous discussion strongly suggests that the asymptotic patterns derived in section 3 are stable under small perturbations. This is due to the fact that in our current analysis there are only two possible sources of instability. The first of these corresponds to a shift on the shock location when the initial value is slightly changed, and has been discussed above. The second one corresponds to a change in the location of the point $\bar{x}_1$ where $\bar{G}(\bar{x}_1) = \frac{4\pi}{3\chi}$ (see figure 3.1) when such a perturbation is applied. A quick glance at the picture of characteristics in figure 3.2 reveals that, in order to determine the values of $G(\xi, \tau)$ for large times, we only need to know these in a small neighbourhood of $\xi = \bar{x}_1$. That region has been studied at the beginning of section 3.2, where it has been shown that the dynamics therein is dominated by equation (3.20). Such an equation has a neutral eigenvalue, all the others being stable (cf (3.21)). The neutral eigenvalue is associated to the existence of a continuum set of stationary profiles (3.16), that depend on the parameter K there. Such an eigenvalue can be cancelled by means of a change of the constant K in (3.16), which induces a change in the value of the corresponding point $\bar{x}_1$. Once this cancellation has been performed, only the stable eigenvalues remain in (3.21), and the structure of characteristic curves of (3.14) yields that $G(\xi, \tau) \to \bar{G}(\xi)$ as $\tau \to \infty$, away from the shocks.

5. Concluding remarks

We have described a class of radial solutions of the reaction-diffusion system (1.1) that blow up at the origin in a finite time T, where $T > 0$ is arbitrary. When written in suitable rescaled variables, the solution profile approaches towards an imploding shock wave (which for any $t < T$ is smoothed out by the effect of diffusion) as blow-up unfolds. Such a type of behaviour is of a hydrodynamical type, and is akin to a blow-up mechanism for nonlinear Fokker–Planck equations which was recently studied by rigorous methods in [4].

We next remark briefly on the way in which the mass is distributed in the solutions obtained before. In terms of the variables Φ, y, τ, we have shown that the asymptotic profiles are of the form:

$$\Phi(y, \tau) \sim \frac{8\pi}{3\chi}e^{\frac{\tau}{2}} \cdot \mathbf{1}_{\{y \geqslant e^{\frac{\tau}{6}}\}}$$

where $\mathbf{1}_{\{y \geqslant e^{\frac{\tau}{6}}\}}$ denotes a function that is equal to one when $y \geqslant e^{\frac{\tau}{6}}$, and is zero elsewhere. As to the local mass function, we then have that:

$$M(r, t) = e^{-\frac{\tau}{2}}\Phi(y, \tau) \sim \frac{8\pi}{3\chi} \cdot \mathbf{1}_{\{r \geqslant (T-t)^{\frac{1}{3}}\}}. \qquad (5.1)$$

Recalling (3.3), we obtain a corresponding estimate for $u(r,t)$, namely:

$$u(r,t) \sim \frac{1}{4\pi r^2}\frac{\partial M}{\partial r} \sim \frac{2}{3\chi r^2}\frac{\partial}{\partial r}(\mathbf{1}_{\{r\geqslant (T-t)^{\frac{1}{3}}\}}) = \frac{2}{3\chi r^2}\delta(r-(T-t)^{\frac{1}{3}}). \tag{5.2}$$

It follows from (5.1) and (5.2) that our solutions will have a mass which is sharply concentrated at distances of order $\mathrm{O}((T-t)^{\frac{1}{3}})$. On the other hand, our analysis shows that the region near the unfolding shock has a width $\delta\xi$ of order $\delta\xi = \mathrm{O}(\mathrm{e}^{-\frac{\tau}{3}})$ (cf (3.26)). In terms of the variable r, this gives a width $\delta r \sim \mathrm{e}^{-\frac{\tau}{2}}R(\tau)\delta\xi$ of order:

$$\delta r = \mathrm{O}((T-t)^{\frac{2}{3}}).$$

This shows that $u(r,t) = \mathrm{O}((T-t)^{-\frac{4}{3}})$ in such region, and in the original variables our solutions look as indicated in figure 1.1.

Another remark concerns the amount of mass that eventually collapses at the origin. Suppose that we now take $R(\tau) = C\mathrm{e}^{\frac{\tau}{6}}$, $C \neq 1$, instead of (3.12). The constant K in (3.16) then remains unchanged, but the corresponding asymptotic formula for the mass now reads:

$$M(0,t) \sim \frac{8\pi}{3\chi}C^3$$

which can be made arbitrary.

The manner of the blow-up just described is quite different to that analysed in [8] for the same system when $N = 2$. In particular, while a minimal mass is required for blow-up to occur in two dimensions, no such restriction appears when $N = 3$. In both cases, however, a completely nonlinear analysis had to be performed to compute the blow-up profiles. This is a natural fact, since these profiles develop near blow-up, and therefore cannot be determined by means of a weakly nonlinear analysis as that used, for instance, to predict the initiation of the instability which will eventually lead to the formation of singularities.

A final remark concerns the methods employed in this paper. We have obtained our solutions by means of matched asymptotic expansions techniques. We followed such an approach in [9] to show that, when $N = 2$, the complete system (1.1*a*), (1.2) has solutions exhibiting chemotactic collapse that are quite similar to those obtained in [8] for the simplified model (1.1). Rigorous proofs of the existence of such solutions were provided in [10] for the complete system, thus completing the analysis already performed in [8] for system (1.1). The basic elements in the rigorous proofs in [8, 10] are as follows.

(i) Deriving first the sought-for patterns by asymptotic methods.

(ii) Proving then that, if we start from data whose profiles are close enough to those obtained in (i), the flow associated to the system will drive solutions towards the desired blow-up structure as time passes.

The last step above is fulfilled by means of a topological fixed point argument, which is explained in detail in [8, 10]. We expect that such a proof can also be provided in our case here (actually, we have considered a closely related situation in [4]). However, we have chosen to present only the formal argument in this note. We believe that, besides a considerable simplification in the presentation, this approach enables us to insist on what we consider to be the main elements of such result.

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