Arch. Math. 69 (1997) 333–337
0003-889X/97/040333-05 $ 2.50/0

Archiv der Mathematik

On operators which attain their norm at extreme points

By

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Abstract. We introduce a geometric property on Banach spaces, the E-property, that is implied by the λ-property and that implies the Bade property, although these properties are different. By mean of the E-property we characterize the topological dimension of compact metric spaces. If $(X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces and $p\in[1,+\infty)$ we relate the E-property of $(\oplus X_n)_p$ with the E-property of every X_n. Finally, if K is a compact Hausdorff space and X is a Banach space, we study the E-property on the dual space of $C(K,X)$.

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Let X be a Banach space and let X^* be its dual space. We denote by B_X the closed unit ball of X and by S_X its unit sphere. The set of extreme points of B_X will be denoted by Ext B_X. We denote by $\overline{\text{co}}\,(\text{Ext}\,B_X)$ the closed convex hull of Ext B_X, and by s-co$\,(\text{Ext}\,B_X)$ the sequentially-convex hull of Ext$\,B_X$: i.e. $x\in$ s-co$\,(\text{Ext}\,B_X)$ if and only if there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of positive real numbers with $\sum_{i=1}^{\infty}\alpha_i=1$ and a sequence $(e_n)_{n\in\mathbb{N}}$ in Ext$\,B_X$ such that $x=\sum_{i=1}^{\infty}\alpha_i e_i$. We have that co$\,(\text{Ext}\,B_X)\subset$ s-co$\,(\text{Ext}\,B_X)\subset\overline{\text{co}}\,(\text{Ext}\,B_X)$ and if X is a finite-dimensional space these three sets are coincident with B_X.

X is said to have the Bade property ([5]) if $\overline{\text{co}}\,(\text{Ext}\,B_X)=B_X$. The following concepts were introduced by R. M. Aron and R. H. Lohman [3]. Let $x\in B_X$. An ordered triple (e,y,λ) in Ext $B_X\times B_X\times(0,1]$ is amenable to x if $x=\lambda e+(1-\lambda)y$. X is said to have the λ-property if every $x\in B_X$ admits an amenable triple. X has the λ-property if and only if $B_X=$ s-co$\,(\text{Ext}\,B_X)$ ([4]).

Let K be a compact Hausdorff space. We denote by $C(K,X)$ the space of continuous X-valued functions on K endowed with the sup norm. When $X=\mathbb{R}$ we denote $C(K)=C(K,X)$. The following conditions are equivalent ([5] and [7]): 1) $C(K)$ has the Bade property. 2) $C(K)$ has the λ-property. 3) K is 0-dimensional.

Mathematics Subject Classification (1991): Primary 46B20; Secondary 46E15.

In [3] it is proved that if X has the λ-property then X has the Bade property and that if K is the unit ball of $\mathbb{C}$ then the space $C(K,\mathbb{C})$ has the Bade property but lacks the λ-property.

It is not hard to prove that the following conditions are equivalent: 1) X has the Bade property. 2) Every continuous linear form $f: X \to \mathbb{R}$ is such that $\|f\| = \sup\{f(e) : e \in \operatorname{Ext} B_X\}$. 3) If Y is a Banach space and $T: X \to Y$ is a continuous linear mapping then $\|T\| = \sup\{\|T(e)\| : e \in \operatorname{Ext} B_X\}$.

Theorem 1. *Let X be a Banach space. The two following conditions are equivalent:*

1. *For every continuous linear form $f: X \to \mathbb{R}$ such that $\|f\| = 1$ and $f(x) = 1$, for some $x \in S_X$, there exists an $e \in$ Ext B_X such that $f(e) = 1$.*
2. *For every Banach space Y and every continuous linear mapping $T: X \to Y$ such that $\|T\| = \|T(x)\|$, for some $x \in B_X$, there exists an $e \in \operatorname{Ext} B_X$ such that $\|T\| = \|T(e)\|$.*

Proof. Let us assume that 1) is true and let us suppose that $T: X \to Y$ is a continuous linear mapping such that $\|T\| = 1$ and that $x_0 \in S_X$ is such that $\|T(x_0)\| = 1$. Let $y_0^* \in S_{X^*}$ be such that $y_0^*(T(x_0)) = 1$. We have $y_0^* T \in S_{X^*}$ and, by hypothesis, there exists an $e_0 \in \operatorname{Ext} B_X$ such that $y_0^*(T(e_0)) = 1$. Hence, $1 \geqq \|T(e_0)\| \geqq y_0^*(T(e_0)) = 1$ and therefore $\|T(e_0)\| = 1$.

Definition 2. Let X be a Banach space. We will say that X has the E-property if X verifies either of the two equivalent conditions that appear in Theorem 2.1.

Theorem 3. *Let X be a Banach space and let us consider the following conditions:* 1) *X has the λ-property.* 2) *X has the E-property.* 3) *X has the Bade property. We have that* 1) $\Rightarrow$ 2) $\Rightarrow$ 3).

Proof. 1) $\Rightarrow$ 2). Let $f: X \to \mathbb{R}$ be a continuous linear form such that $\|f\| = 1$ and $f(x) = 1$ for some $x \in S_X$. If $(e, y, \lambda) \in \operatorname{Ext} B_X \times B_X \times (0,1]$ is an amenable triple to x then $x = \lambda e + (1-\lambda) y$, hence $f(e) = 1$.

2) $\Rightarrow$ 3). Let $f: X \to \mathbb{R}$ be as before. We are going to check that $\sup\{f(e) : e \in \operatorname{Ext} B_X\} = 1$. Let $\varepsilon > 0$. By Bishop-Phelps Theorem ([8]) there exist a $z^* \in S_{X^*}$ and a $z \in S_X$ such that $\|f - z^*\| < \varepsilon$ and $|z^*(z)| = 1$. By hypothesis, there exists an $e_0 \in \operatorname{Ext} B_X$ with $|z^*(e_0)| = 1$. Therefore, we have $|f(e_0)| \geqq |z^*(e_0)| - |f(e_0) - z^*(e_0)| > 1 - \varepsilon$.

Remark 4. Let K be a compact Hausdorff space and let X be a Banach space.

a) As a consequence of Theorem 3, it is easy to prove that the following conditions on a Hausdorff compact space K are equivalent: 1) $C(K)$ has the Bade property. 2) $C(K)$ has the E-property. 3) $C(K)$ has the λ-property. 4) K is 0-dimensional.

b) By [2], the following statements are also equivalent: 1) $C(K,X)^*$ has the Bade property. 2) $C(K)^*$ and X^* have the Bade property. 3) K is dispersed and X^* has the Bade property.

If K is 0-dimensional and K is not dispersed then $X = C(K)$ has the E-property, but X^* does not have the E-property.

Example 5. Let us prove that there exist Banach spaces with the Bade property that lack the E-property.

Let $D = \{z \in \mathbb{C} : |z| \leqq 1\}$. The space $X = C(D,\mathbb{C})$ has the Bade property ([8]). Nevertheless, we will prove now that X does not have the E-property.

If X has the E-property and μ is a positive regular Borel measure with $\|\mu\| = 1$ and whose support is $S_1 = \{z \in \mathbb{C} : |z| = 1\}$, let $l \in X$ be defined by $l(z) = \bar{z}$, for every $z \in D$, and let $x^* \in X^*$ be defined by $x^*(f) = \int l \cdot f d\mu, f \in X$. We can see immediately that $\|x^*\| \leqq 1$ and $x^*(\bar{l}) = \int l \cdot \bar{l} d\mu = \int_{S_1} 1 \cdot d\mu = 1$. Hence, there is a $g \in$ Ext B_X (i. e. $|g(z)| = 1$ for every $z \in D$ ([8])) such that $x^*(g) = 1$. Therefore $\int l \cdot g d\mu = 1$ and $l \cdot g = 1$ μ-almost everywhere in S_1. Since $\bar{l}$ and g are continuous functions, we have that $g(s) = \bar{l}(s)$ holds for $s \in S_1$. This proves that g must then be a retract from the unit disc D onto its frontier, which is impossible.

Remark 6. In [6] it is proved that:

a. If K is a compact Hausdorff space and X is an n-dimensional strictly convex space then the following conditions are equivalent: i) $C(K, X)$ has the λ-property, ii) every continuous mapping $g : F \to S_X$ defined on a closed subset F of K has a continuous extension from K into S_X, iii) $\dim(K) \leqq n - 1$, where $\dim(K)$ is the covering dimension of K.

b. If K is a Hausdorff compact space and X is an infinite-dimensional strictly convex space then i) and ii) are both true.

Theorem 7. *Let K be a compact metric space and let $\mathbb{R}^n$ be the n-dimensional euclidean space. If $C(K, \mathbb{R}^n)$ has the E-property then* $\dim(K) \leqq n - 1$.

Proof. Let $F \subset K$ be a closed set and let $g : F \to S_{\mathbb{R}^n}$ be a continuous mapping. Let $\hat{g} : K \to B_{\mathbb{R}^n}$ be a norm one extension of g and let $T : C(K, \mathbb{R}^n) \to \mathbb{R}$ be the mapping defined, for $f \in C(K, \mathbb{R}^n)$, by $T(f) = \int_K f(t) \cdot \hat{g}(t) d\mu$, where $\mu \in C(K)^*$ is such that $\mu \geqq 0$, $\|\mu\| = 1$ and supp $\mu = F$ and where we denote by $f(t) \cdot \hat{g}(t)$ the usual inner product in $\mathbb{R}^n$ of $f(t)$ and $\hat{g}(t)$. It is easy to check that $\|T\| = T(\hat{g}) = \int_F 1 d\mu = 1$. There exists an $e \in \text{Ext}\, B_{C(K,\mathbb{R}^n)}$ (i.e. $\|e(t)\| = 1$ for every $t \in K$) such that $T(e) = 1$. Hence, $1 = \int_F e(t) \cdot g(t) d\mu$ and $e(t) \cdot g(t) = 1$, for every $t \in F$. Since $\|e(t)\| = \|g(t)\| = 1$, $e(t) = g(t)$, for every $t \in F$, and e is a continuous extension of g from K into $S_{\mathbb{R}^n}$.

Remark 8. As a consequence of Theorem 7 we have that:

a) $C(B_{\mathbb{R}^k}, \mathbb{R}^n)$ has the E-property if and only if $k \leqq n - 1$.

b) If K is a metric compact space and $X = \mathbb{R}^n$ then $C(K, X)$ has the λ-property if and only if $C(K, X)$ has the E-property. Nevertheless, we do not know if that equivalence is also valid if K is an arbitrary compact Hausdorff space and X is an arbitrary Banach space.

Theorem 9. *If X is a reflexive Banach space then X has the E-property.*

Proof. Let $f : X \to \mathbb{R}$ be a continuous linear form with $\|f\| = 1$ and let $M = \{x \in B_X : f(x) = 1\}$. Then $M \neq \emptyset$ and M is a weakly compact convex set. By the Krein-Milman Theorem there exists an extreme point $e \in M$. It is now easy to check that $e \in \text{Ext}\, B_X$.

Example 10. There are Banach spaces with the E-property that lack the λ-property.

If, for every $n \in \mathbb{N}$, we consider the space $X_n = (\mathbb{R}^n, \| \|_1)$, then every X_n has the λ-property and $X = (\oplus X_n)_p$, $1 < p < +\infty$, is a reflexive Banach space. However, X does not have the λ-property ([1] and [9]). Hence, X has the E-property but lacks the λ-property.

Theorem 11. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces and $p \in [1, +\infty)$. $X = (\oplus X_n)_p$ has the E-property if and only if X_n has the E-property, for every $n \in \mathbb{N}$.*

Proof. We will consider that $p > 1$. We have that $X^* = (\oplus X_n^*)_q$, where $1/p + 1/q = 1$. Let us suppose that X has the E-property. Let $n \in \mathbb{N}$, let $z_n^* \in S_{X_n^*}$ and let $z_n \in S_{X_n}$ be such that $z_n^*(z_n) = 1$. Let $\varphi \in X^*$ be defined as the sequence $\varphi = (x_i^*)_{i\in\mathbb{N}}$ where $x_n^* = z_n^*$ and $x_i^* = 0$ for $i \neq n$. Obviously $\|\varphi\| = \|z_n^*\| = 1$. If $(x_i)_{i\in\mathbb{N}} \in X$ is such that $x_n = z_n$ and $x_i = 0$ for $i \neq n$ then $\varphi((x_i)_{i\in\mathbb{N}}) = z_n^*(z_n) = 1$ and, therefore, there exists a sequence $(e_n)_{n\in\mathbb{N}} \in \operatorname{Ext} B_X$ which satisfies $\varphi((e_n)_{n\in\mathbb{N}}) = z_n^*(e_n) = 1$. In other words, $e_n \in \operatorname{Ext} B_{X_n}$.

If, for every $n \in \mathbb{N}$, the Banach space X_n has the E-property, let us consider the sequences $\varphi = (x_n^*)_{n\in\mathbb{N}}$, in S_{X^*}, and $(x_n)_{n\in\mathbb{N}}$, in S_X, such that $\varphi((x_n)_{n\in\mathbb{N}}) = \sum\limits_{i=1}^{\infty} x_i^*(x_i) = 1$. It is easy to check that

$$1 = \left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \leqq \sum_{i=1}^{\infty} |x_i^*(x_i)| \leqq \sum_{i=1}^{\infty} ||x_i^*||\, ||x_i|| \leqq \left(\sum_{i=1}^{\infty} ||x_i^*||^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{\infty} ||x_i||^p\right)^{\frac{1}{p}} = 1.$$

Therefore if $x_n \neq 0$ we have that $|x_n^*(x_n)| = \|x_n^*\|\,\|x_n\|$ and, by hypothesis, there exists an $e_n \in \operatorname{Ext} B_{X_n}$ such that $x_n^*(e_n) = \|x_n^*\|$. Let $(z_n)_{n\in\mathbb{N}}$ be the sequence defined by $z_n = \|x_n\| e_n$, if $x_n \neq 0$, and $z_n = 0$, if $x_n = 0$. It is easy to check that $(z_n)_{n\in\mathbb{N}} \in \operatorname{Ext} B_X$ and $\varphi((z_n)_{n\in\mathbb{N}}) = \sum\limits_{n=1}^{\infty} x_n^*(z_n) = \sum\limits_{n=1}^{\infty} \|x_n^*\| \cdot \|x_n\| = 1$. This proves the theorem if $p > 1$.

Now we will consider that $p = 1$. We have that $X^* = (\oplus X_n^*)_\infty$. If X has the E-property then we can show, similarly to the case $p > 1$, that X_n has the E-property, for every $n \in \mathbb{N}$. Conversely, let us suppose that X_n has the E-property, for every $n \in \mathbb{N}$. Let $\varphi = (x_n^*)_{n\in\mathbb{N}} \in S_{X^*}$ be such that there exists a sequence $(x_n)_{n\in\mathbb{N}} \in S_X$ with $\varphi((x_n)_{n\in\mathbb{N}}) = 1$. We have that

$$1 = \sum_{n=1}^{\infty} x_n^*(x_n) \leqq \sum_{n=1}^{\infty} |x_n^*(x_n)| \leqq \sum_{n=1}^{\infty} \| x_n \| = 1.$$

Hence, if $m \in \mathbb{N}$ and $x_m \neq 0$, we have that $x_m^*\left(\dfrac{x_m}{\| x_m \|}\right) = 1$ and $\| x_m^* \| = 1$. By hypothesis, there exists an $e'_m \in \operatorname{Ext} B_{X_m}$ such that $x_m^*(e'_m) = 1$. Let $(e_n)_{n\in\mathbb{N}}$ be the sequence defined by $e_m = e'_m$ and $e_n = 0$ if $n \neq m$. Obviously $(e_n)_{n\in\mathbb{N}} \in \operatorname{Ext} B_X$ and $\varphi((e_n)_{n\in\mathbb{N}}) = x_m^*(e'_m) = 1$.

Let T be a set and let X be a Banach space. We consider the space $l_1(T, X) = \{(x_t)_{t\in T} : \sum\limits_{t\in T} \|x_t\| < +\infty\}$, endowed with the norm $\|(x_t)_{t\in T}\| = \sum\limits_{t\in T} \|x_t\|$. It is easy to check that $(x_t)_{t\in T} \in \operatorname{Ext} B_{l_1(T,X)}$ if and only if there exists a $t_0 \in T$ such that $x_{t_0} \in \operatorname{Ext} B_X$ and $x_t = 0$, for $t \in T \setminus \{t_0\}$.

Theorem 12. *The space $l_1(T, X)$ has the E-property if and only if X has the E-property.*

Proof. Let us suppose that X has the E-property. For every $f \in l_1(T, X)^*$ and $t_0 \in T$ let $f_{t_0} : X \to \mathbb{R}$ denote the continuous linear form defined, for $x \in X$, by $f_{t_0}(x) = f(x^{t_0})$, where $x^{t_0} \in l_1(T, X)$ is defined by $x_{t_0}^{t_0} = x$ and $x_t^{t_0} = 0$, if $t \neq t_0$. We have $f((x_t)_{t\in T}) = \sum\limits_{t\in T} f_t(x_t)$ and $||f|| = \sup\,\{||f_t|| : t \in T\}$. Let us suppose now that there exists $(x_t)_{t\in T} \in S_{l_1(T,X)}$ such that $f((x_t)_{t\in T}) = ||f|| = 1$. We have that $1 = \sum\limits_{t\in T} f_t(x_t) \leqq \sum\limits_{t\in T} \|f_t\| \cdot \|x_t\| \leqq \sum\limits_{t\in T} \|x_t\| = 1.$

Hence, $f_t(x_t) = \|x_t\|$ and $\|f_t\| = \|x_t\|$, for every $t \in T$. Let $p \in T$ be such that $x_p \neq 0$. We have $f_p\left(\frac{x_p}{\|x_p\|}\right) = \|f_p\| = 1$ and, by the hypothesis, there exists an $e \in \operatorname{Ext} B_X$ such that $f_p(e) = 1$. Therefore, $e^p \in \operatorname{Ext} B_{l_1(T,X)}$ and $f(e^p) = f_p(e) = 1$.

Conversely, let us suppose that $l_1(T,X)$ has the E-property and let $g : X \to \mathbb{R}$ be a continuous linear form which verifies that $g(z) = \|g\| = 1$, for $z \in S_X$. Let $p \in T$ and let us define $f : l_1(T,X) \to \mathbb{R}$ by $f((x_t)_{t\in T}) = g(x_p)$. We have also $\|f\| = 1$ and $f(z^p) = g(t) = 1$. Hence, there exists a sequence $(e_t)_{t\in T} \in \operatorname{Ext} B_{l_1(T,X)}$ such that $f((e_t)_{t\in T}) = g(e_p) = 1$. It is clear that $e_p \in \operatorname{Ext} B_X$.

Remark 13. Recall that if K is a compact Hausdorff space and X is a Banach space then K is dispersed if and only if $C(K,X)^* \simeq l_1(K,X^*)$. From Theorem 12 and Remark 4 we can derive the following corollary.

Corollary 14. *Let X be a Banach space and let K be a compact Hausdorff space. The following conditions are equivalent:* 1) *K is dispersed and X^* has the E-property.* 2) *$C(K)^*$ and X^* have the E-property.* 3) *$C(K,X)^*$ has the E-property.*

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Eingegangen am 2. 8. 1996*)

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*) Eine Neufassung ging am 6. 3. 1997 ein.