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Nonclassical symmetries of a porous medium equation with absorption

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Abstract. In this paper new symmetry reductions and exact solutions are presented for the porous medium equation with absorption $u_t = (u^n)_{xx} + g(x)u^m$. Those spatial forms for which the equation can be reduced to an ordinary differential equation are studied. The symmetry reductions and exact solutions presented are derived by using the nonclassical method developed by Bluman and Cole and are unobtainable by Lie classical method.

1. Introduction

The quasilinear parabolic equation

$$u_t = (u^n)_{xx} + g(x)u^m \tag{1}$$

with $n \neq 0$ serves as a simple mathematical model for various physical problems. Perhaps its most common use at the present time is to describe the flow of liquids in porous media or the transport of thermal energy in plasma. Here, we suppose that the diffusivity and absorption term have a power-law dependence on concentration $u(x, t)$ where n and m are constants. The second term on the right-hand side of (1) describes volumetric absorption, which in the case of plasma is caused by radiation to which plasma is transparent.

While the spatial dependent factors in (1) are usually assumed to be constant, there is no fundamental reason to assume so. Actually, allowing for their spatial dependence enables one to incorporate additional factors into the study which may play an important role. For instance, in a porous medium this may account for intrinsic factors such as medium contamination with another material or in plasma, this may express the impact that solid impurities arising from the walls have on the enhancement of the radiation channel.

The importance of the effect of space-dependent parts on the overall dynamics of (1) is well known. When $g(x) = 0$ equation (1) becomes

$$u_t = (u^n)_{xx}. \tag{2}$$

A complete group classification for the nonlinear heat equation (2) was derived by Ovsiannikov [51–53] by considering the PDE as a system of PDEs, and by Bluman [8, 11]. A classification for Lie–Bäcklund symmetries was obtained by Bluman and Kumei [9].

The basic idea of any similarity solution is that assumed by a functional form of the solution enables a PDE to be reduced to an ODE. The majority of known exact solutions of (2) turns out to be similarity solutions, even though originally they might have been derived, say by a separation of variable technique, or as travelling wave solutions. The main known

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exact solutions of nonlinear diffusion (2) are summarized by Hill [30]. In [30–32], Hill and Hill deduced a number of first integrals for stretching similarity solutions of the nonlinear diffusion equation, and of general high-order nonlinear evolution equations, by two different integration procedures.

King [39] obtained approximate solutions to the porous medium equation (2), integral results for the multidimensional nonlinear diffusion equation [40], and determine [38] new results by generalizing known instantaneous source and dipole solutions of N-dimensional radially nonlinear diffusion equations. He also applied generalized Bäcklund transformations and obtained a number of equivalence transformations to derive links between a large number of different types of nonlinear diffusion equations [41, 44]. By using local and nonlocal symmetries, some exact solutions which are not similarity solutions of (2) for special values of n [42, 44] were obtained.

Nonlinear diffusion with absorption arises in many areas of science and engineering. It occurs in the spatial diffusion processes where the physical structure of the medium changes with concentration. The same PDE also arises in the context of nonlinear heat conduction with a source term. For example, materials undergoing heating by microwave radiation exhibit thermal conductivities and body heating which are strongly dependent on temperature.

For $g(x)$ = constant, exact solutions and first integrals are obtained by Hill in [33], by the technique of separation of variables and the use of invariant one-parameter group transformations to reduce the governing PDE to various ODEs. For two of the equations so obtained, first integrals were deduced which subsequently give rise to a number of explicit simple solutions. Nonlinear diffusion with absorption is characterized by phenomena such as 'blow up', 'extinction', and 'waiting time' behaviour. The indices n and m encompass a wide range of this physical behaviour. For example, Kalashnikov [35] showed that $u(x, t) \equiv 0$ for all x after a finite time provided that $n > 1$ and $0 < m < 1$, a phenomenon referred to as 'extinction'.

A well known exact solution of (1) applying for $m = 2-n$ is due to Kersner [37]. Gurtin and MacCamy [28] proposed a transformation which reduces (1) with $g(x)$ = constant and $m = 1$ to (2). However, in general, the background details necessary to obtain solutions of (1) with $m = 1$ via this transformation and (2) are about the same as those required to obtain the solutions directly from (1).

In [23] Galaktionov presented a new technique of 'separation of variables' for constructing new exact solutions of the nonlinear heat conduction equations with a source, which are reduced to equations with quadratic nonlinearities. Most of the solutions thus constructed are not invariant under point transformations groups and Lie–Bäcklund groups. The proposed method was first implemented in [6] to construct an exact solution of equation (1) with $g(x) \equiv C > 0$ and $m = n$. In [24] a method is proposed to obtain exact blow-up solutions for nonlinear heat conduction equations with source.

Several references for the classification of Lie and Lie–Bäcklund symmetries for heat equations in homogeneous and nonhomogeneous medium, are also listed in [34].

Classical and nonclassical symmetries of the nonlinear equation (1), with $n = 1$, and $g(x)$ = constant were considered by Clarkson and Mansfield [18], by using the method of differential Grobner bases, and by Arrigo *et al* [4]. They obtained several new exact solutions.

In [25] a group classification problem was solved for equation (1) with a convective term, by studying those spatial forms which admit the classical symmetry group. Both the symmetry group and the spatial dependence were found through consistent application of the Lie-group formalism. In [9] Bluman introduced a method to find a new class of symmetries for a PDE when it can be written in a conserved form. These symmetries are neither point nor Lie–Bäcklund symmetries, they are nonlocal symmetries which are called *potential* symmetries. Potential symmetries were obtained in [26] for the porous medium equation.

Motivated by the fact that symmetry reductions for many PDEs are known that are not obtained by using the classical Lie group method, there have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole developed the nonclassical method to study the symmetry reductions of the heat equation; Clarkson and Mansfield [19] presented an algorithm for calculating the determining equations associated with the nonclassical method. The basic idea of the method is that the PDE (1) is augmented with the invariance surface condition

$$pu_x + qu_t - r = 0 \tag{3}$$

which is associated with the vector field

$$V = p(x,t,u)\frac{\partial}{\partial x} + q(x,t,u)\frac{\partial}{\partial t} + r(x,t,u)\frac{\partial}{\partial u}. \tag{4}$$

By requiring that both (1) and (3) are invariant under the transformation with infinitesimal generator (4) one obtains an overdetermined, nonlinear system of equations for the infinitesimals $p(x,t,u)$, $q(x,t,u)$, $r(x,t,u)$. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is in general, larger than for the classical method as in this method one requires only the subset of solutions of (1) and (3) to be invariant under the infinitesimal generator (4). However, the associated vector fields do not form a vector space. These methods were generalized by Olver and Rosenau [48, 49] to include 'weak symmetries', 'side conditions' or 'differential constraints', although their methods are too general to be practical.

All similarity reductions obtained from the optimal system of subalgebras by Lie classical symmetries were obtained in [25], therefore although most papers studying nonclassical symmetries include the classical ones, in this paper we consider nonclassical symmetries of equation (1), which are unobtainable by Lie classical method and find conditions on $g(x)$ as well as the special values of n and m for which these reductions can be derived.

In each section we list the functions $g(x)$ and the parameters n and m for which we obtain nonclassical symmetries. We also report the reduction obtained as well as some new exact solutions.

2. Nonclassical symmetries

To apply the nonclassical method to (1) we require (1) and (3) to be invariant under the infinitesimal generator (4). In the case $q \neq 0$, without loss of generality, we may set $q(x,t,u) = 1$. The nonclassical method applied to (1) gives rise to the following determining equations for the infinitesimals

$$\frac{\partial^2 p}{\partial u^2}u - n\frac{\partial p}{\partial u} + \frac{\partial p}{\partial u} = 0 \tag{5}$$

$$-n\frac{\partial^2 r}{\partial u^2}u^{n-1} + 2n\frac{\partial^2 p}{\partial u\partial x}u^{n-1}(n-n^2)\frac{\partial r}{\partial u}u^{n-2} + (n^2-n)ru^{n-3} - 2p\frac{\partial p}{\partial u} = 0 \tag{6}$$

$$-\left(2n\frac{\partial^2 r}{\partial u\partial x} - n\frac{\partial^2 p}{\partial x^2}\right)u^{n-1} + (2n-2n^2)\frac{\partial r}{\partial x}u^{n-2} - 3g\frac{\partial p}{\partial u}u^m + \frac{(n-1)pr}{u} + 2\frac{\partial p}{\partial u}r$$

$$-2p\frac{\partial p}{\partial x}-\frac{\partial p}{\partial t}=0 \tag{7}$$

$$-n\frac{\partial^2 r}{\partial x^2}u^{n-1}+\left(g\frac{\partial r}{\partial u}-2g\frac{\partial p}{\partial x}-\frac{\mathrm{d}g}{\mathrm{d}x}p\right)u^m+g(n-r)u^{m-1}+2\frac{\partial p}{\partial x}r=0. \tag{8}$$

Solutions of this system depend in a fundamental way on the values of n, m and on the function $g(x)$. By solving (5) we obtain

$$p=p_2(x,t)u^n-\frac{p_1(x,t)}{n}$$

and we can distinguish the following cases depending on n and m.

2.1. Case 1: $n\neq 0,-1,-\frac{1}{2}$

Solving (6), we obtain

$$r=a_1u^{n+2}+a_2u^{n+1}+\frac{r_2}{u^{n-1}}+a_3u^2+r_1u$$

where

$$a_1=-\frac{2{p_2}^2}{(n+1)(2n+1)}\qquad a_2=\frac{1}{n}\frac{\mathrm{d}p_2}{\mathrm{d}x}\qquad a_3=\frac{2p_1p_2}{n(n+1)}.$$

Substituting p and r into (7) and (8), we obtain that p_1, p_2, r_1, r_2 and $g(x)$ are related by two conditions and we can now distinguish the following subcases depending on n and m.

2.1.1. Case 1a: $m\neq n+2,n,n+1,2,1,0,1-2n,2-n,1-n$. For $g(x)$ following a power law, we recover the classical symmetries, that appear in [25].

2.1.2. Case 1b: $n=1$, $m\neq 0,1,2$. It follows that $p_2=0$, $r_2=0$ and p_1, r_1, and g are related by the following conditions

$$2\frac{\partial r_1}{\partial x}+\frac{\partial^2 p_1}{\partial x^2}+2p_1\frac{\partial p_1}{\partial x}-\frac{\partial p_1}{\partial t}=0 \tag{9}$$

$$(1-m)gr_1+2g\frac{\partial p_1}{\partial x}+p_1\frac{\mathrm{d}g}{\mathrm{d}x}=0 \tag{10}$$

$$\frac{\partial r_1}{\partial t}-\frac{\partial^2 r_1}{\partial x^2}-2r_1\frac{\partial p_1}{\partial x}=0. \tag{11}$$

Even though the previous equations are too complicated to be solved in general, special solutions will be considered. Choosing $p_1=p_1(x)$ and solving (9) we obtain

$$r_1=-\tfrac{1}{2}(p_1'+p_1^2+r_3) \tag{12}$$

with $r_3=r_3(t)$. Setting $r_3=k_1$, and solving (10) yields

$$g=c\exp\left[\frac{(1-m)}{2}\left(\int p_1+k_1\int\frac{1}{p_1}\right)\right](p_1)^{-\frac{(m+3)}{2}} \tag{13}$$

with $c=$ constant. By substituting (12) into (11), multiplying by p_1 and integrating once with respect to x, it follows that $p_1(x)$ must satisfy the following condition

$$2p_1p_1''-(p_1')^2+4p_1^2p_1'+p_1^4+2k_1p_1^2+2k_2=0. \tag{14}$$

Hence, for any function $p_1 = p_1(x)$ satisfying (14), where g is given by (13) and k_1 and k_2 are arbitrary constants, we obtain the nonclassical symmetry reduction

$$z = \int \frac{\mathrm{d}x}{p_1(x)} + t \qquad u(x,t) = \exp\left[\frac{1}{2}\left(\int p_1 + k_1(z-t)\right)\right]\sqrt{p_1}h(z). \tag{15}$$

Setting $k_1 = k_2 = 0$ and making

$$p_1 = \frac{w^2(x)}{\int w^2(x)\,\mathrm{d}x} \tag{16}$$

(14) can be written as

$$w''(x) = 0. \tag{17}$$

Consequently

$$w = c_1x + c_2 \qquad p_1 = \frac{3w^2}{x\alpha(x)} \qquad \text{and} \qquad g(x) = \frac{cx^2\alpha(x)^2}{kw(x)^{m+3}}$$

with

$$\alpha(x) = (c_1^2x^2 + 3c_1c_2x + 3c_2^2) \qquad \text{and} \qquad k = 3^{(m+3)/2}.$$

Hence, the nonclassical symmetry reduction becomes

$$z = \frac{c_2c_3^2}{3w(x)} + \frac{x^2}{6} + \frac{c_3x}{3} - t \qquad u = \sqrt{3}h(z)w(x)$$

where $c_3 = \frac{c_2}{c_1}$ and $h(z)$ satisfies the following ODE

$$h'' + ch^m = 0 \tag{18}$$

whose solution for $m \neq -1$ is

$$\int (m+1)^{1/2}[c(k_3(m+1) + h^{m+1})]^{-1/2}\,\mathrm{d}h = 2^{1/2}(z + k_4)$$

for $m = -1$ is given by

$$\sqrt{\pi}\,\mathrm{erf}\sqrt{-k3 - \log h}\,\mathrm{e}^{-k3} = \sqrt{2}\sqrt{c}(z + k4)$$

and for $m = 3$ is solvable in terms of elliptic functions.

2.1.3. Case 1c : n = 1, m = 2. It follows that $p_2 = 0$ and p_1, r_1, r_2 and g are related by the following conditions

$$-2\frac{\partial r_1}{\partial x} - \frac{\partial^2 p_1}{\partial x^2} - 2p_1\frac{\partial p_1}{\partial x} + \frac{\partial p_1}{\partial t} = 0 \tag{19}$$

$$-gr_1 + 2g\frac{\partial p_1}{\partial x} + \frac{\mathrm{d}g}{\mathrm{d}x}p_1 = 0 \tag{20}$$

$$-2gr_2 - \frac{\partial^2 r_1}{\partial x^2} + \frac{\partial r_1}{\partial t} - 2\frac{\partial p_1}{\partial x}r_1 = 0 \tag{21}$$

$$-\frac{\partial^2 r_2}{\partial x^2} + \frac{\partial r_2}{\partial t} - 2\frac{\partial p_1}{\partial x}r_2 = 0. \tag{22}$$

Although in general we are not able to solve these equations, special solutions will be considered. Choosing $p_1 = p_1(x)$, from these equations we obtain that r_1 and $g(x)$ are respectively given by (12) and (13) with $m = 2$ and setting $k_1 = 0$, r_2 adopts the following form

$$r_2 = \frac{1}{4p_1 g}\frac{\mathrm{d}\phi}{\mathrm{d}x} \tag{23}$$

where

$$\phi = p_1\frac{\mathrm{d}^2 p_1}{\mathrm{d}x^2} - \frac{1}{2}\left(\frac{\mathrm{d}p_1}{\mathrm{d}x}\right)^2 + 2p_1^2\frac{\mathrm{d}p_1}{\mathrm{d}x} + \frac{p_1^4}{2}.$$

We obtain that for any function $p_1 = p_1(x)$ satisfying the following equation

$$4\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}{p_1}^2 + 4\frac{\mathrm{d}\phi}{\mathrm{d}x}\left(\frac{\mathrm{d}p_1}{\mathrm{d}x} + {p_1}^2\right)p_1 + \phi^2 = k \tag{24}$$

nonclassical reductions can be derived integrating the characteristic equation.

Setting $r_2 = 0$, the similarity variable and similarity solution adopt the form obtained in the former case by (15) and h satisfies (18) with $m = 2$.

We remark that nonclassical symmetries for (1) with $n = 1$ and $g =$ constant were derived by Clarkson in [17] and that for $m \neq 3$ and $m \neq 1$ he did not find any further symmetries besides the classical ones.

2.1.4. Case 1d: $n = 1$, $m = 1$. Then $p_2 = 0$ and p_1, r_1, and g must be related by the following conditions

$$-2\frac{\partial r_1}{\partial x} - \frac{\partial^2 p_1}{\partial x^2} - 2p_1\frac{\partial p_1}{\partial x} + \frac{\partial p_1}{\partial t} = 0 \tag{25}$$

$$-\frac{\partial^2 r_1}{\partial x^2} + \frac{\partial r_1}{\partial t} - 2\frac{\partial p_1}{\partial x}r_1 + 2g\frac{\partial p_1}{\partial x} + \frac{\partial g}{\partial x}p_1 = 0 \tag{26}$$

and

$$-\frac{\partial^2 r_2}{\partial x^2} + \frac{\partial r_2}{\partial t} - 2\frac{\partial p_1}{\partial x}r_2 - gr_2. \tag{27}$$

The previous equations are too difficult to be solved in general, nevertheless special solutions will be considered. Choosing $p_1 = p_1(x)$ we obtain

$$p = -p_1 \qquad r = -\tfrac{1}{2}(p_1' + p_1^2 + r_3)u + r_2.$$

If $r_3 = k_1$, it follows that

$$2p_1p_1'' - (p_1')^2 + 4p_1^2p_1' + p_1^4 + 2k_1{p_1}^2 + 2k_2 + 2gp_1^2 = 0. \tag{28}$$

Now for any arbitrary function $g(x)$, if p_1 satisfies (28) we can derive a nonclassical symmetry reduction. Setting $r_2 = k_1 = k_2 = 0$, if p_1 adopts the form given by (16), then (28) becomes

$$w_{xx} + gw = 0. \tag{29}$$

Hence, for any arbitrary function $g(x)$, $w(x)$ and $p_1(x)$ can be respectively derived from (29) and (16), and the nonclassical symmetry reduction is given by (15).

2.1.5. Case 1e: $m = n + 2$. The only case in which we obtain symmetries different from the classical ones is for $n = 1$, $m = 3$ which is included in the former case.

For $m = n$ we obtain nonclassical symmetries in the following cases.

2.1.6. Case 1f: $n = \frac{1}{2}$, $m = \frac{1}{2}$. In this case $p_2 = 0$, $p = -2p_1$ and p_1, r_1, r_2 and g must be related by the following conditions

$$p_1 r_2 - \frac{\frac{\partial r_1}{\partial x}}{2} - \frac{\partial^2 p_1}{\partial x^2} = 0 \tag{30}$$

$$p_1 r_1 - 8 p_1 \frac{\partial p_1}{\partial x} + 2 \frac{\partial p_1}{\partial t} = 0 \tag{31}$$

$$\frac{\partial r_1}{\partial t} + \frac{{r_1}^2}{2} - 4 \frac{\partial p_1}{\partial x} r_1 = 0 \tag{32}$$

$$\frac{\partial r_2}{\partial t} + r_1 r_2 - 4 \frac{\partial p_1}{\partial x} r_2 - \frac{\frac{\partial^2 r_1}{\partial x^2}}{2} + 4g \frac{\partial p_1}{\partial x} + 2 \frac{\mathrm{d}g}{\mathrm{d}x} p_1 = 0 \tag{33}$$

$$-\frac{\partial^2 r_2}{\partial x^2} + {r_2}^2 - g r_2 = 0. \tag{34}$$

Despite the fact that the former equations are complicated to be solved in general, special solutions will be obtained. Choosing $p_1 = p_1(x)$ we can distinguish the following cases.

(1) If $p_1 \neq 0$ then solving (30) and (31) we obtain

$$r = 8 \frac{\mathrm{d}p_1}{\mathrm{d}x} u + \frac{5}{p_1} \frac{\mathrm{d}^2 p_1}{\mathrm{d}x^2} \sqrt{u}.$$

Substituting p_1, r_1 and r_2 into (33) and (34) we obtain that

$$g = \frac{2p_1''}{p_1} - \frac{6(p_1')^2}{p_1^2} + \frac{k_1}{2p_1^2}$$

and p_1 must satisfy the following equation

$$p_1^2 p_1''' - 4 p_1 p_1' p_1'' + \frac{k_1}{2} p_1' - k_2 = 0.$$

Setting $k_2 = 0$, dividing by p_1^6 and integrating once with respect to x leads to

$$10 p_1 p_1'' - 10 k_3 p_1^5 - k_1 = 0$$

whose solution is given by

$$\sqrt{(5)} \int (k_1 \log(p_1) + 2k_3 p_1^5 + k_4)^{-\frac{1}{2}} \, \mathrm{d}p_1 = x + k_5.$$

Hence, we obtain the nonclassical reduction

$$z = t - \int \frac{\mathrm{d}x}{2p_1(x)} \qquad u = \frac{\left(h(z) - 5 \frac{\mathrm{d}p_1}{\mathrm{d}x}\right)^2}{16 {p_1}^4}.$$

Setting $k_1 = k_4 = 0$, p_1 and g adopt the following form

$$p_1 = \frac{k_6}{(\alpha(x))^{2/3}} \qquad g = -\frac{4}{9(\alpha(x))^2}$$

with $\alpha(x) = x + k_5$. Consequently we obtain the nonclassical symmetry reduction

$$u = \frac{1}{16 k_6^4} \alpha^{8/3} \left(\frac{10 k_6}{3 \alpha^{5/3}} + h(z) \right)^2 \qquad z = t - \frac{3}{10 k_6} \alpha^{5/3}$$

where $h(z)$ satisfies the ODE

$$h' + h^2 = c$$

whose solutions are

$$h = \frac{\sqrt{c}[k\exp(2\sqrt{c}z) + 1]}{k\exp(2\sqrt{c}z) - 1} \qquad \text{if } c > 0$$

$$h = -\sqrt{-c}\tan[\sqrt{-c}(z + k)] \qquad \text{if } c < 0.$$

(2) If $p_1 = r_1 = 0$, then choosing $r_2 = r_2(x)$, we obtain that for any arbitrary function $g(x)$, if r_2 satisfies $gr_2 + r_2'' - r_2^2 = 0$, then r adopts the following form $r = r_2\sqrt{u}$. Hence, we obtain the nonclassical symmetry reduction

$$u = \left(\frac{\mathrm{tr}_2}{2} + h(x)\right)^2$$

where $h(x)$ satisfies the following ODE

$$r_2 h - h'' - gh = 0.$$

For example, if

$$g = \tan(x) - 2\sec^2(x)$$

an exact solution is

$$u = \tfrac{1}{4}((2k_1 x + t + 2k_2)\tan(x) + 2k_1)^2.$$

2.1.7. Case 1g: $m = -\frac{1}{3}$, $n = -\frac{1}{3}$. In this case $p_2 = r_2 = 0$, and p_1, r_1, and g must be related by the following conditions

$$\frac{2}{9}\frac{\partial r_1}{\partial x} + \frac{\partial^2 p_1}{\partial x^2} = 0 \tag{35}$$

$$4p_1 r_1 + 18 p_1 \frac{\partial p_1}{\partial x} + 3\frac{\partial p_1}{\partial t} = 0 \tag{36}$$

$$\frac{\partial r_1}{\partial t} + \frac{4{r_1}^2}{3} + 6\frac{\partial p_1}{\partial x} r_1 = 0 \tag{37}$$

$$\frac{1}{3}\frac{\partial^2 r_1}{\partial x^2} - 6g\frac{\partial p_1}{\partial x} - 3\frac{\partial g}{\partial x} p_1 = 0. \tag{38}$$

Solving (35) and (36) we obtain

$$r_1 = -\frac{9}{2}\frac{\partial p_1}{\partial x} + r_3 \qquad p_1 = c(x)\exp\left(-\frac{4}{3}\int r_3\,\mathrm{d}t\right)$$

with $r_3 = r_3(t)$. If we make $r_3(t) = -\frac{3b'(t)}{4b(t)}$, p and r become

$$p = b(t)c(x) \qquad r_1 = -\frac{9b}{2}\frac{\mathrm{d}c}{\mathrm{d}x} - \frac{3}{4b}\frac{\mathrm{d}b}{\mathrm{d}t}.$$

Substituting them into (37) and (38), we derive that $b = -\frac{1}{k_1(t+k_2)}$ and for any arbitrary function $g(x)$, making $c(x) = d(x)^2$, we have that $d(x)$ must satisfy the following ODE

$$d'' + gd + \frac{k_3}{2d^3} = 0.$$

Hence, we obtain the nonclassical symmetry reduction

$$z = -\frac{1}{k_1}\log(t + k_2) - \int\frac{\mathrm{d}x}{\mathrm{d}^2\,(x)} \qquad u = \frac{h(z)(-k_1 t - k_1 k_2)^{\frac{3}{4}}}{d^3}.$$

The ODE to which (1) is reduced, after making $h(z) = \frac{1}{y(z)^3}$ is

$$4y^4\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + 12\frac{\mathrm{d}y}{\mathrm{d}z} + 3k_1 y = 0.$$

2.1.8. Case 1h: $m = 1 - n$. Besides the classical symmetries we obtain that for $g(x) = k_1x + k_2$, $p = 0$ and $r = (k_1x + k_2)u^{1-n}$ the similarity solution adopts the form

$$u = [n(t + h(x))(k_1x + k_2)]^{\frac{1}{n}}$$

where $h(x)$ satisfies

$$(k_1x + k_2)h'' + 2k_1h' = 0.$$

A solution for this equation is $h = \frac{k_3x+k_4}{k_1x+k_2}$, that leads to the exact solution

$$u = [n((k_1t + k_3)x + k_2t + k_4)]^{\frac{1}{n}}.$$

2.2. Case 2: $n = -1$.

In this case solving the determining equations for the infinitesimals we obtain

$$p = p_1(x, t) \qquad r = r_2u^2 + r_1u$$

where p_1, r_1 and r_2 are related by the following conditions

$$-2p_1r_2u^3 - \left(2p_1r_1 + 2p_1\frac{\partial p_1}{\partial x} + \frac{\partial p_1}{\partial t}\right)u^2 - 2\frac{\mathrm{d}r_1}{\mathrm{d}x} - \frac{\mathrm{d}^2p_1}{\mathrm{d}x^2} = 0$$

$$-gmr_2u^{m+2} - \left(gmr_1 + gr_1 + 2g\frac{\partial p_1}{\partial x} + \frac{\mathrm{d}g}{\mathrm{d}x}p_1\right)u^{m+1} + 2{r_2}^2u^4 + \left(\frac{\mathrm{d}r_2}{\mathrm{d}t} + 4r_1r_2 + 2\frac{\partial p_1}{\partial x}r_2\right)u^3 + \left(\frac{\mathrm{d}r_1}{\mathrm{d}t} + 2{r_1}^2 + 2\frac{\partial p_1}{\partial x}r_1\right)u^2 + \frac{\mathrm{d}^2r_2}{\mathrm{d}x^2}u + \frac{\mathrm{d}^2r_1}{\mathrm{d}x^2} = 0.$$

We can now distinguish the following subcases.

(1) If $r_2 = 0$ and $g(x) = k(x + k_2)^{\frac{m-3}{2}}$ we recover the classical symmetries that appear in [25].

(2) If $p_1 = 0$, $m = 2$ and $g(x) = k_1x + k_2$ then $r = (k_1x + k_2)u^2$ and we obtain the nonclassical symmetry reduction

$$u = -\frac{1}{(k_1x + k_2)t + h(x)}$$

where $h = k_3x + k_4$.

2.3. Case 3: $n = -\frac{1}{2}$

In this case solving the determining equations for the infinitesimals we obtain

$$p = p_1(x, t) \qquad r = r_2u^{3/2} + r_1u$$

where p_1, r_1 and r_2 are related by the following conditions

$$-3p_1r_2u^2 - \left(3p_1r_1 + 4p_1\frac{\partial p_1}{\partial x} + 2\frac{\partial p_1}{\partial t}\right)u^{\frac{3}{2}} - \frac{\mathrm{d}r_1}{\mathrm{d}x} - \frac{\mathrm{d}^2p_1}{\mathrm{d}x^2} = 0$$

$$-2gmr_2u^{m+1} - \left(2gmr_1 + gr_1 + 4g\frac{\partial p_1}{\partial x} + 2\frac{\mathrm{d}g}{\mathrm{d}x}p_1\right)u^{m+\frac{1}{2}} + 3{r_2}^2u^{\frac{5}{2}} + 2\left(\frac{\mathrm{d}r_2}{\mathrm{d}t} + 3r_1r_2 + 2\frac{\partial p_1}{\partial x}r_2\right)u^2 + \left(2\frac{\mathrm{d}r_1}{\mathrm{d}t} + 3{r_1}^2 + 4\frac{\partial p_1}{\partial x}r_1\right)u^{\frac{3}{2}} + \frac{\mathrm{d}^2r_2}{\mathrm{d}x^2}\sqrt{u} + \frac{\mathrm{d}^2r_1}{\mathrm{d}x^2} = 0.$$

We can now distinguish the following cases.

(a) If $r_2 = 0$, and $g(x) = k(x + k_2)^{\frac{m-3}{2}}$ we recover the classical symmetries appearing in [25].

(b) If $p_1 = 0$, $m = \frac{3}{2}$ and $g(x) = k_1 x + k_2$ then we obtain $r = (k_1 x + k_2)u^{\frac{3}{2}}$ and so we obtain the nonclassical symmetry reduction

$$u = -\frac{1}{[(k_1 x + k_2)t + h(x)]^2} \qquad \text{where } h = k_3 x + k_4.$$

(c) If $p_1 = 0$, $m = 1$ and $g(x) = c$, besides the classical symmetries, we obtain $r = \frac{c}{1-\alpha(t)^{-3}} u$, with $\alpha(t) = \exp\left(\frac{c}{2}(t + d)\right)$ hence we obtain the nonclassical symmetry reduction

$$u = (1 + \alpha(t) + \alpha(t)^2)^{2/3} (\alpha(t) - 1)^{2/3} h(x)$$

where $h(x)$ satisfies the following ODE

$$2hh'' - 3h'^2 - 4ch^{7/2} = 0.$$

3. Concluding remarks

In this paper we have derived the nonclassical symmetries of the quasilinear parabolic equation (1) by using a method due to Bluman and Cole [11]. Recognizing the importance of the space-dependent parts on the overall dynamics of (1), we have studied those spatial forms as well as the different choices for the constants n and m for which equation (1) admits the nonclassical symmetry group. We have then constructed new invariant solutions, as well as new ODEs to which (1) is reduced. These new solutions are unobtainable by the method of Lie classical symmetries.

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