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SIMPSON POINTS IN PLANAR PROBLEMS WITH LOCATIONAL CONSTRAINTS. THE POLYHEDRAL-GAUGE CASE

EMILIO CARRIZOSA, EDUARDO CONDE,
MANUEL MUÑOZ-MÁRQUEZ AND JUSTO PUERTO

In this paper we address the problem of finding *Simpson* points in planar models with locational constraints when distances are measured by *polyhedral gauges*. Making use of the results we have stated in a previous paper, we show here the existence of a finite set of points in the plane, independent of the weights associated with the users, that contains at least a Simpson point.

Connection between Simpson points (as a result of a voting process) and *Weber* points (as the outcome of a planning process) are explored. It is shown by means of an example that the existing relations for the unconstrained case are no longer true when locational constraints are imposed. In order to reconcile both voting and planning processes, a biobjective problem is described, for which we construct a finite dominating set.

1. Introduction. Let A be a finite set of points (users) in $\mathbb{R}^2$; associated with each $a \in A$, let $\mu(\{a\})$ be a positive weight, and let ν^a be a gauge (see Durier 1990, Durier and Michelot 1985, Michelot 1993, Rockafellar 1970). Given a nonempty set S in $\mathbb{R}^2$, the Simpson function (Carrizosa et al. 1993b, Durier 1989) gives to each feasible $x \in S$ the value $W_S(x)$,

$$W_S(x) = \max_{y \in S} \mu(\{a \in A\colon \nu^a(y-a) < \nu^a(x-a)\})$$

where μ is defined as

$$\mu(B) = \sum_{b \in B} \mu(\{b\}).$$

In this paper, exploiting the general properties given in Carrizosa (1992) and Carrizosa et al. (1993b), we address the problem of finding the Simpson set $\Sigma(A, S)$ (the set of optimal solutions to problem $\min_{x \in S} W_S(x)$) under the following assumptions:

–Each ν^a is a polyhedral gauge in $\mathbb{R}^2$.

–The feasible set S is a polyhedron.

We use the same methodology than in Carrizosa et al. (1993b), which has a very long tradition in the location field (see, e.g., Francis 1963, Hakimi 1964, 1990, Thisse, Ward and Wendell 1984): we show the existence of a finite set—independent of the weights—that contains at least an optimal solution.

The rest of the paper is organized as follows: §2 reviews some properties of polyhedral gauges, and extends the definition of *elementary convex sets* and *intersection points* (see Durier 1990, Durier and Michelot 1985) to constrained problems. Section 3 presents a localization result, namely: the set of intersection points contains

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at least a Simpson point. In §4 we show by means of an example that the existing connections between Simpson and Weber points for the unconstrained case do not remain valid when locational constraints are imposed. In order to reconcile the processes of voting and planning, a biobjective problem is proposed, for which the set of intersection points is a dominator.

In order to avoid redundancy, we have not repeated in detail the concepts and interpretations used in this paper. We refer the reader to Carrizosa et al. (1993b) for further details.

2. Polyhedral gauges and elementary convex sets. We recall (see, e.g., Durier 1990, Durier and Michelot 1985) that a function γ is said to be a *polyhedral gauge* if there exists a polytope B (the unit ball of γ) containing the origin in its interior such that

$$\gamma(x) = \inf\left\{t > 0: \frac{1}{t}x \in B\right\} \quad \forall x \in \mathbb{R}^2.$$

Let $\{e^1, \dots, e^n\}$ be the set of extreme points of B. Then, B consists of the convex combinations of $\{e^1, \dots, e^n\}$:

$$B = \left\{x \in \mathbb{R}^2 : x = \sum_{i=1}^{m} \lambda_i e^i, \text{ for some } \lambda_1, \dots, \lambda_m \geq 0, \sum_{i=1}^{m} \lambda_i = 1\right\}.$$

With that, for all $x \in \mathbb{R}^2$,

$$\begin{aligned}\gamma(x) &= \inf\left\{t > 0: \frac{1}{t}x = \sum_{i=1}^{m} \lambda_i e^i;\ \sum_{i=1}^{m} \lambda_i = 1;\ \lambda_i \geq 0\ \forall i\right\}\\ &= \min\left\{t \geq 0: x = \sum_{i=1}^{m} \mu_i e^i;\ \sum_{i=1}^{m} \mu_i = t;\ \mu_i \geq 0\ \forall i\right\}\\ &= \min\left\{\sum_{i=1}^{m} \mu_i : \sum_{i=1}^{m} \mu_i e^i = x;\ \mu_i \geq 0\ \forall i\right\}.\end{aligned}$$

Hence, γ can be evaluated by solving the linear program above, the dual of which is:

$$\max\{\langle u, x\rangle : \langle u, e^i\rangle \leq 1\ \forall i = 1, \dots, m\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^2$. Let B^0 denote the *polar* of B:

$$\begin{aligned}B^0 &= \{u: \langle u, z\rangle \leq 1\ \forall z \in B\}\\ &= \{u: \langle u, e^i\rangle \leq 1\ \forall i = 1, \dots, m\}.\end{aligned}$$

B^0 is also a polytope, whose set of extreme points will be denoted by $\text{ext}(B^0)$. Hence, we can obtain an equivalent expression of γ in terms of $\text{ext}(B^0)$:

$$\gamma(x) = \max_{v \in \text{ext}(B^0)} \langle v, x\rangle \qquad \forall x \in \mathbb{R}^2.$$

Expression above implies a result that has been widely observed and successfully used in the literature: every polyhedral gauge γ is a piecewise linear function.

Indeed, denoting for each $v \in \text{ext}(B^0)$ by $N_{B^0}(v)$ the normal cone to B^0 at v, one has that there exists a finite family $\mathscr{Q}(\gamma)$ of polyhedra, $\mathscr{Q}(\gamma) = \{N_{B^0}(v)\colon v \in \text{ext}(B^0)\}$ that covers $\mathbb{R}^2$, and, within each of these polyhedra $N_{B^0}(v)$, γ is a linear function.

With these ideas in mind, we can extend to the constrained case the definitions of *elementary convex sets* and *intersection points*, given by Durier and Michelot (Durier 1990, Durier and Michelot 1985) for the unconstrained case. First, recall that, see, e.g. Brondsted (1983), given a closed convex set X, a convex subset F of X is said to be a face of X iff for any two distinct points $x, y \in X$ such that the open segment with endpoints x, y intersects F then the whole closed segment with endpoints x, y is contained in F.

DEFINITION 2.1. Let A be a nonempty finite set of points in $\mathbb{R}^2$, let $\{\nu^a\}_{a \in A}$ be a set of polyhedral gauges, and let S be a nonempty polyhedron.

• A nonempty subset C of S is said to be an *elementary convex set in S* (e.c.s. in S) iff for each $a \in A$ there exists $Q^a \in \mathscr{Q}(\nu^a)$ such that C is a face of $S \cap \bigcap_{a \in A}(a + Q^a)$.

• A point $x \in S$ is said to be an *intersection point in S* iff x is an extreme point of some e.c.s. C in S.

REMARK. From the definition, it follows that a set C is an e.c.s. in $\mathbb{R}^2$ iff C is an e.c.s. following the definition of Durier and Michelot (1985). Analogously, a point x is an intersection point in $\mathbb{R}^2$ iff x is an intersection point following the definition of Durier and Michelot (1985).

REMARK. From the definition of faces of a polyhedron and, since a nonempty intersection of e.c.s. in $\mathbb{R}^2$ is an e.c.s. in $\mathbb{R}^2$ (see Durier 1990), it follows that the nonempty intersection of elementary convex sets in S is also an e.c.s. in S.

REMARK. The set I of intersection points in S has a very manageable form: given a convex set X, denote by $\text{ext}(X)$ the set of its extreme points; for each $a \in A$, let B_a denote the unit ball of ν^a; given $a, e \in \mathbb{R}^2$, $e \neq 0$, denote by $r(a,e)$ the closed ray with apex at a in the direction of e, i.e.,

$$r(a,e) = \{x\colon x = a + \lambda e \text{ for some } \lambda \geq 0\}.$$

With that notation, as a direct application of the relations between normal cones to B^0 and faces of B (see Durier and Michelot 1985), it follows that the set I of intersection points in S consists of the points in

- $A \cap S$.
- $\text{ext}(S)$.
- $\text{ext}\{S \cap r(a,e)\colon a \in A,\ e \in \text{ext}(B_a)\}$.
- $\text{ext}\{S \cap r(a^\iota, e^\iota) \cap r(a^j, e^j)\colon a^\iota, a^j \in A,\ e^\iota \in \text{ext}(B_{a^\iota}),\ e^j \in \text{ext}(B_{a^j}) i \neq j,\}$.

In particular, if $|\text{ext}(B_a)| = O(r)\ \forall a \in A$, then $|I| = O(|A|^2 r^2 + |\text{ext}(S)|)$.

3. A localization result. The purpose of this section is to show that, for any system of weights $\{\mu(\{a\})\colon a \in A\}$, the set I of intersection points in S contains at least a Simpson point, i.e., $I \cap \Sigma(A,S) \neq \varnothing$. The methodology used is very similar to that used in Carrizosa (1992) and Carrizosa et al. (1993b): First, we describe geometrically the set of *weakly efficient points* (see Carrizosa et al. 1993b for details), and exploit the fact that the set $\Sigma(A,S)$ of Simpson points can be represented as intersections of sets of weakly efficient points.

Given $B \subset A$, $B \neq \emptyset$, we denote by $WE(B,S)$ the set of weakly efficient solutions to the point-objective problem

$$\min_{x \in S} \left(\nu^b(x-b) \right)_{b \in B}$$

i.e.,

$$WE(B,S) = \left\{ x \in S\colon \text{no } y \in S \text{ verifies } \nu^b(y-b) < \nu^b(x-b) \text{ for each } b \in B \right\}.$$

Let us first give some results about $WE(B,S)$ when B is an arbitrary nonempty subset of A. Observe that such results are in most cases direct extensions of analogous properties given in Durier (1990) for the unconstrained case. Given a set X, let $\mathrm{ri}(X)$ denote its *relative interior*; see, e.g. Brondsted (1983).

LEMMA 3.1. *For any $x \in S$, there exists an e.c.s. F in S such that $x \in \mathrm{ri}(F)$.*

PROOF. Let $x \in S$; by Theorem 6.2 in Durier (1990), there exists an e.c.s. C in $\mathbb{R}^2$ such that $x \in \mathrm{ri}(C)$. On the other hand, by Corollary 5.7 in Brondsted (1983), there exists a face H of S such that $x \in \mathrm{ri}(H)$. Hence, $C \cap H$ is an e.c.s. in S, and $x \in \mathrm{ri}(C) \cap \mathrm{ri}(H)$, thus (see page 25 of Brondsted 1983) $x \in \mathrm{ri}(C \cap H)$. □

LEMMA 3.2. *Let $x \in WE(B,S)$, and let F be an e.c.s. in S such that $x \in \mathrm{ri}(F)$. Then, $F \subset WE(B,S)$.*

PROOF. Let $x \in WE(B,S)$. By Theorem 1 in Lowe et al. (1984), there exist $\{w_b\}_{b \in B}$, $w_b \geq 0$ for each $b \in B$ with $\sum_{b \in B} w_b = 1$ such that

$$\sum_{b \in B} w_b \nu^b(x-b) \leq \sum_{b \in B} w_b \nu^b(y-b) \qquad \forall y \in S.$$

In particular, if we define the function $g\colon y \to g(y) = \sum_{b \in B} w_b \nu^b(y-b)$, one has:

$$g(x) = \min_{y \in S} g(y) = \min_{y \in F} g(y).$$

The function g is affine in F, F is a polyhedron, and g attains its minimum in F at a point x in $\mathrm{ri}(F)$. Hence, g is constant in F, which implies that any point $y^* \in F$ solves a Weber problem in S, namely:

$$g(y^*) = g(x) = \min_{y \in S} g(y)$$

thus $y^* \in WE(B,S)$ for each $y^* \in F$. Hence, $F \subset WE(B,S)$, as asserted. □

THEOREM 3.1. *$WE(B,S)$ is a connected union of bounded elementary convex sets.*

PROOF. Nonemptiness of $WE(B,S)$ can be proved as follows: consider the function $g\colon x \to g(x) = \sum_{b \in B} \nu^b(x-b)$. Since g is convex (thus continuous), has compact level sets and S is closed, there must exist some $x^* \in S$ such that $g(x^*) = \min_{x \in S} g(x)$. Such x^* is obviously a weakly efficient solution. Furthermore, $WE(B,S)$ is also bounded; indeed, $WE(B,S) \subset \bigcup_{b \in B} \{x \in S\colon\ \nu^b(x) \leq \nu^b(x^*)\}$, which is a bounded set. In order to show that $WE(B,S)$ is a union of elementary convex sets in S, we have to show that, for any $x \in WE(B,S)$, there exists an e.c.s. F in S such that $x \in F \subset WE(B,S)$. Given $x \in WE(B,S)$, by Lemma 3.1, there exists an e.c.s. F in S such that $x \in \mathrm{ri}(F)$. By Lemma 3.2, $F \subset WE(B,S)$. As $WE(B,S)$ is bounded, F is necessarily bounded. Finally, connectedness follows from the general results in Warburton (1983), and the result holds. □

We can give a geometrical description of the set $\Sigma(A, S)$ of Simpson points and state a localization result:

COROLLARY 3.1. *$\Sigma(A, S)$ is not empty, and can be represented as a finite union of bounded elementary convex sets in S.*

PROOF. Since W_S takes a finite number of different values, $\Sigma(A, S)$ is nonempty. Furthermore, if we let $\alpha = \min_{x \in S} W_S(x)$, it follows from Theorem 3.1 in Carrizosa et al. (1993b) that

$$\Sigma(A, S) = \bigcap_{\mu(B) > \alpha} WE(B, S);$$

thus, by Theorem 3.1, the result holds. □

COROLLARY 3.2. *The set I of intersection points in S contains at least a Simpson point.*

PROOF. By Corollary 3.1, there exists an e.c.s. F in S such that $F \subset \Sigma(A, S)$. Furthermore, as $\Sigma(A, S) \subset WE(A, S)$, and $WE(A, S)$ is bounded, F is also bounded, thus has at least an extreme point x. By the definition of the set I of intersection points, it follows that $x \in I$, and the result holds. □

Hence, in order to find a Simpson point, one can follow the process below:

(1) Find the set I of intersection points in S.

(2) Evaluate W_S at each $x \in I$ following the algorithm described in Carrizosa (1992) or Carrizosa et al. (1993b).

(3) Take as Simpson point a point $x \in I$ with smallest value $W_S(x)$.

As an illustration, consider the following example:

EXAMPLE 3.1. Let $A = \{a, b, c\}$, with $a = (0, 4)$, $b = (-2, 1)$, $c = (0, 0)$, associated with the following gauges and weights:

ν^a is the l_∞ norm, i.e., $\nu^a(x) = \max(|x_1|, |x_2|)$.

ν^b is the l_1 norm, i.e., $\nu^b(x) = |x_1| + |x_2|$.

ν^c is the asymmetric gauge whose unit ball is the triangle with vertices $(1, -1), (0, 1), (-1, -1)$. In other words,

$$\nu^c(x) = \max(\langle e_1, x\rangle, \langle e_2, x\rangle, \langle e_3, x\rangle),$$

with $e_1 = (2, 1)$, $e_2 = (0, -1)$ and $e_3 = (0, -1)$, the extreme points of the ball dual to ν^c.

$\mu(\{a\}) = 2/9$; $\mu(\{b\}) = 1/3$; $\mu(\{c\}) = 4/9$.

Let S be the square $[1, 4] \times [-4, 2]$. See Figure 1.

The set I of intersection points in S is easily shown to consist of ten points, whose coordinates and values of W_S are given in Table 1.

Hence, z^{10} is a Simpson point; moreover, one can see that z^{10} is the *unique* Simpson point (i.e., $\Sigma(A, S) = \{z^{10}\}$. Indeed, if there were some $z \neq z^{10}$, with $z \in \Sigma(A, S)$, then, there would exist some bounded e.c.s. F in S such that $z \in \mathrm{ri}(F) \subset F \subset \Sigma(A, S)$. Then, all the extreme points of F (and at least one of them is different to z^{10}) would be both Simpson points and intersection points in S, which is false (see Table 1). Hence, $\Sigma(A, S) = \{z^{10}\}$, as asserted. □

4. Simpson and Weber points. Perhaps the most popular and extensively studied problem in location theory is the constrained *Weber problem*, which seeks (see, e.g. Love, Morris and Wesolowsky 1988, Plastria 1993), a point $x^* \in S$ (a *Weber point*)

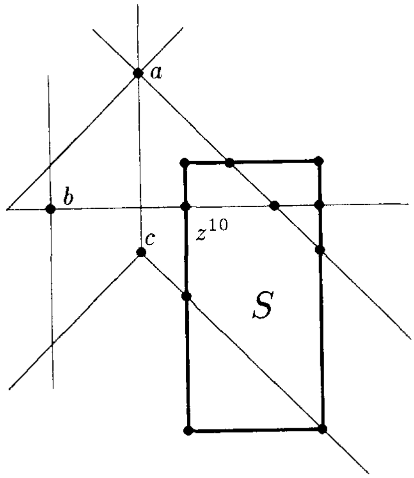

FIGURE 1. The intersection points in S.

TABLE 1

W_S at the intersection points in S.

Coordinates	W_S
(3, 1)	1
(1, 2)	7/9
(2, 2)	7/9
(4, 2)	1
(4, 1)	1
(4, 0)	1
(4, −4)	1
(1, −4)	1
(1, −1)	5/9
(1, 1)	4/9

that minimizes the average distance from the users to this point. In other words, if we define the function M,

$$M: x \to M(x) = \sum_{a \in A} \mu(\{a\}) \nu^a(x - a),$$

the Weber points are the minimizers in S of M.

Connections between Weber points (as outcome of a planning process) and the outcome of a voting process have received considerable attention for both networks and planar topologies (see, e.g., Hansen and Thisse 1981, Labbé 1985 for networks, Demange 1983, Durier 1989 for planar models) because of their obvious and important implications in real world locational decision-making: if both processes give the same output x, then, locating the facility at x not only maximizes the efficiency of the system, but also attains the highest user satisfaction (measured by the consensus obtained by x in terms of votes).

In this sense, some successful results have been obtained. For the planar case, the most general one (valid under less restrictive assumptions than those stated here) is due to Durier (1989):

THEOREM 4.1. *Suppose that $\nu^a = \nu$ for each $a \in A$. Then, any point $x \in \mathbb{R}^2$ with $W_{\mathbb{R}^2}(x) \leq 1/2$ is also a Weber point.*

Hence, as soon as there exists some point $x \in \mathbb{R}^2$ with $W_{\mathbb{R}^2}(x) \leq 1/2$ (i.e., x is a *Condorcet point*), then we have that x is also a Weber point in $\mathbb{R}^2$.

This result is, however, rather limited, since Condorcet points are unlikely to exist (Demange 1983). A significant exception is the l_1 norm case: if $\nu^a = \nu$ for each $a \in A$, and ν is the l_1 norm, then it is well known (see, e.g., Demange 1983, Durier 1989) that Condorcet points exist whatever the set A is. Hence, in the l_1 norm case, every Simpson point is a Weber point.

Unfortunately, things may be rather different when the feasible region S is a proper subset of $\mathbb{R}^2$:

• The theorem above is no longer true: there might exist points $x \in S$ with $W_S(x) \leq 1/2$ not being Weber points.

• In the l_1 norm case, although the existence of Condorcet points in constrained problems has been proved (see Carrizosa et al. 1993a), there does not seem to be a direct connection between Condorcet and Weber points.

As an illustration, consider the following example:

EXAMPLE 4.1. Let ν be the l_1 norm, let $A = \{a^1, a^2, a^3, a^4, a^5\}$, with

$$a^1 = (0,2), \qquad a^2 = (1,3), \qquad a^3 = (2,1), \qquad a^4 = (3,5), \qquad a^5 = (4,4),$$

and let the weight associated with each a^i equal $1/5$. If S is the triangle with vertices $(0,0), (2,0), (0,1)$, it is easy to check that

• $\Sigma(A, S) = \{(2,0)\}$, and $W_S(2,0) = 2/5 < 1/2$.

• The point $(1, 1/2)$ is the *unique* Weber solution in S. □

In real world location problems, decision-makers might be interested in locating the facility at a point $x \in S$ with a very high system efficiency, but, at the same time, with a negligible social reaction against. In terms of competitive location (see, e.g., Hakimi 1990), making both values small means having low transportation costs and capturing at the same time a high portion of the market.

In order to deal with these two (and as shown in the example above) criteria, two classical approaches may be followed:

(1) The constraint approach.

(2) The *bicriterion* approach.

In the constraint approach, one of the two objectives is taken as objective function, and the other one is considered as constraint. Hence, one could consider as feasible the set those points $x \in S$ such that at most a portion α will be against, and one minimizes M:

$$(CP_1(\alpha)) \qquad \begin{aligned} \min \quad & M(x), \\ \text{s.t.} \quad & W_S(x) \leq \alpha, \\ & x \in S. \end{aligned}$$

On the other hand, one might control the efficiency of the system (by considering as feasible only those points $x \in S$ with $M(x) \leq t$) and will minimize, among these points, the votes against the location. This leads to solving the optimization problem $(CP_2(t))$:

$$(CP_2(t)) \qquad \begin{aligned} \min \quad & W_{S(t)}(x), \\ \text{s.t.} \quad & x \in S(t), \end{aligned}$$

with $S(t) = \{x \in S\colon M(x) \leq t\}$.

Finally, in the bicriterion approach, one considers the biobjective (BP) problem of simultaneous minimization of M and W_S:

$$(BP) \qquad \min_{x \in S} (M(x), W_S(x)).$$

The results developed in the section above enable us to conclude that the set I of intersection points helps to solve these problems:

THEOREM 4.2. *For each* α, $0 < \alpha \leq 1$ *such that* $(CP_1(\alpha))$ *is feasible, the set* I *contains at least an optimal solution to* $(CP_1(\alpha))$.

PROOF. We first show that we can restrict ourselves to the case $\alpha < 1$. Indeed, if $\alpha = 1$, the set $\{x \in S\colon W_S(x) \leq \alpha\}$ equals S, thus $(CP_1(1))$ becomes

$$\min_{x \in S} M(x).$$

As M has compact level sets and S is closed, it follows that there exists some $x^* \in S$, optimal solution to $(CP_1(1))$. Furthermore, such x^* must be in $WE(A, S)$, thus, by Theorem 3.1 of Carrizosa et al. (1993b), $W_S(x^*) < 1$. Hence, $(CP_1(1))$ is equivalent to

$$\begin{aligned} \min \quad & M(x), \\ \text{s.t.} \quad & W_S(x) \leq \alpha^*, \end{aligned}$$

with $\alpha^* = \max\{W_S(x)\colon x \text{ minimizes } M\} < 1$. Hence, we restrict ourselves to the case $\alpha < 1$. Let $x \in S$ such that $W_S(x) \leq \alpha$. By Theorem 3.1 in Carrizosa et al. (1993b), one has that

$$x \in WE(B, S) \quad \text{for each } B \subset A \text{ with } \mu(B) > W_S(x).$$

Furthermore, since $W_S(x) \leq \alpha < 1$, $x \in WE(A, S)$. By Lemma 3.1 there exists an e.c.s. F in S such that $x \in \text{ri}(F)$. We will show that an extreme point of F dominates x.

By Lemma 3.2,

$$\begin{aligned} x &\in \text{ri}(F) \\ &\subset F \\ &\subset \bigcap_{\mu(B) > W_S(x)} WE(B, S). \end{aligned}$$

This implies that F is bounded (F is a subset of the bounded set $WE(A, S)$), and

$$W_S(y) \leq W_S(x) \qquad \forall y \in F.$$

Now, consider the optimization problem $\min_{y \in F} M(y)$; since F is an e.c.s. in S, each ν^a is linear in F, so M has the same property. Furthermore, F is a polyhedron, and, as shown above, is bounded. Hence, there exists an extreme point x^* of F such that $M(x^*) \leq M(x)$. With that, we have shown:

(1) $x^* \in I$.
(2) $M(x^*) \leq M(x)$.
(3) $W_S(x^*) \leq \alpha$ (because $x^* \in F \subset \{y\colon W_S(y) \leq \alpha\}$).

Hence, the result holds. □

Concerning problem $(CP_2(t))$, it should be observed that the corresponding set $S(t)$ is a polyhedron, so the results obtained in the previous section can be used to find an optimal solution (one just has to inspect the intersection points in $S(t)$, which would be an easy task when the shape of the gauge is simple enough to enable the explicit construction of $S(t)$ at a low computational cost).

Theorem 4.2 can be used to show that I is also a dominating set for the biobjective problem (BP):

COROLLARY 4.1. *For every $x \in S$, there exists $x^* \in I$ such that*

$$W_S(x^*) \le W_S(x),$$

$$M(x^*) \le M(x).$$

PROOF. Just take $\alpha = W_S(x)$ in the theorem above. □

5. Conclusions. Exploiting the results obtained in Carrizosa et al. (1993b), we have addressed in this paper the problem of finding Simpson points in planar models with locational constraints when distances are measured by polyhedral (and possibly different for the different points) gauges. The most remarkable result states that a finite set of points (the set I of intersection points) contains a Simpson point. Hence, the problem is reduced to finding I and evaluating the points in I, a task that can be done using the algorithm proposed by the authors in Carrizosa et al. (1993b).

We show with an example that the weak existing links between Weber and Simpson points may disappear when locational constraints are introduced. In order to reconcile both voting and planning, we propose some optimization models, which can be solved with the tools developed in this paper.

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E. Carrizosa: Facultad de Matemáticas, Universidad de Sevilla, Tarfia s/n, 41012 Sevilla, Spain; e-mail: ecarriz@obelix.cica.es

E. Conde: Facultad de Matemáticas, Universidad de Sevilla, Tarfia s/n, 41012 Sevilla, Spain; e-mail: educon@obelix.cica.es

M. Muñoz-Márquez: Dpto. de Matemáticas, Universidad de Cádiz, Polígono de San Pedro, Puerto Real, 11510 Cádiz, Spain: e-mail: manuel@obelix.cica.es

J. Puerto: Facultad de Matemáticas, Universidad de Sevilla, Tarfia s/n, 41012 Sevilla, Spain; e-mail: puerto@cica.es