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A classification of integrable quasiclassical deformations of algebraic curves*

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Abstract
A previously introduced scheme for describing integrable deformations of algebraic curves is completed. Lenard relations are used to characterize and classify these deformations in terms of hydrodynamic-type systems. A general solution of the compatibility conditions for consistent deformations is given and expressions for the solutions of the corresponding Lenard relations are provided.

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1. Introduction

Algebraic curves find important applications in the theory of integrable systems [1–3]. They are particularly relevant [4–7] in the study of the zero-dispersion limit of integrable systems and the analysis of Whitham equations. In [6, 7] Krichever formulated a general method to characterize dispersionless integrable systems underlying the deformations of algebraic curves in the Whitham averaging method. A different scheme to determine integrable deformations of algebraic curves $\mathcal{C}$ of the form

$$F(p,k) := p^N - \sum_{n=1}^{N} u_n(k)p^{N-n} = 0 \tag{1}$$

was introduced in [8–11]. Here the coefficients (*potentials*) are assumed to be general polynomials in k. Our previous work focused on curves of degrees $N = 2$ and 3, and the aim of the present paper is to complete the analysis by considering the general case of algebraic curves of arbitrary degree N.

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The method proposed in [8–11] applies for finding deformations $\mathcal{C}(x, t)$ of (1) such that the branches of the multiple-valued function $p(k) = (p_1(k), \ldots, p_N(k))^T$ determined by (1) obey an equation of the form

$$\partial_t p_i = \partial_x \left(\sum_{r=1}^{N} a_r(k, u(k)) p_i^{N-r} \right), \qquad a_r \in \mathbb{C}[k], \tag{2}$$

where a_r are functions of k and $u(k) = (u_1(k), \ldots, u_N(k))$. As a consequence of (2) the potentials $u(k)$ satisfy an evolution equation of hydrodynamic type and the problem is to determine expressions for a_r such that (2) is consistent with the polynomial dependence of u on the variable k. That is to say, if $(d_1, \ldots, d_N)$ are the degrees of the polynomials $(u_1(k), \ldots, u_N(k))$, then degree$(\partial_t u_n) \leqslant d_n$ must be satisfied for all n. At this point a Lenard relation allows us to formulate a sufficient condition for the consistency of (2) in terms of a system of inequalities involving the degrees d_n only. Thus we are led to the problem of determining the degrees satisfying the consistency condition (*consistent degrees*) for each N. In [9] it was found that for $N = 2$ the consistent degrees (d_1, d_2) are characterized by the inequality $d_1 \leqslant d_2 + 1$. For $N = 3$ there is only a finite set of consistent degrees given by [11]:

$$\begin{array}{cccccc} (0,0,1) & (0,1,0) & (0,1,1) & (0,1,2) & (1,0,0) & (1,0,1) \\ (1,1,0) & (1,1,1) & (1,1,2) & (1,2,1) & (1,2,2) & (1,2,3). \end{array} \tag{3}$$

In the present work, we complete these results. Thus, it is first shown that for $N = 4$ the set of consistent degrees is

$$\begin{array}{cccc} (0,0,0,1) & (0,0,1,0) & (0,0,1,1) & (0,1,0,0) \\ (0,1,0,1) & (0,1,1,0) & (0,1,1,1) & (0,1,1,2), \end{array} \tag{4}$$

and then it is proved that for $N \geqslant 5$ the consistent degrees $(d_1, \ldots, d_N)$ are given by

$$d_i = 0, \quad i = 1, 2, \ldots, N-3, \qquad d_{N-2}, \quad d_{N-1}, \quad d_N \leqslant 1. \tag{5}$$

We note the fact that no compatible degrees $d_i \geqslant 2$ arise for $N \geqslant 5$. This implies that for $N \geqslant 5$ the algebraic curves satisfying the consistency conditions have zero genus since they are obviously rational ones. In contrast for $N = 4$ and $N = 2, 3$ (see also [8–11]) the cases involving consistent degrees equal or higher than 2 (equal or higher than 3 for $N = 2$) generically correspond to algebraic curves with non-zero genus. Hence, the degree $N = 5$ represents a threshold for a change in the properties of algebraic curves. This feature is reminiscent of the statement of the classical Abel theorem [12].

By substituting the branches p_i by their Laurent series in k into (2), infinite series of conservation laws follow. It means that the deformations of (1) supplied by our method are integrable. In fact, the corresponding hydrodynamic systems satisfied by the potentials $u_n(k)$ represent the quasiclassical (dispersionless) limits of the standard integrable models arising from the compatibility between generalized (energy-dependent) spectral problems

$$\left(\partial_x^N - \sum_{n=1}^{N} u_n(k, x) \partial_x^{N-n} \right) \psi = 0, \tag{6}$$

and equations of the form

$$\partial_t \psi = \left(\sum_{r=1}^{N} a_r(k, x, t) \partial_x^{N-r} \right) \psi. \tag{7}$$

The work is organized as follows. We first outline our method in section 2. Then section 3 is devoted to determine and classify the curves (1) which admit deformations consistent with the degrees of their potentials. Finally, in section 4 we characterize the hydrodynamic-type systems which govern these deformations.

2. Deformations of algebraic curves

In order to write equation (2) in terms of the potentials u_n we introduce the *power sums*

$$\mathcal{P}_s = \frac{1}{s}\left(p_1^s + \cdots + p_N^s\right), \qquad s \geqslant 1. \tag{8}$$

One can relate potentials and power sums through Newton recurrence formulae, the solution of which is given by Waring's formula [13]

$$\mathcal{P}_s = \sum_{1\leqslant i\leqslant s}^{(s)} \frac{1}{i}(u_1 + \cdots + u_N)^i, \tag{9}$$

where the superscript (s) in the summation symbol indicates that only the terms of weight s are retained, with the weights being defined as

$$\text{weight}\left[u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_N^{\alpha_N}\right] := \sum_{j=1}^{N} j\alpha_j. \tag{10}$$

Using these variables, equation (2) can be rewritten as [10, 11]

$$\partial_t \mathbf{u} = J_0 \mathbf{a}, \tag{11}$$

where

$$J_0 = T^T V^T \partial_x \cdot V, \qquad \mathbf{u} = (u_1, u_2, \ldots u_N)^T, \qquad \mathbf{a} = (a_N, a_{N-1}, \ldots, a_1)^T,$$

$$T := \begin{pmatrix} 1 & -u_1 & \cdots & -u_{N-1} \\ 0 & 1 & \cdots & -u_{N-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \qquad V := \begin{pmatrix} 1 & p_1 & \cdots & p_1^{N-1} \\ 1 & p_2 & \cdots & p_2^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_N & \cdots & p_N^{N-1} \end{pmatrix}.$$

The elements of J_0 can be easily written in terms of the power sums as

$$\begin{aligned}
(J_0)_{11} &= N\partial_x, \\
(J_0)_{i1} &= (i-1)\mathcal{P}_{i-1}\partial_x - \sum_{l=2}^{i-1} u_{i-l}\mathcal{P}_{l-1}\partial_x - N u_{i-1}\partial_x, \qquad \text{if} \quad i \neq 1, \\
(J_0)_{ij} &= (i+j-2)\mathcal{P}_{i+j-2}\partial_x + (j-1)\mathcal{P}_{i+j-2,x} \\
&\quad - \sum_{k=1}^{i-1} u_{i-k}[(k+j-2)\mathcal{P}_{k+j-2}\partial_x + (j-1)\mathcal{P}_{k+j-2,x}], \qquad \text{if} \quad j \neq 1.
\end{aligned} \tag{12}$$

The problem now is to determine expressions for $\mathbf{a}$ (in (11)) depending on k and $\mathbf{u}$, such that the flow (11) is consistent with the polynomial dependence of $\mathbf{u}$ on the variable k. That is to say, if $d_n := \text{degree}(u_n)$ are the degrees of the coefficients u_n as polynomials in k, then

$$\text{degree}(J_0\mathbf{a})_n \leqslant d_n, \qquad n = 1, \ldots N,$$

must be satisfied. The strategy [9–11] for finding consistent deformations is to solve Lenard-type relations

$$J_0 \mathbf{r} = 0, \qquad \mathbf{r} := (r_1, \ldots, r_N)^\top, \qquad r_i \in \mathbb{C}((k)), \tag{13}$$

and take $\mathbf{a} := \mathbf{r}_+$, where $(\cdot)_+$ and $(\cdot)_-$ indicate the parts of non-negative and negative powers in k, respectively. Now from the identity

$$J_0\mathbf{a} = J_0\mathbf{r}_+ = -J_0\mathbf{r}_-,$$

it is clear that a sufficient condition for the consistency of (11) is that

$$\max_{m=1,\ldots,N}\{\text{degree}(J_0)_{nm}\} \leqslant d_n + 1, \qquad n = 1, \ldots, N. \tag{14}$$

This condition for consistency only depends on the curve (1) and does not refer to the particular solution of the Lenard relation

In the subsequent discussion we will use an important result concerning the branches $p_i(k)$: let $\mathbb{C}((\lambda))$ denote the field of Laurent series in λ with at most a finite number of terms with positive powers, then we have [14, 15]:

Newton Theorem. *There exists a positive integer l such that the N branches*

$$p_j(z) := (p_j(k))|_{k=z^l} \tag{15}$$

are elements of $\mathbb{C}((z))$. Furthermore, if $F(p,k)$ is irreducible as a polynomial over the field $\mathbb{C}((k))$ then $l_0 = N$ is the least permissible l and the branches $p_j(z)$ can be labelled so that

$$p_j(z) = p_N(\epsilon^j z), \qquad \epsilon := \exp\left(\frac{2\pi\imath}{N}\right).$$

Notation convention. *Henceforth, given an algebraic curve $\mathcal{C}$ we will denote by z the variable associated with the least positive integer l_0 for which the substitution $k = z^{l_0}$ implies $p_j \in \mathbb{C}((z)), \forall j$. We refer to l_0 as the Newton exponent of $\mathcal{C}$.*

It was proved in [10, 11] that the solution of the Lenard relation $J_0\mathbf{r} = 0$ is given by

$$\mathbf{r} = T\nabla_{\mathbf{u}}R, \qquad R = \sum_{i=1}^{N} g_i(z)p_i, \qquad \nabla_{\mathbf{u}}R = \left(\frac{\partial R}{\partial u_1}, \ldots, \frac{\partial R}{\partial u_N}\right)^T, \tag{16}$$

with $g_i \in \mathbb{C}((z))$. The problem of choosing the functions g_i such that $R \in \mathbb{C}((k))$ (and consequently $\mathbf{r} \in \mathbb{C}((k))$) was solved in [11] by introducing the element σ_0 of the Galois group of the curve

$$\sigma_0(p_j)(z) := p_j(\epsilon_0 z), \qquad \epsilon_0 := \exp\left(\frac{2\pi\imath}{l_0}\right). \tag{17}$$

Thus it is clear that the requirement of $R \in \mathbb{C}((k))$ is equivalent to the invariance of R under σ_0, i.e.

$$R(\epsilon_0 z, \sigma_0\boldsymbol{p}) = R(z, \boldsymbol{p}). \tag{18}$$

The scheme now consists in using the *Lagrange resolvents* [12]

$$\mathcal{L}_i := \sum_{j=1}^{N}(\epsilon^i)^j p_j, \qquad i = 1, 2, \ldots, N, \tag{19}$$

to construct functions R satisfying (18) and such that $R \in \mathbb{C}((k))$.

The case $N = 3$ was completely solved in [11]. There arise twelve possible choices (3) which are classified in terms of σ_0 and l_0 according to table 1 and the invariant functions R in (16) are given by

$$\begin{aligned} &l_0 = 3, \qquad R = zf_1(z^3)\mathcal{L}_1 + z^2 f_2(z^3)\mathcal{L}_2 + f_3(z^3)\mathcal{L}_3, \\ &l_0 = 2, \qquad R = f_1(z^2)(\mathcal{L}_1 + \mathcal{L}_2) + zf_2(z^2)(\mathcal{L}_1 - \mathcal{L}_2) + f_3(z^2)\mathcal{L}_3 \\ &l_0 = 1, \qquad R = f_1(z)\mathcal{L}_1 + f_2(z)\mathcal{L}_2 + f_3(z)\mathcal{L}_3, \end{aligned} \tag{20}$$

with f_1, f_2 and f_3 being arbitrary analytic functions of k.

Table 1. Classification of (3) according to σ_0 and l_0.

σ_0	l_0	(d_1, d_2, d_3)
$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_1 \end{pmatrix}$	3	(0, 0, 1) (0, 1, 2)
$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_1 & p_3 \end{pmatrix}$	2	(0, 1, 0) (0, 1, 1) (1, 0, 0) (1, 1, 2)
$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$	1	(1, 0, 1) (1, 1, 0) (1, 1, 1) (1, 2, 1) (1, 2, 2) (1, 2, 3)

3. Solutions of the consistency condition

Let us first consider condition (14) for $N = 4$. Taking into account (12) we find that the elements of J_0 are given by

$$
\begin{aligned}
(J_0)_{11} &= 4\partial_x, \\
(J_0)_{12} &= u_1\partial_x + u_{1x}, \\
(J_0)_{13} &= \left(u_1^2 + 2u_2\right)\partial_x + \left(u_1^2 + 2u_2\right)_x, \\
(J_0)_{14} &= \left(u_1^3 + 3u_1u_2 + 3u_3\right)\partial_x + \left(u_1^3 + 3u_1u_2 + 3u_3\right)_x, \\
(J_0)_{21} &= -3u_1\partial_x, \\
(J_0)_{22} &= 2u_2\partial_x + u_{2x}, \\
(J_0)_{23} &= (u_1u_2 + 3u_3)\partial_x + 2(u_2u_{1x} + u_{3x}), \\
(J_0)_{24} &= \left(u_1^2u_2 + 2u_2^2 + u_1u_3 + 4u_4\right)\partial_x + 3(u_{4x} + u_2u_{2x} + u_2u_1u_{1x} + u_3u_{1x}), \\
(J_0)_{31} &= -2u_2\partial_x, \\
(J_0)_{32} &= 3u_3\partial_x + u_{3x}, \\
(J_0)_{33} &= (4u_4 + u_1u_3)\partial_x + 2(u_{4x} + u_3u_{1x}), \\
(J_0)_{34} &= \left(u_1u_4 + 2u_2u_3 + u_1^2u_3\right)\partial_x + 3(u_4u_{1x} + u_3u_1u_{1x} + u_3u_{2x}), \\
(J_0)_{41} &= -u_3\partial_x, \\
(J_0)_{42} &= 4u_4\partial_x + u_{4x}, \\
(J_0)_{43} &= u_1u_4\partial_x + 2u_4u_{1x}, \\
(J_0)_{44} &= \left(u_1^2u_4 + 2u_2u_4\right)\partial_x + 3u_4(u_1u_{1x} + u_{2x}).
\end{aligned}
$$

Thus, the compatibility condition (14) reduces to

$$
\begin{aligned}
&d_1 = 0, \qquad d_2 \leqslant 1, \qquad d_3 \leqslant 1, \\
&d_4 \leqslant d_2 + 1, \qquad d_4 \leqslant d_3 + 1,
\end{aligned}
$$

which leads to the proposition

Proposition 1. *For $N = 4$ the degrees (d_1, d_2, d_3, d_4) satisfying the compatibility condition (14) are*

$$
\begin{aligned}
&(0, 0, 0, 1), \quad (0, 0, 1, 0), \quad (0, 0, 1, 1), \quad (0, 1, 0, 0), \\
&(0, 1, 0, 1), \quad (0, 1, 1, 0), \quad (0, 1, 1, 1), \quad (0, 1, 1, 2).
\end{aligned} \tag{21}
$$

In order to derive our general result for $N \geqslant 5$, we start by proving

Proposition 2. *For each $N \in \mathbb{N}$ ($N \geqslant 5$) the degrees*

$$d_i = 0, \qquad i = 1, 2, \ldots, N-3, \qquad d_{N-2}, d_{N-1}, d_N = 0, 1, \tag{22}$$

satisfy the compatibility condition (14).

Proof. We extend recursively the definition of the weights (10) by

$$\text{weight}\big[\big(\partial_x^n u_j\big)\mathrm{P}(\mathbf{u}, \mathbf{u}_x, \ldots)\big] = j + \text{weight}[\mathrm{P}(\mathbf{u}, \mathbf{u}_x, \ldots)],$$

where $\mathrm{P}(\mathbf{u}, \mathbf{u}_x, \ldots)$ denotes any differential polynomial in $\mathbf{u}$. Taking into account (9) and (12), we find that the elements of J_0 are weight homogeneous with respect to the scaling

$$(u_1, u_2, \ldots, u_N) \to (\lambda u_1, \lambda^2 u_2, \ldots, \lambda^N u_N),$$

and their weights are given by

$$\text{weight}[(J_0)_{ik}] = i + k - 2.$$

For the case $i + k < 2N - 2$ we have weight$[(J_0)_{ik}] < 2N - 4$ and, as a consequence, if the indices (i, k) satisfy $i + k < 2N - 2$ then $(J_0)_{ik}$ does not involve neither terms of the form $u_{N-2}^{j+1}, u_{N-1}^{j+1}, u_N^{j+1}, u_{N-2}^j u_{N-1}^l, u_{N-2}^j u_N^l, u_{N-1}^j u_N^l$, $j, l \geqslant 1$ nor similar terms containing derivatives. Thus,

$$\begin{aligned}\text{degree}[(J_0)_{ik}] \leqslant \max\{&[d_1, \ldots, d_{N-3}], d_{N-2} + [d_1, \ldots, d_{N-3}],\\ &d_{N-1} + [d_1, \ldots, d_{N-3}], d_N + [d_1, \ldots, d_{N-3}]\},\end{aligned} \tag{23}$$

where $[d_1, \ldots, d_{N-3}]$ stands for degrees of terms appearing in $(J_0)_{ik}$ which are linear combination of d_1,...,d_{N-3} with entire coefficients.

Now we examine the remaining elements $(J_0)_{ik}$, i.e.

$$(i, k) \in \{(N-2, N), (N-1, N-1), (N-1, N), (N, N-2), (N, N-1), (N, N)\}.$$

- weight$[(J_0)_{N-2,N}] = 2N - 4$, so that $(J_0)_{N-2,N}$ may contain terms of the form $u_{N-2}^2, u_{N-2}u_{N-2,x}$ and we have

  $$\begin{aligned}\text{degree}[(J_0)_{N-2,N}] \leqslant \max\{&[d_1, \ldots, d_{N-3}], d_{N-2} + [d_1, \ldots, d_{N-3}],\\ &d_{N-1} + [d_1, \ldots, d_{N-3}], d_N + [d_1, \ldots, d_{N-3}], 2d_{N-2}\}.\end{aligned} \tag{24}$$

- weight$[(J_0)_{N-1,N-1}] = 2N-4$. This weight allows the presence of terms such as $u_{N-2}^2\partial_x$ and $u_{N-2}u_{N-2,x}$, which arise multiplied by the coefficients:

  $$\text{coeff}\left[(2N-4)\mathcal{P}_{2N-4}\partial_x, u_{N-2}^2\partial_x\right] = N - 2,$$

  $$\text{coeff}\left[u_{N-k-1}(N+k-3)\mathcal{P}_{N+k-3}\partial_x, u_{N-2}^2\partial_x\right] = \begin{cases} N-2 & \text{if } k = 1,\\ 0 & \text{if } k \neq 1,\end{cases} \quad \Rightarrow \quad \text{coeff}\left[(J_0)_{N-1N-1}, u_{N-2}^2\partial_x\right] = 0.$$

  $$\text{coeff}\left[(N-2)\mathcal{P}_{2N-4,x}, u_{N-2}u_{N-2x}\right] = N - 2,$$

  $$\text{coeff}\left[u_{N-k-1}(N-2)\mathcal{P}_{N+k-3,x}, u_{N-2}u_{N-2x}\right] = \begin{cases} N-2 & \text{if } k = 1,\\ 0 & \text{if } k \neq 1,\end{cases} \quad \Rightarrow \quad \text{coeff}\left[(J_0)_{N-1N-1}, u_{N-2}u_{N-2x}\right] = 0.$$

  Thus, $(J_0)_{N-1N-1}$ does not contain terms in $u_{N-2}^2, u_{N-2}u_{N-2x}$ and consequently

  $$\begin{aligned}\text{degree}[(J_0)_{N-2,N}] \leqslant \max\{&[d_1, \ldots, d_{N-3}], d_{N-2} + [d_1, \ldots, d_{N-3}], d_{N-1}\\ &+ [d_1, \ldots, d_{N-3}], d_N + [d_1, \ldots, d_{N-3}]\}.\end{aligned} \tag{25}$$

- weight$[(J_0)_{N-1,N}] = 2N-3$. Terms of the form $u_{N-2}^2 u_1$, $u_{N-2}u_{N-1}$, or similar terms containing derivatives may arise. A direct computation, similar to that in the previous case, proves that there are no terms $u_{N-2}^2 u_1$, $u_{N-2}^2 u_{1,x}$, $u_{N-2}u_{N-2,x}u_1$ in $(J_0)_{N-1,N-1}$. Then we have that

$$\begin{aligned}\text{degree}[(J_0)_{N-1,N}] \leqslant \max\{&[d_1,\ldots,d_{N-3}], d_{N-2}+[d_1,\ldots,d_{N-3}],\\ &d_{N-1}+[d_1,\ldots,d_{N-3}], d_N+[d_1,\ldots,d_{N-3}], d_{N-2}+d_{N-1}\}.\end{aligned} \tag{26}$$

- weight$[(J_0)_{N,N-2}] = 2N-4$. A direct computation shows that there are no terms u_{N-2}^2, $u_{N-2}u_{N-2,x}$ in $(J_0)_{N,N-2}$, so that

$$\begin{aligned}\text{degree}[(J_0)_{N,N-2}] \leqslant \max\{&[d_1,\ldots,d_{N-3}], d_{N-2}+[d_1,\ldots,d_{N-3}],\\ &d_{N-1}+[d_1,\ldots,d_{N-3}], d_N+[d_1,\ldots,d_{N-3}]\}.\end{aligned} \tag{27}$$

- weight$[(J_0)_{N,N-1}] = 2N-3$. One can see that $(J_0)_{N,N-1}$ has no terms $u_{N-2}^2 u_1$, $u_{N-2}u_{N-1}$ or similar terms containing derivatives. Consequently

$$\begin{aligned}\text{degree}[(J_0)_{N,N-2}] \leqslant \max\{&[d_1,\ldots,d_{N-3}], d_{N-2}+[d_1,\ldots,d_{N-3}],\\ &d_{N-1}+[d_1,\ldots,d_{N-3}], d_N+[d_1,\ldots,d_{N-3}]\}.\end{aligned} \tag{28}$$

- weight$[(J_0)_{NN}] = 2N-2$. This element may involve terms $u_{N-2}u_N$, $u_{N-2x}u_N$ or $u_{N-2}u_{Nx}$. On the other hand, it can be checked, as in the previous cases, that terms $u_{N-2}^2 u_2$, $u_{N-2}^2 u_1^2$, $u_{N-2}u_{N-1}u_1$, u_{N-1}^2 or similar ones containing derivatives cannot arise. Consequently

$$\begin{aligned}\text{degree}[(J_0)_{N,N-2}] \leqslant \max\{&[d_1,\ldots,d_{N-3}], d_{N-2}+[d_1,\ldots,d_{N-3}],\\ &d_{N-1}+[d_1,\ldots,d_{N-3}], d_N+[d_1,\ldots,d_{N-3}], d_{N-2}+d_N\}.\end{aligned} \tag{29}$$

In summary, by taking into account (23)–(29), we conclude that (14) is satisfied provided that

$$\begin{aligned}&[d_1,\ldots,d_{N-3}] \leqslant 1, && 2d_{N-2} \leqslant d_{N-2}+1,\\ &d_{N-2}+[d_1,\ldots,d_{N-3}] \leqslant 1, && d_{N-2}+d_{N-1} \leqslant d_{N-1}+1,\\ &d_{N-1}+[d_1,\ldots,d_{N-3}] \leqslant 1, && d_{N-2}+d_N \leqslant d_N+1.\\ &d_N+[d_1,\ldots,d_{N-3}] \leqslant 1, &&\end{aligned} \tag{30}$$

Thus, any choice of the degrees verifying

$$d_i = 0, \qquad i = 1, 2, \ldots, N-3, \qquad d_{N-2}, d_{N-1}, d_N \leqslant 1$$

satisfies (30) and in consequence it verifies (14). □

We next show that (22) constitutes the complete set of degrees satisfying (14).

Proposition 3. *For each $N \in \mathbb{N}$ ($N \geqslant 5$) the compatibility condition (14) implies*

$$d_i = 0, \quad i = 1, 2, \ldots, N-3, \qquad d_{N-2}, d_{N-1}, d_N \leqslant 1.$$

Proof. The cases N even or odd must be considered separately. Suppose first that $N = 2M$ with $M \in \mathbb{N}(M \geqslant 3)$. From (12) we have that

$$(J_0)_{12M} = (2M-1)\mathcal{P}_{2M-1}\partial_x + (2M-1)\mathcal{P}_{2M-1,x}.$$

Thus, it is clear that $(J_0)_{12M}$ contains terms in

$$\begin{aligned}&u_1^{2M-1}\partial_x, && u_j^2 u_1^{2M-2j-1}\partial_x, \quad j = 2,\ldots,M-1,\\ &u_{2M-1}\partial_x, && u_{2M-2}u_1\partial_x,\end{aligned}$$

and consequently, condition (14) with $n = 1$ implies that

$$(2M-1)d_1 \leqslant d_1 + 1, \qquad 2d_j + (2M-2j-1)d_1 \leqslant d_1 + 1, \quad j = 2, \ldots, M-1,$$
$$d_{2M-1} \leqslant d_1 + 1, \qquad d_{2M-2} + d_1 \leqslant d_1 + 1,$$

or equivalently

$$d_j = 0, \quad j = 1, 2, \ldots, M-1, \qquad d_{2M-2}, d_{2M-1} \leqslant 1. \tag{31}$$

By taking now $i = 2l$, $j = 2M (l < M)$ in (12) we have that

$$(J_0)_{2l2M} = 2(l+M-1)\mathcal{P}_{2(l+M-1)}\partial_x + (2M-1)\mathcal{P}_{2(l+M-1),x} - \sum_{k=1}^{2l-1} u_{2l-k}[(k+2M-2)\mathcal{P}_{k+2M-2}\partial_x + (2M-1)\mathcal{P}_{k+2M-2,x}].$$

Then, we have that $(J_0)_{22M}$ contains a term $u_{2M}\partial_x$ so that

$$d_{2M} \leqslant d_2 + 1.$$

Since according to (31) $(M \geqslant 3) d_2 = 0$, we have that

$$d_{2M} \leqslant 1. \tag{32}$$

On the other hand, we also see that $(J_0)_{2l2M}$ contains a term $u^2_{l+M-1}\partial_x$. Hence, condition (14) with $n = 2l$ implies

$$2d_{l+M-1} \leqslant d_{2l} + 1, \qquad \text{for each} \quad l < M. \tag{33}$$

Now from (33) we deduce the following.

- By setting $l = 1$ in (33), we get $2d_M \leqslant d_2 + 1$, but $d_2 = 0$ so that $d_M = 0$. Thus,
$$M \geqslant 3 \Rightarrow d_j = 0, \qquad j = 1, 2, \ldots, M.$$
- Suppose that $M \geqslant 4$, and put $l = 2$ into (33), then we have that $2d_{M+1} \leqslant d_4 + 1$. But under our hypothesis $d_4 = 0$, so that
$$M \geqslant 4 \Rightarrow d_j = 0, \qquad j = 1, 2, \ldots, M+1.$$
- Suppose that $M \geqslant 5$, and put $l = 3$ into (33), then $2d_{M+2} \leqslant d_6 + 1$. Again, under our actual hypothesis $d_6 = 0$, we have that
$$M \geqslant 5 \Rightarrow d_j = 0, \qquad j = 1, 2, \ldots, M+2.$$

Let us now use induction to prove

$$M \geqslant k+3 \Rightarrow d_j = 0, \qquad j = 1, 2, \ldots, M+k. \tag{34}$$

We have already proved (34) for $k = 1, 2$. Assume that it holds for $k \leqslant k_0 - 1$ and let us check it for $k = k_0$.

Take $M \geqslant k_0 + 3$ and put $l = k_0 + 1$ in (33), then we have that

$$2d_{M+k_0} \leqslant d_{2k_0+2} + 1.$$

As $2k_0 + 2 \leqslant M + k_0 - 1$ it follows that $d_{2k_0+2} = 0$, so that $d_{M+k_0} = 0$ which proves (34).

Finally, for a given M, take $k = M - 3$, then

$$d_j = 0, \qquad j = 1, 2, \ldots, 2M-3.$$

Hence, by taking (31) and (32) into account, we have proved that (14) implies

$$d_j = 0, \quad j = 1, 2, \ldots, 2M-3, \qquad d_{2M-2}, d_{2M-1}, d_{2M} \leqslant 1.$$

We consider now the case $N = 2M + 1$ with $M \in \mathbb{N}$ $(M \geqslant 2)$. From (12)

$$(J_0)_{12M+1} = 2M\mathcal{P}_{2M}\partial_x + 2M\mathcal{P}_{2M,x}.$$

Consequently $(J_0)_{12M+1}$ contains terms in

$$u_1^{2M}\partial_x, \qquad u_j^2 u_1^{2M-2j}\partial_x, \quad j = 2, \ldots, M, \qquad u_{2M}\partial_x, \qquad u_{2M-1}u_1\partial_x,$$

and condition (14) with $n = 1$ implies that

$$2Md_1 \leqslant d_1 + 1, \qquad 2d_j + (2M - 2j)d_1 \leqslant d_1 + 1, \quad j = 2, \ldots, M,$$
$$d_{2M} \leqslant d_1 + 1, \qquad d_{2M-1} + d_1 \leqslant d_1 + 1,$$

or equivalently

$$d_j = 0, \quad j = 1, 2, \ldots, M, \qquad d_{2M-1}, d_{2M} \leqslant 1. \tag{35}$$

On the other hand, by setting $i = 2l + 1$, $j = 2M + 1 (l < M)$ in (12) we have that

$$(J_0)_{2l+12M+1} = 2(l + M)\mathcal{P}_{2(l+M)}\partial_x + 2M\mathcal{P}_{2(l+M),x} - \sum_{k=1}^{2l} u_{2l+1-k}[(k + 2M - 1)\mathcal{P}_{k+2M-1}\partial_x + 2M\mathcal{P}_{k+2M-1,x}].$$

Thus, $(J_0)_{2l+12M+1}$ contains the term $u_{M+l}^2\partial_x$, so that condition (14) with $n = 2l + 1$ implies

$$2d_{M+l} \leqslant d_{2l+1} + 1. \tag{36}$$

By putting $l = 1, 2, 3$ in (36) it follows:

- For $l = 1$ we have that $2d_{M+1} \leqslant d_3 + 1$. Thus,
$$M \geqslant 3 \Rightarrow d_j = 0, \qquad j = 1, 2, \ldots, M + 1.$$
- For $l = 2$ it follows that $2d_{M+2} \leqslant d_5 + 1$. Consequently
$$M \geqslant 4 \Rightarrow d_j = 0, \qquad j = 1, 2, \ldots, M + 2.$$
- For $l = 3$ inequality (36) reads $2d_{M+3} \leqslant d_7 + 1$ so that
$$M \geqslant 5 \Rightarrow d_j = 0, \qquad j = 1, 2, \ldots, M + 3.$$

Let us now use induction to show that

$$M \geqslant k + 2 \Rightarrow d_j = 0, \qquad j = 1, 2, \ldots, M + k. \tag{37}$$

We have proved (37) for $k = 1, 2, 3$. Suppose that it holds for $k \leqslant k_0 - 1$ and let us check it for $k = k_0$. Take $M \geqslant k_0 + 2$ and $l = k_0$ in (36), we find

$$2d_{M+k_0} \leqslant d_{2k_0+1} + 1.$$

But $2k_0 + 1 \leqslant M + k_0 - 1$, then $d_{2k_0+1} = 0$, $d_{M+k_0} = 0$ and (37) follows. Thus, for a given M, if we take $k = M - 2$ we have that

$$d_j = 0, \qquad j = 1, 2, \ldots, 2M - 2. \tag{38}$$

Finally, from the expression

$$(J_0)_{22M+1} = (2M + 1)\mathcal{P}_{2M+1}\partial_x + 2M\mathcal{P}_{2M+1,x} - u_1[2M\mathcal{P}_{2M}\partial_x + 2M\mathcal{P}_{2M,x}],$$

we have that (14) implies $d_{2M+1} \leqslant d_2 + 1$ and consequently $d_{2M+1} \leqslant 1$. This fact, together with (35) and (38) lead us to

$$d_j = 0, \quad j = 1, 2, \ldots, 2M - 2, \qquad d_{2M-1}, d_{2M}, d_{2M+1} \leqslant 1. \qquad \square$$

From propositions 2 and 3 it follows that

Theorem. *For each $N \in \mathbb{N}$ $(N \geqslant 5)$ the degrees $(d_1, \ldots, d_N)$ satisfy the compatibility condition (14) if and only if*

$$d_i = 0, \quad i = 1, 2, \ldots, N - 3, \qquad d_{N-2}, d_{N-1}, d_N \leqslant 1. \tag{39}$$

4. Hierarchies of consistent deformations

Our next task is to classify all the compatible cases in terms of the corresponding Newton exponent and the element σ_0 (17) of the Galois group of the curve.

We start by considering the case $N \geqslant 5$. In order to find l_0 and σ_0 for each one of the seven nontrivial choices (39), we study the asymptotic behaviour of the N branches $p_i, i = 1, 2, \ldots, N$ as $k \to \infty$. By writing the potentials as

$$u_n = \sum_{j=0}^{d_n} u_{nj} k^j$$

we have

- $(0, \ldots, 0, 0, 0, 1)$. In this case (1) can be written as

$$k = \frac{1}{u_{N1}} \left(p^N - \sum_{l=1}^{N} u_{l0} p^{N-l} \right),$$

so that

$$p_j^N \sim u_{N1} k \qquad \text{as} \quad k \to \infty, \quad j = 1, 2, \ldots, N.$$

Consequently, $p_j \in \mathbb{C}\left(\left(k^{\frac{1}{N}}\right)\right)$, $j = 1, 2, \ldots, N$ and

$$l_0 = N, \qquad \sigma_0 = \begin{pmatrix} p_1 & p_2 & \cdots & p_{N-1} & p_N \\ p_2 & p_3 & \cdots & p_N & p_1 \end{pmatrix}.$$

- $(0, \ldots, 0, 0, 1, 0)$. Now, (1) takes the form

$$k = \frac{1}{u_{N-11}} \left(p^{N-1} - \sum_{l=1}^{N} u_{l0} p^{N-l-1} - \frac{u_{N0}}{p} \right).$$

Thus, the roots satisfy

$$p_j^{N-1} \sim u_{N-11} k \qquad \text{as} \quad k \to \infty, \quad j = 1, 2, \ldots, N-1,$$
$$p_N \sim -\frac{u_{N0}}{u_{N-11}} \frac{1}{k} \qquad \text{as} \quad k \to \infty,$$

and we find

$$l_0 = N-1, \qquad \sigma_0 = \begin{pmatrix} p_1 & p_2 & \cdots & p_{N-1} & p_N \\ p_2 & p_3 & \cdots & p_1 & p_N \end{pmatrix}.$$

- $(0, \ldots, 0, 0, 1, 1)$. From (1) we can write

$$k = \sum_{j=0}^{N-1} c_j p^j + \frac{c_{-1}}{u_{N-11} p + u_{N1}},$$

for certain coefficients c_j, $j = -1, 0, 1, \ldots, N-1$. Hence

$$p_j^{N-1} \sim \frac{1}{c_{N-1}} k \qquad \text{as} \quad k \to \infty, \quad j = 1, 2, \ldots, N-1,$$
$$p_N \sim -\frac{u_{N1}}{u_{N-11}} + \frac{c_{-1}}{u_{N-11}} \frac{1}{k} \qquad \text{as} \quad k \to \infty,$$

so that

$$l_0 = N-1, \qquad \sigma_0 = \begin{pmatrix} p_1 & p_2 & \cdots & p_{N-1} & p_N \\ p_2 & p_3 & \cdots & p_1 & p_N \end{pmatrix}.$$

- $(0,\ldots,0,1,0,0)$. Equation (1) of the curve implies

$$k=\frac{1}{u_{N-21}}\left(p^{N-2}-\sum_{l=1}^{N-2}u_{l0}p^{N-l-2}+\frac{u_{N-10}}{p}+\frac{u_{N0}}{p^2}\right).$$

Then,

$$\begin{aligned} p_j^{N-2}&\sim u_{N-21}k &&\text{as}\quad k\to\infty,\quad j=1,2,\ldots,N-2,\\ p_j^2&\sim\frac{u_{N0}}{u_{N-21}}\frac{1}{k} &&\text{as}\quad k\to\infty,\quad j=N-1,N.\end{aligned}$$

Thus, the corresponding Galois group element is given by

$$\sigma_0=\begin{pmatrix}p_1 & p_2 & \cdots & p_{N-2} & p_{N-1} & p_N\\ p_2 & p_3 & \cdots & p_1 & p_N & p_{N-1}\end{pmatrix},$$

and the Newton exponent is

$$l_0=\begin{cases}N-2 & \text{if } N \text{ is even},\\ 2(N-2) & \text{if } N \text{ is odd}.\end{cases}$$

- $(0,\ldots,0,1,1,0)$. From (1) we have

$$k=\sum_{j=0}^{N-2}c_jp^j+\frac{d_1}{p-b_1}+\frac{d_2}{p},$$

for certain coefficients c_j, $j=0,1,\ldots,N-2$, b_1 and d_k, $k=1,2$. The branches satisfy

$$\begin{aligned} p_j^{N-2}&\sim\frac{1}{c_{N-2}}k &&\text{as}\quad k\to\infty,\quad j=1,2,\ldots,N-2,\\ p_{N-1}&\sim b_1+\frac{d_1}{k} &&\text{as}\quad k\to\infty,\\ p_N&\sim\frac{d_2}{k} &&\text{as}\quad k\to\infty,\end{aligned}$$

so that

$$l_0=N-2,\qquad \sigma_0=\begin{pmatrix}p_1 & p_2 & \cdots & p_{N-2} & p_{N-1} & p_N\\ p_2 & p_3 & \cdots & p_1 & p_{N-1} & p_N\end{pmatrix}.$$

- $(0,\ldots,0,1,0,1)$ and $(0,\ldots,0,1,1,1)$. In these cases (1) implies

$$k=\sum_{j=0}^{N-2}c_jp^j+\frac{d_1}{p-b_1}+\frac{d_2}{p-b_2},$$

for certain coefficients c_j, b_k, d_k, $j=0,1,\ldots,N-2$; $k=1,2$. Therefore

$$\begin{aligned} p_j^{N-2}&\sim\frac{1}{c_{N-2}}k &&\text{as}\quad k\to\infty,\quad j=1,2,\ldots,N-2,\\ p_{N-1}&\sim b_1+\frac{d_1}{k} &&\text{as}\quad k\to\infty,\\ p_N&\sim b_2+\frac{d_2}{k} &&\text{as}\quad k\to\infty,\end{aligned}$$

so that

$$l_0=N-2,\qquad \sigma_0=\begin{pmatrix}p_1 & p_2 & \cdots & p_{N-2} & p_{N-1} & p_N\\ p_2 & p_3 & \cdots & p_1 & p_{N-1} & p_N\end{pmatrix}.$$

Table 2. Classification of (39) according to σ_0 and l_0.

σ_0	l_0	$(d_1, \ldots, d_N)$
$\begin{pmatrix} p_1 & p_2 & \cdots & p_{N-1} & p_N \\ p_2 & p_3 & \cdots & p_N & p_1 \end{pmatrix}$	N	$(0, \ldots, 0, 0, 0, 1)$
$\begin{pmatrix} p_1 & p_2 & \cdots & p_{N-1} & p_N \\ p_2 & p_3 & \cdots & p_1 & p_N \end{pmatrix}$	$N-1$	$(0, \ldots, 0, 0, 1, 0)$ $(0, \ldots, 0, 0, 1, 1)$
$\begin{pmatrix} p_1 & \cdots & p_{N-2} & p_{N-1} & p_N \\ p_2 & \cdots & p_1 & p_{N-1} & p_N \end{pmatrix}$	$N-2$	$(0, \ldots, 0, 1, 1, 0)$ $(0, \ldots, 0, 1, 1, 1)$ $(0, \ldots, 0, 1, 0, 1)$
$\begin{pmatrix} p_1 & \cdots & p_{N-2} & p_{N-1} & p_N \\ p_2 & \cdots & p_1 & p_N & p_{N-1} \end{pmatrix}$	$N-2$ if N even $2(N-2)$ if N odd	$(0, \ldots, 0, 1, 0, 0)$

These results are summarized in table 2.

From the general theorem proved in the previous section and the explicit expressions given above it is obvious that for $N \geqslant 5$ the consistency conditions (39) are satisfied by rational curves only.

We end this section by completing the previous table for $N = 4$. Only the special set of degrees (0, 1, 1, 2) remains to be analysed. The corresponding branches can be expanded as

$$p_i = a_{i1}k^{\frac{1}{2}} + a_{i0} + \frac{a_{i-1}}{k^{\frac{1}{2}}} + \cdots, \qquad i = 1, 2, 3, 4,$$

where

$$a_{i0} = \frac{{a_{i1}}^2 u_{10} + u_{31}}{4a_{i1}^2 - 2u_{21}},$$

$$\begin{aligned} a_{i-1} = \frac{1}{8a_{i1}\left(2a_{i1}^2 - u_{21}\right)^3}\big[& a_{i1}^6\left(6u_{10}^2 + 16u_{20}\right) \\ & + a_{i1}^4\left(-5u_{10}^2u_{21} + 4u_{10}u_{31} + 16(-u_{20}u_{21} + u_{41})\right) \\ & - 2a_{i1}^2\left(-2u_{20}u_{21}^2 + 3u_{10}u_{21}u_{31} + u_{31}^2 + 8u_{21}u_{41}\right) \\ & + u_{21}\left(-u_{31}^2 + 4u_{21}u_{41}\right)\big], \\ & \vdots \qquad \vdots \end{aligned}$$

and a_{i1}, $i = 1, 2, 3, 4$ are the solutions of the equation

$$a_1^4 - u_{21}a_1^2 - u_{42} = 0.$$

By labelling its solutions so that $a_{21} = -a_{11}$, $a_{41} = -a_{31}$, we obtain

$$p_2(z) = p_1(-z), \qquad p_4(z) = p_3(-z), \qquad k = z^2.$$

Thus it follows that

$$l_0 = 2, \qquad \sigma_0 = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & p_4 & p_3 \end{pmatrix}.$$

The results for the case $N = 4$ are summarized in table 3.

We note that except for the case (0, 1, 1, 2) the curves satisfying the consistency condition for $N = 4$ are rational ones.

Let us now turn our attention to the problem of obtaining the hierarchy of integrable deformations (11). It is required to determine the function R of the form (16) satisfying the invariance condition (18). In view of (18) we discuss the different cases according to the corresponding element σ_0 of the Galois group of the curve.

Table 3. Classification of (4) according to σ_0 and l_0.

σ_0	l_0	(d_1, d_2, d_3, d_4)
$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_3 & p_4 & p_1 \end{pmatrix}$	4	(0, 0, 0, 1)
$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_3 & p_1 & p_4 \end{pmatrix}$	3	(0, 0, 1, 0) (0, 0, 1, 1)
$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & p_3 & p_4 \end{pmatrix}$	2	(0, 1, 1, 0) (0, 1, 1, 1) (0, 1, 0, 1)
$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & p_4 & p_3 \end{pmatrix}$	2	(0, 1, 0, 0) (0, 1, 1, 2)

- $\sigma_0 = \begin{pmatrix} p_1 & p_2 & \cdots & p_{N-1} & p_N \\ p_2 & p_3 & \cdots & p_N & p_1 \end{pmatrix}$.

From tables 1, 2 and 3 we have that $l_0 = N$, $\left(\epsilon_0 = \epsilon = \mathrm{e}^{\frac{2\pi\imath}{N}}\right)$. For $N \geqslant 4$ the only choice of degrees corresponding to σ_0 is $(0, \dots, 0, 0, 0, 1)$. We look for functions $R_k = \sum_{j=1}^{N} \alpha_j p_j$ such that $\sigma_0(R_k) = \epsilon_0^{N-k} R_k$, $k = 0, 1, \dots, N-1$. It is easy to check that

$$\sigma_0(R_k) = \alpha_N p_1 + \sum_{j=2}^{N} \alpha_{j-1} p_j,$$

so that the condition $\sigma_0(R_k) = \epsilon_0^{N-k} R_k$ implies that

$$\alpha_{j-1} = \epsilon_0^{N-k} \alpha_j, \quad j = 2, \dots N-1, N;$$
$$\alpha_N = \epsilon_0^{N-k} \alpha_1.$$

This system admits the nontrival solutions

$$\alpha_j = \epsilon_0^{(N-k)(N-j)} \alpha_N = \epsilon_0^{jk} \alpha_N.$$

Thus the functions R of the form (16) which satisfy (18) can be written as

$$R = \sum_{k=0}^{N-1} z^k f_k(z^N) \sum_{j=1}^{N} \epsilon_0^{jk} p_j, \tag{40}$$

with $f_k \in \mathbb{C}((z^N))$, $k = 0, 1, \dots, N-1$. Taking into account that $\epsilon_0 = \epsilon$ and recalling (19), we see that the functions R can also be written in terms of the Lagrange resolvents as

$$R = f_0(z^N)\mathcal{L}_N + \sum_{k=1}^{N-1} z^k f_k(z^N)\mathcal{L}_k,$$

which coincides with the first equation for $N = 3$ in (20).

- $\sigma_0 = \begin{pmatrix} p_1 & \cdots & p_{N-2} & p_{N-1} & p_N \\ p_2 & \cdots & p_{N-1} & p_1 & p_N \end{pmatrix}$.

The corresponding Newton exponent is $l_0 = N-1$ $\left(\epsilon_0 = \mathrm{e}^{\frac{2\pi\imath}{N-1}}\right)$ and for $N \geqslant 4$ the degrees of the potentials are $(0, \dots, 0, 0, 1, 0)$ and $(0, \dots, 0, 0, 1, 1)$. In this case we have that $\sigma_0(p_N) = p_N$, or equivalently $p_N \in \mathbb{C}((k))$. Moreover, we need $N-1$ additional functions R verifying the invariance condition (18). Proceeding as in the previous case we look for functions of the form

$$R_k = \sum_{j=1}^{N-1} \alpha_j p_j, \qquad \text{such that} \qquad \sigma_0(R_k) = \epsilon_0^{N-1-k} R_k, \quad k = 0, 1, \ldots, N-2.$$

Since the action of σ_0 on the function R_k is given by

$$\sigma_0(R_k) = \alpha_N p_1 + \sum_{j=2}^{N-1} \alpha_{j-1} p_j,$$

the condition $\sigma_0(R_k) = \epsilon_0^{N-1-k} R_k$ leads to

$$\alpha_{j-1} = \epsilon_0^{N-1-k} \alpha_j, \quad j = N-1, N-2 \ldots, 2$$
$$\alpha_{N-1} = \epsilon_0^{N-1-k} \alpha_1,$$

so that $\alpha_j = \epsilon_0^{(N-1-k)(N-1-j)} \alpha_N = \epsilon_0^{jk} \alpha_N$, and

$$R = \sum_{k=0}^{N-2} z^k f_k(z^{N-1}) \sum_{j=1}^{N-1} \epsilon_0^{jk} p_j + f_{N-1}(z^{N-1}) p_N. \tag{41}$$

Example. For $N = 4$

$$R = f_0(z^3)(p_1 + p_2 + p_3) + z f_1(z^3)\left(e^{\frac{2\pi i}{3}} p_1 + e^{\frac{4\pi i}{3}} p_2 + p_3\right)$$
$$+ z^2 f_2(z^3)\left(e^{\frac{4\pi i}{3}} p_1 + e^{\frac{2\pi i}{3}} p_2 + p_3\right) + f_3(z^3) p_4.$$

- $\sigma_0 = \begin{pmatrix} p_1 & \cdots & p_{N-2} & p_{N-1} & p_N \\ p_2 & \cdots & p_1 & p_{N-1} & p_N \end{pmatrix}.$

In this case σ_0, $l_0 = N-2$, $\left(\epsilon_0 = e^{\frac{2\pi\imath}{N-2}}\right)$. For $N \geqslant 4$ it corresponds to the sets of degrees $(0, \ldots, 0, 1, 0, 1)$, $(0, \ldots, 0, 1, 1, 0)$ and $(0, \ldots, 0, 1, 1, 1)$. Note that $p_{N-1}, p_N \in \mathbb{C}((k))$. Let us look for functions

$$R_k = \sum_{j=1}^{N-2} \alpha_j p_j, \quad \text{verifying} \quad \sigma_0(R_k) = \epsilon_0^{N-2-k} R_k, k = 0, 1, \ldots, N-3.$$

We find that

$$\alpha_{j-1} = \epsilon_0^{N-2-k} \alpha_j, \quad j = N-2, N-3, \ldots, 2$$
$$\alpha_{N-2} = \epsilon_0^{N-2-k} \alpha_1,$$

then $\alpha_j = \epsilon_0^{(N-2-k)(N-2-j)} \alpha_{N-2} = \epsilon_0^{jk} \alpha_{N-2}$, and

$$R = \sum_{k=0}^{N-3} z^k f_k(z^{N-2}) \sum_{j=1}^{N-2} \epsilon_0^{jk} p_j + f_{N-2}(z^{N-2}) p_{N-1} + f_{N-1}(z^{N-2}) p_N. \tag{42}$$

- $\sigma_0 = \begin{pmatrix} p_1 & \cdots & p_{N-2} & p_{N-1} & p_N \\ p_2 & \cdots & p_1 & p_N & p_{N-1} \end{pmatrix}.$

This element corresponds to the sets of degrees $(0, \ldots, 0, 1, 0, 0)$ and, in the particular case $N = 4$, to the special choice $(0, 1, 1, 2)$ too. From the discussion in section 3 it follows that the Newton exponent of σ_0 depends on whether N is even or odd.

- N even: $l_0 = N-2$ $\left(\epsilon_0 = e^{\frac{2\pi\imath}{N-2}}\right)$. It is easy to see that $p_{N-1}+p_N \in \mathbb{C}((k))$ and $\sigma_0(-p_{N-1}+p_N) = -(-p_{N-1}+p_N)$. On the other hand, since σ_0 acts on p_j, $j=1,2,\ldots,N-2$ and ϵ_0 coincides with the previous one, we have again that

$$R_k = \sum_{j=1}^{N-2} \epsilon_0^{jk} p_j, \qquad k = 0,1,\ldots,N-3,$$

satisfy $\sigma_0(R_k) = \epsilon_0^{N-2-k} R_k$. Thus R is now given by

$$R = \sum_{k=0}^{N-3} z^k f_k(z^{N-2}) \sum_{j=1}^{N-2} \epsilon_0^{jk} p_j + z^{\frac{N-2}{2}} f_{N-2}(z^{N-2})(p_{N-1} - p_{N-1}) \\ + f_{N-1}(z^{N-2})(p_{N-1}+p_N). \tag{43}$$

Example. For $N = 4$

$R = f_0(z^2)(p_1+p_2) + zf_1(z^2)(-p_1+p_2) + zf_2(z^2)(-p_3+p_4) + f_3(z^2)(p_3+p_4)$.

$\circ N$ odd: $l_0 = 2(N-2)$ $\left(\epsilon_0 = e^{\frac{\pi\imath}{N-2}}\right)$. Again in this case $p_{N-1}+p_N \in \mathbb{C}((k))$ and $\sigma_0(-p_{N-1}+p_N) = -(-p_{N-1}+p_N)$. Moreover, if we look for functions $R_k = \sum_{j=1}^{N-2} \alpha_j p_j$ such that

$$\sigma_0(R_k) = \epsilon_0^{2(N-2-k)} R_k, \quad k = 0,\ldots,N-3,$$

by proceeding as in the previous cases, we find that $\alpha_j = \epsilon_0^{2(N-2-k)(N-2-j)} \alpha_{N-2} = \epsilon_0^{2jk} \alpha_{N-2}$, so that

$$R = \sum_{k=0}^{N-3} z^{2k} f_k(z^{2(N-2)}) \sum_{j=1}^{N-2} \epsilon_0^{2jk} p_j + z^{N-2} f_{N-2}(z^{2(N-2)})(p_N - p_{N-1}) \\ + f_{N-1}(z^{2(N-2)})(p_{N-1}+p_N). \tag{44}$$

Example. For $N = 5$

$$R = f_0(z^6)(p_1+p_2+p_3) + z^2 f_1(z^6)\left(e^{\frac{2\pi i}{3}} p_1 + e^{\frac{4\pi i}{3}} p_2 + p_3\right) + z^4 f_2(z^6)\left(e^{\frac{4\pi i}{3}} p_1 + e^{\frac{2\pi i}{3}} p_2 + p_3\right) \\ + z^3 f_3(z^6)(-p_4+p_5) + f_4(z^6)(p_4+p_5).$$

Thus, the integrable deformations (11), (16) are determined by the expressions of R in (40), (41), (42), (43) or (44) depending on σ_0 and the Newton exponent l_0.

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References

[1] Novikov S P, Manakov S V, Pitaevski L P and Zakharov V E 1984 *Theory of Solitons* (New York: Plenum)
[2] Belokolos E D, Bobenko A I, Enolski V Z, Its A R and Matveev V B 1994 *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Berlin: Springer)
[3] Dubrovin B and Novikov S 1989 *Russ. Math. Surv.* **44** 35
[4] Flaschka H, Forest M G and Mclauglin D W 1980 *Commun. Pure Appl. Math.* **33** 739
[5] Dubrovin B A 1992 *Commun. Math. Phys.* **145** 415
[6] Krichever I M 1988 *Funct. Anal. Appl.* **22** 206
[7] Krichever I M 1994 *Commun. Pure. Appl. Math.* **47** 437

[8] Kodama Y and Konopelchenko B G 2002 *J. Phys. A: Math. Gen.* **35** L489–500
[9] Konopelchenko B G and Martínez Alonso L 2004 *J. Phys. A: Math. Gen.* **37** 7859
[10] Kodama Y, Konopelchenko B G and Martínez Alonso L 2005 *Theor. Math. Phys.* **144** 961
[11] Kodama Y, Konopelchenko B G, Martínez Alonso L and Medina E 2005 *J. Math. Phys.* **46** 113502
[12] van der Waerden B L 1991 *Algebra* vol 1 (Berlin: Springer)
[13] Redei L 1967 *Introduction to Algebra* vol 1 (Oxford: Pergamon)
[14] Walker R Y 1978 *Algebraic Curves* (Berlin: Springer)
[15] Abhyankar S S 1990 Algebraic geometry for scientists and engineers *Mathematical Surveys and Monograps* vol 35 (Providence, RI: American Mathematical Society)