Characterization of the Sources in Convolutive Mixtures: A Cumulant-Based Approach

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Abstract. This paper addresses the characterization of independent and non-Gaussian sources in a linear mixture. We present an eigensystem based approach to determine the number of independent components in the signal received by a single sensor. The temporal structure of the sources is also characterized using fourth-order statistics.

1 Introduction

In many situations, we observe the superposition of an unknown number of signals and noise when studying a physical phenomenon of interest. In mathematical form, the observed signal $x(n)$ can be written as

$$x(n) = \sum_{i=1}^{N} s_i(n) + r(n) \tag{1}$$

where $s_i(n)$ denotes the signal emitted by the i-th source and $r(n)$ stands for additive noise. Single-Sensor Source Separation is the problem of estimating the source signals $s_i(n)$ from $x(n)$; it is a challenging, still unsolved, problem (excepting the cases in which the source signals have non-overlapping spectra).

The aim of this research was to characterize the source signals $s_i(n)$ on the basis of the properties of higher-order statistics. The use of higher-order cumulants offers two main advantages: first of all, they are not affected by additive Gaussian noise. Secondly, cumulants are linear in the addition of independent variables. The latter property is very useful when considering mixtures like (1). Other results could be used to complement BSS of convolutive [3, 5, 8] or single-channel mixtures [1, 2, 4, 9].

This paper is organized as follows. In Section 2, we state some relevant hypothesis and fix notation. Section 3 presents a new cumulant matrix which collects useful information on the temporal structure of the source signals $s_i(t)$. Section 4 discusses some applications of the main theoretical results. Section 5 presents numerical experiments. Finally, Section 6 is devoted to the conclusions.

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2 Model Assumptions and Notation

To begin with, we suppose the following hypotheses:

H1. Each source can be modeled as a moving-average (MA) process of order L, i.e.,

$$s_i(n) = \sum_{k=0}^{L} h_i(k)\, w_i(n-k) \quad (2)$$

for $i = 1, \ldots, N$, where the excitation sequences $\{w_i(n)\}_{i=1}^{N}$ are non-Gaussian, zero-mean, i.i.d. processes with variance σ_i^2 and kurtosis κ_i.

H2. The source signals $\{s_i(n)\}_{i=1}^{N}$ are statistically independent among themselves.

H3. The additive noise $r(n)$ is stationary, normally distributed and independent from the sources.

H4. We assume that $N \leq L+1$.

Hypotheses **H1**–**H3** can be usually assumed in practice. We will need hypothesis **H4** later on.

For purposes of notation, given any process $\{z(n)\}$ we define its covariance as

$$c_2^z(l) \stackrel{def}{=} cum(z(n), z(n+l)) \quad (3)$$

and the fourth-order cumulant [10] of $\{z(n)\}$ as

$$c_4^z(l_1, l_2, l_3) \stackrel{def}{=} cum(z(n), z(n+l_1), z(n+l_2), z(n+l_3)) \quad (4)$$

Note that, thanks to **H1**–**H3**, the cumulants (3) and (4) of $\{x(n)\}$ and the source signals $\{s_i(n)\}$ are well-defined.

3 Cumulant Matrix

Let $\mathbf{M}$ be the $(L+1) \times (L+1)$ symmetric cumulant matrix whose (i,j)-entry is given by

$$< \mathbf{M} >_{ij} = m_x(i-1, j-1) \quad (5)$$

where we have defined

$$\boxed{m_x(p,q) = \textstyle\sum_{k=-L}^{L-p} c_4^x(k, k+p, q)} \quad (6)$$

This matrix has a very particular structure: it is shown in **Appendix A** that

$$\mathbf{M} = \sum_{i=1}^{N} \frac{\kappa_i}{\sigma_i^4} \mathbf{c}_2^{s_i}\, \mathbf{c}_2^{s_i\,T} = \mathbf{C}\,\mathbf{D}\,\mathbf{C}^T \quad (7)$$

where $\mathbf{c}_2^{s_i}$ is the $(L+1)\times 1$ vector whose k-th entry is $c_2^{s_i}(k-1)$, $\mathbf{C}$ is the matrix whose columns are the covariance vectors $\mathbf{c}_2^{s_i}$, i.e.,

$$\mathbf{C} = (\mathbf{c}_2^{s_1} | \ldots | \mathbf{c}_2^{s_N})$$

and $\mathbf{D}$ is the $N \times N$ diagonal matrix whose entries are the fourth-order normalized cumulants κ_i/σ_i^4. It is supposed from now on that:

H5. The covariance vectors $\mathbf{c}_2^{s_1}, \ldots, \mathbf{c}_2^{s_N}$ are linearly independent (i.e., matrix $\mathbf{C}$ is full column rank).

Hypothesis **H5** is reasonable when the sources have different physical origins (In particular, sources with the same power spectra[1] are excluded). It follows that:

Property 1. The rank of matrix $\mathbf{M}$ is N [6].

This is interesting in the sense that *property 1* can be used to estimate the number N of sources.

The following property characterizes the covariances:

Property 2. Vectors $\mathbf{c}_2^{s_i}$ are a linear combination of the eigenvectors of $\mathbf{M}$ associated with nonzero eigenvalues.

Proof is given in **Appendix B**. Let $\mathbf{v}_1, \ldots, \mathbf{v}_N$ be those eigenvectors of $\mathbf{M}$ associated with nonzero eigenvalues. We cannot infer from (7) that $\mathbf{v}_1, \ldots, \mathbf{v}_N$ equal the covariances $\mathbf{c}_2^{s_1}, \ldots, \mathbf{c}_2^{s_N}$: as a matter of fact, the eigenvectors $\mathbf{v}_1, \ldots,$ $\mathbf{v}_N$ are orthogonal (since $\mathbf{M}$ is symmetric) whereas $\mathbf{c}_2^{s_1}, \ldots, \mathbf{c}_2^{s_N}$ are usually not orthogonal. Mathematically, *property 2* only implies that there must exists an invertible $N \times N$ matrix $\mathbf{P}$ that relates $\mathbf{v}_1, \ldots, \mathbf{v}_N$ and $\mathbf{c}_2^{s_1}, \ldots, \mathbf{c}_2^{s_N}$ as follows:

$$(\mathbf{v}_1 | \ldots | \mathbf{v}_N)\ \mathbf{P} = (\mathbf{c}_2^{s_1} | \ldots | \mathbf{c}_2^{s_N}) \tag{8}$$

Unfortunately, matrix $\mathbf{P}$ is completely unknown *a priori* and cannot be found in practice.

Finally, it can be stated that:

Property 3. Vectors $\mathbf{c}_2^{s_i}$ are orthogonal to all eigenvectors of $\mathbf{M}$ associated with zero eigenvalues.

Property 3 is an immediate consequence of *property 2*.

4 Applications

The generative model (1) appears in convolutive BSS and in the single-sensor BSS problem. In both cases:

A1. *Property 1* can be used to estimate the number N of sources.

A2. The covariance $\mathbf{c}_2^{s_i}$ of each estimated source $s_i(n)$ must satisfy *properties 2* and *3*. Hence, constraints can be derived from both properties that may be used to prevent BSS algorithms from converging to spurious solutions. For example, *property 3* implies that:

$$\mathbf{c}_2^{s_i\,T}\,\mathbf{U} = \mathbf{0}$$

[1] Power spectra is the Fourier transform of the covariance.

for $i = 1, \dots, N$, where

$$\mathbf{U} = (\mathbf{u}_1 | \dots | \mathbf{u}_{L+1-N})$$

and $\mathbf{u}_1, \dots, \mathbf{u}_{L+1-N}$ are the eigenvectors of $\mathbf{M}$ associated with zero eigenvalues.

A3. If $N = 1$, it follows from (7) that the eigenvector associated with the unique nonzero eigenvalue of $\mathbf{M}$ is equal to the covariance vector $\mathbf{c}_2^{s_1}$ of the unique source $s_1(n)$ (up to a multiplicative constant). Thus, after estimating $\mathbf{c}_2^{s_1}$, we can filter the data $x(n)$ with a Wiener filter to enhance the source signal $s_1(n)$ from the noise $r(n)$ [7].

Note that, due to the presence of the noise $r(n)$, A1–A3 would not be feasible if only second-order statistics were used. By contrasts, the fourth-order cumulant function (6) is not affected by additive Gaussian noise.

5 Example

Let us consider a mixture of four signals plus noise

$$x(n) = \sum_{i=1}^{4} s_i(n) + r(n),$$

where $s_i(n) = h_i(n) * w_i(n)$ ('$*$' denotes 'convolution'), $\{w_i(n)\}_{i=1}^{4}$ are not-gaussian leptokurtic (i.e. with positive kurtosis) signals, each of which is obtained by raising to the third power a different i.i.d. Gaussian process and the noise $\{r(n)\}$ is i.i.d. and Gaussian. All of them are normalized to have unity variance. The impulse responses of the MA filters are randomly chosen as:

$$h_1(n) = \delta(n) + 0.34\,\delta(n-1) + 0.12\,\delta(n-2) - 0.41\,\delta(n-3) + 0.65\,\delta(n-4) + 0.75\,\delta(n-5),$$

$$h_2(n) = \delta(n) - 0.35\,\delta(n-1) - 0.29\,\delta(n-2) - 0.28\,\delta(n-3) - 0.28\,\delta(n-4) + 0.66\,\delta(n-5),$$

$$h_3(n) = \delta(n) + 0.65\,\delta(n-1) + 0.52\,\delta(n-2) - 0.41\,\delta(n-3) + 0.55\,\delta(n-4) + 0.14\,\delta(n-5),$$

and

$$h_4(n) = \delta(n) - 0.50\,\delta(n-1) - 0.31\,\delta(n-2) - 0.81\,\delta(n-3) - 0.60\,\delta(n-4) + 0.24\,\delta(n-5),$$

Figure 1 shows the power spectral density of each source, calculated using Welch's method with a Hanning window.

We used 7000 samples of $x(n)$ to estimate matrix $\mathbf{M}$ with $L = 6$. After each experiment, to normalize and facilitate the comparison, eigenvalues were divided by their maximum value.

The second column of table 1 shows the mean of the normalized eigenvalues, averaged over 1000 independent experiments (the quantity between brackets is the standard deviation). The third column of table 1 shows the 'true' normalized eigenvalues, i.e., the eigenvalues that would be obtained if there were no errors in the estimation of the cumulant matrix $\mathbf{M}$. It is observed in Table 1 that there

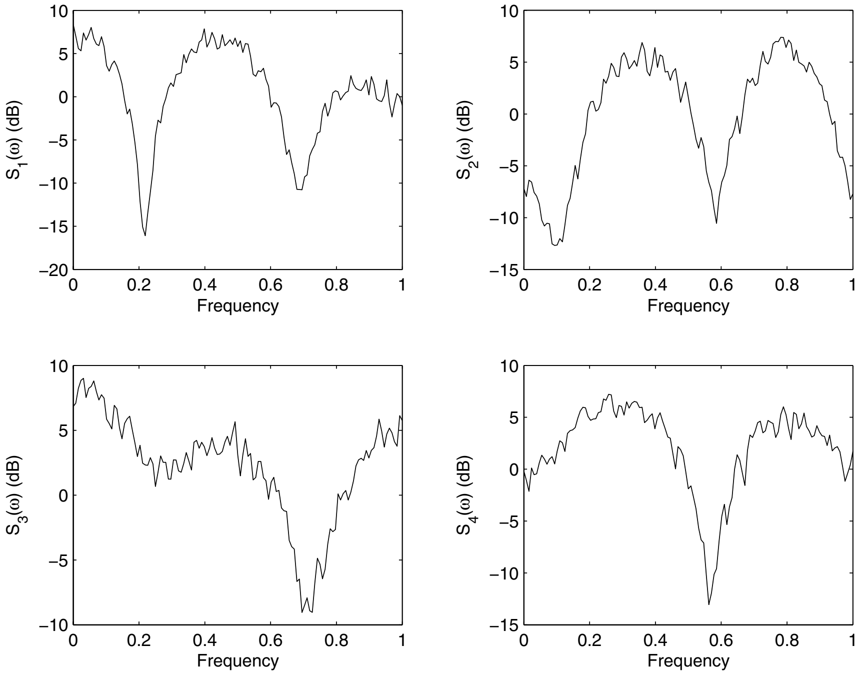

Fig. 1. Power Spectral Density of the four sources $s_1(n), s_2(n), s_3(n), s_4(n)$.

Table 1. Statistics of the eigenvalues of $\mathbf{M}$.

Eigenvalue Number	Mean (Standard Deviation)	True Normalized Eigenvalue
1	1.0000 (0.0000)	1.0000
2	0.1282 (0.0303)	0.1234
3	0.0501 (0.0173)	0.0448
4	0.0186 (0.0113)	0.0155
5	0.0020 (0.0074)	0.0000
6	-0.0107 (0.0074)	0.0000
7	-0.0265 (0.0113)	0.0000

are three clearly-nonzero eigenvalues, indicating that the mixture is composed of at least $N = 3$ non-gaussian sources. We may need an additional criterion to decide whether a fourth-source is present or not (i.e., to decide whether the fourth eigenvalue is zero or not).

Let p_{ij} be the scalar product between the covariance $\mathbf{c}_2^{s_i}$ and the eigenvector of $\mathbf{M}$ that corresponds to the j-th normalized eigenvalue. Table 2 shows the mean value and the standard deviation of p_{ij} for $1 \leq i \leq 4$ and $1 \leq j \leq 7$.

In view of Table 2, p_{ij} seeems to be zero-mean for all i, j. However, the key is the standard deviation: observe that the standard deviations are large for $j = 1, 2, 3$, whereas they are small for $j = 5, 6, 7$. It is inferred that the covariances belong to the subspace spanned by the first, second and third eigenvectors,

Table 2. Statistics of the scalar product between $\mathbf{c}_2^{s_i}$ and the eigenvectors of $\mathbf{M}$.

p_{ij}	$i=1$	$i=2$	$i=3$	$i=4$
$j=1$	-0.1288 (2.45)	-0.1110 (1.7416)	-0.1191 (2.2495)	-0.1379 (2.3996)
$j=2$	-0.0124 (0.78)	0.0154 (0.8566)	-0.0441 (0.6651)	-0.0322 (0.7814)
$j=3$	0.0524 (0.42)	0.0233 (0.5616)	-0.0085 (0.3563)	-0.0576 (0.4778)
$j=4$	-0.0040(0.28)	0.0136 (0.3223)	0.0047 (0.2816)	-0.0370 (0.3133)
$j=5$	-0.0149 (0.21)	-0.0026 (0.2043)	0.0052 (0.2080)	0.0050 (0.2132)
$j=6$	-0.0052 (0.20)	-0.0122 (0.1742)	0.0006 (0.1988)	0.0036 (0.1928)
$j=7$	0.0051 (0.21)	-0.0028 (0.1674)	-0.0010 (0.2523)	0.0053 (0.2036)

whereas they are orthogonal to the subspace spanned by the fifth, sixth and seventh eigenvectors. Both conclusions agree with *properties 2* and *3*, respectively. Again, we may need an additional criterion to decide whether the covariances are orthogonal to the fourth eigenvector or not.

6 Conclusions

We have proposed a method that can be used to estimate the number of sources in a linear mixture and characterize their temporal structure. The method is robust against gaussian noise, since it is based on higher-order statistics.

Appendix A

Since $x(n) = \sum_{i=1}^{N} s_i(n) + r(n)$, it holds that $c_4^x(l_1, l_2, l_3) = \sum_{i=1}^{N} c_4^{s_i}(l_1, l_2, l_3)$. As a consequence

$$m_x(p,q) = \sum_{k=-L}^{L-p} c_4^x(k, k+p, q) = \sum_{i=1}^{N} \sum_{k=-L}^{L-p} c_4^{s_i}(k, k+p, q) \tag{9}$$

Thanks to the multi-linearity property of the cumulants, the covariance of the source $s_i(n)$ can be written as

$$c_{s_i}^2(p) = \sum_{n=0}^{L} h_i(n) h_i(n+p) \sigma_i^2 \tag{10}$$

and the fourth-order cumulant equals

$$c_4^{s_i}(p,q,r) = \sum_{n=0}^{L} h_i(n) h_i(n+p) h_i(n+q) h_i(n+r) \kappa_i \tag{11}$$

Both cumulants can be easily related, as follows: from (10), we get

$$c_2^{s_i}(p)\, c_2^{s_i}(q) = \sum_{n=0}^{L} \sum_{k=0}^{L} h_i(n) h_i(n+p) \sigma_i^2\, h_i(k) h_i(k+q) \sigma_i^2 \tag{12}$$

Using in (12) the change of variables $k' = k - n$ and $n' = n$, we obtain:

$$c_2^{s_i}(p)\, c_2^{s_i}(q) = \sum_{n'=0}^{L} \sum_{k'=-n'}^{L-n'} h_i(n') h_i(n'+k')\, h_i(n'+k'+p)\, h_i(n'+q) \sigma_i^4 \tag{13}$$

Now, it is supposed that $p, q > 0$. Using that the MA coefficients $h_i(n) = 0$ if $n \notin \{0, \ldots, L\}$, it is readily obtained from (13) that

$$c_2^{s_i}(p)\, c_2^{s_i}(q) = \sum_{n'=0}^{L} \sum_{k'=-L}^{L-p} h_i(n') h_i(n'+k')\, h_i(n'+k'+p)\, h_i(n'+q) \sigma_i^4 \tag{14}$$

Then, comparing (11) with (14), it is deduced the following relation between cumulants:

$$c_2^{s_i}(p) c_2^{s_i}(q) = \frac{\sigma_i^4}{\kappa_i} m_{s_i}(p, q), \tag{15}$$

where we have defined

$$m_{s_i}(p, q) = \sum_{k=-L}^{L-p} c_4^{s_i}(k, k+p, q) \tag{16}$$

Then, inserting (16) in (9) and taking into account (15), we finally have:

$$m_x(p, q) = \sum_{i=1}^{N} \frac{\kappa_i}{\sigma_i^4}\, c_2^{s_i}(p) c_2^{s_i}(q) \tag{17}$$

which completes the proof $\square$.

Appendix B

Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{L+1}$ be the eigenvalues of $\mathbf{M}$. Furthermore, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ be the unit-norm eigenvectors associated with $\lambda_1, \ldots, \lambda_N$ and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{L+1-N}$ be the eigenvectors corresponding to $\lambda_{N+1}, \ldots, \lambda_{L+1}$. Observe that $\lambda_{N+1}, \ldots, \lambda_{L+1}$ are all equal to zero since $rank(\mathbf{M}) = N$.

Using the definition of the eigenvalues yields:

$$\mathbf{M}\, \mathbf{v}_i = \lambda_i\, \mathbf{v}_i \tag{18}$$

Substituting (7) into (18) gives:

$$\sum_{i=1}^{N} \frac{\kappa_i}{\sigma_i^4} \mathbf{c}_2^{s_i} \left(\mathbf{c}_2^{s_i\,T}\, \mathbf{v}_i \right) = \lambda_i\, \mathbf{v}_i \tag{19}$$

or, equivalently,

$$\mathbf{v}_i = \sum_{i=1}^{N} \frac{\kappa_i}{\sigma_i^4 \lambda_i} \left(\mathbf{c}_2^{s_i\,T}\, \mathbf{v}_i \right)\, \mathbf{c}_2^{s_i} \tag{20}$$

which means that each eigenvector $\mathbf{v}_i$ can be expressed as a linear combination of the covariances $\mathbf{c}_2^{s_i}$.

Note that vectors $\mathbf{c}_2^{s_i}$ $(1 \leq i \leq N)$ form a basis since they are linearly independent and, hence, span an N dimensional subspace. From this point of view, the preceding identity (20) just means that all eigenvectors $\mathbf{v}_i$ belong to this subspace. But these eigenvectors are orthogonal and therefore also form a basis for the subspace. Consequently, each covariance vector $\mathbf{c}_2^{s_i}$ can be represented as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_N$ as well. This completes the proof □.

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