Semilattices of Groups and Nonstable K-Theory of Extended Cuntz Limits

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Abstract. We give an elementary characterization of those abelian monoids M that are direct limits of countable sequences of finite direct sums of monoids of the form either $(\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}$ or $\mathbb{Z} \sqcup \{0\}$. This characterization involves the Riesz refinement property together with lattice-theoretical properties of the collection of all subgroups of M (viewed as a semigroup), and it makes it possible to express M as a certain submonoid of a direct product $\Lambda \times G$, where Λ is a distributive semilattice with zero and G is an abelian group. When applied to the monoids $V(A)$ appearing in the nonstable K-theory of C*-algebras, our results yield a full description of $V(A)$ for C*-inductive limits A of finite sums of full matrix algebras over either Cuntz algebras $\mathcal{O}_n$, where $2 \leq n < \infty$, or corners of $\mathcal{O}_\infty$ by projections, thus extending to the case including $\mathcal{O}_\infty$ earlier work by the authors together with K. R. Goodearl.

1. Introduction

The goal of this paper is the full elucidation of the nonstable K-theory of a class of C*-algebras called *extended Cuntz limits*, defined as the C*-inductive limits of sequences of finite direct sums of full matrix algebras over the Cuntz algebras $\mathcal{O}_n$ and over nonzero corners of $\mathcal{O}_\infty$ by projections. (We recall the definition of the latter for the information of non-C*-algebraic readers: for $2 \leq n < \infty$, the Cuntz algebra $\mathcal{O}_n$, introduced in

[6], is the unital C*-algebra generated by elements $s_1, \ldots, s_n$ with relations $s_i^* s_j = \delta_{ij}$ and $\sum_{i=1}^n s_i s_i^* = 1$. Further, $\mathcal{O}_\infty$ is the unital C*-algebra defined by generators s_i, $i \in \mathbb{N}$ and relations $s_i^* s_j = \delta_{ij}$.) Hence our work is a continuation of [11] (where the case of $\mathcal{O}_\infty$ is not covered), so it provides an analogue, for extended Cuntz limits, of the description of the range of the invariant for separable AF C*-algebras (namely, ordered K_0) by Elliott [9] and Effros, Handelman, and Shen [8].

We begin by sketching the source of the problem and giving a precise formulation. Most of the remainder of the paper is purely semigroup-theoretic, except for the applications to C*-algebras in the final two sections.

In [17], Rørdam gives a K-theoretic classification of even Cuntz limits (i.e., C*-inductive limits of sequences of finite direct sums of matrix algebras over $\mathcal{O}_n$s with n even). The invariant which Rørdam used for his classification is equivalent, in the unital case, to the pair $(V(A), [1_A])$ where $V(A)$ denotes the (additive, commutative) monoid of Murray–von Neumann equivalence classes of projections (self-adjoint idempotents) in matrix algebras over a C*-algebra A, and $[1_A]$ is the class in $V(A)$ of the unit projection in A (cf. [4, Sections 4.6, 5.1, and 5.2]). Thus, the unital case of the classification states that if A and B are unital even Cuntz limits, then $A \cong B$ if and only if $(V(A), [1_A]) \cong (V(B), [1_B])$, that is, there is a monoid isomorphism $V(A) \to V(B)$ sending $[1_A]$ to $[1_B]$ (cf. [17, Theorem 7.1]). Subsequently, Lin and Phillips [15] extended Rørdam's classification result, by including not only $\mathcal{O}_n$s with n even, but also nonzero corners over $\mathcal{O}_\infty$ (i.e., *extended* even Cuntz limits). While the authors and Goodearl were writing [11], Rørdam communicated to us [18] that his classification can be extended to all Cuntz limits by applying the work of Kirchberg [14] and Phillips [16]. By the same reason, Lin and Phillips' classification result can be enlarged to all extended Cuntz limits.

Most of the paper is devoted to the proof of a semigroup-theoretical result, namely Theorem 6.6, that provides an 'internal' characterization of direct limits of sequences of finite sums of monoids of the form either $(\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}$ or $\mathbb{Z} \sqcup \{0\}$. It turns out that the hard core of the proof of Theorem 6.6 consists of a lattice-theoretical statement about homomorphisms from finite distributive lattices to subgroup lattices of abelian groups, see Theorem 5.4.

2. Lattices and Abelian Groups

A *lattice* is a structure $(L, \leq, \vee, \wedge)$, where $(L, \leq)$ is a partially ordered set in which every pair $\{x, y\}$ of elements admits a join, $x \vee y$, and a meet, $x \wedge y$. The *zero* (resp., *unit*) of a lattice L are its smallest (resp., largest) element if it exists, then denoted by 0 (resp., 1). We say that L is *complete*, if every subset X of L has a supremum, denoted by $\bigvee X$. For elements

a and b in L, let $a \prec b$ hold, if $a < b$ and there exists no $x \in L$ such that $a < x < b$. A nonzero element p in L is *join-irreducible*, if p is not the join of two smaller elements. In case L is finite, there exists a largest element of L smaller than p, denoted by p_* (so $p_* \prec p$). We denote by $J(L)$ the set of all join-irreducible elements of L. For $a \in L$, we denote by $J_L(a)$, or $J(a)$ if L is understood, the set of all join-irreducible elements of L below a. It is well-known that if L is finite, then $a = \bigvee J(a)$ for all $a \in L$.

We say that L is *distributive* (resp., *modular*), if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (resp., $x \geq z$ implies that $x \wedge (y \vee z) = (x \wedge y) \vee z$), for all $x, y, z \in L$.

For an abelian group G, we denote by $\operatorname{Sub} G$ the lattice of all subgroups of G, ordered by inclusion. It is well-known that $\operatorname{Sub} G$ is a modular lattice. For subgroups A and B of an abelian group G, we abbreviate

$$
\begin{array}{lll}
A \leqslant B, & \text{if} & A \text{ is a subgroup of } B, \\
A \leqslant_{\text{pure}} B, & \text{if} & A \text{ is a pure subgroup of } B, \\
A \leqslant^{\oplus} B, & \text{if} & A \text{ is a direct summand of } B.
\end{array}
$$

In particular, $A \leqslant^{\oplus} B \Longrightarrow A \leqslant_{\text{pure}} B \Longrightarrow A \leqslant B$.

We shall denote disjoint union by the symbol $\sqcup$.

3. Equivalence of Projections in C*-Algebras

We shall denote by $M \sim N$ the Murray–von Neumann equivalence of self-adjoint, idempotent matrices M and N over a C*-algebra A, that is, $M \sim N$ if there exists a matrix X such that $M = X^* X$ and $N = X X^*$ (in particular, $X = X X^* X$). We denote by $[M]$ the $\sim$-equivalence class of a matrix M, and by $V(A)$ the abelian monoid of $\sim$-equivalence classes of self-adjoint, idempotent matrices over A, with addition defined by $[M] + [N] = \left[\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \right]$. The monoid $V(A)$ encodes the so-called *nonstable K-theory* of A. We shall use the following basic lemma.

LEMMA 3.1 (folklore). *Let p be a projection (i.e., a self-adjoint, idempotent element) in a C*-algebra A and let $\alpha, \beta \in V(A)$. If $[p] = \alpha + \beta$, then there are projections $a, b \in pAp$ such that $p = a + b$, $[a] = \alpha$, and $[b] = \beta$.*

(Observe that the given conditions imply that $ab = ba = 0$.)

Proof. Let $M \in M_k(A)$ and $N \in M_l(A)$ be self-adjoint, idempotent matrices such that $\alpha = [M]$ and $\beta = [N]$. By assumption, there exists $X \in M_{k+l,1}(A)$ such that

$$
p = X^* X, \quad \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} = X X^*.
$$

Write $X = \begin{pmatrix} U \\ V \end{pmatrix}$ with $U \in M_{k,1}(A)$ and $V \in M_{l,1}(A)$. Hence $p = U^*U + V^*V$, $M = UU^*$, $N = VV^*$, while $UV^* = 0$. From $XX^*X = X$ it follows that $UU^*U = U$ and $VV^*V = V$. Therefore, $a = U^*U$ and $b = V^*V$ are as required.   □

LEMMA 3.2 (folklore). *Let A be a C*-algebra. Then:*

(i) *For every $n \geq 1$, $V(M_n(A)) \cong V(A)$.*
(ii) *If A is separable, then $V(A)$ is countable.*
(iii) *If A is unital and has real rank zero, then given a nonzero projection $p \in A$, $V(pAp) \cong V(A)|_{[p]}$ (the order-ideal of $V(A)$ generated by $[p] \in V(A)$).*

Proof. (i), (ii). See [4, p. 28].
(iii). By [2, Theorem 7.2], A is a unital exchange ring. By [2, p. 111], $V(A)|_{[p]} = V(ApA)$. Thus, as $V(pAp) \cong V(ApA)$ by [1, Proof of Lemma 7.3], the result holds.   □

It is routine to check that for any unital C*-algebra A, the class $[1_A]$ is an order-unit in $V(A)$, and that the canonical isomorphism $V(M_m(A)) \to V(A)$ sends $[1_{M_m(A)}]$ to $m[1_A]$. Observe that the isomorphism in Lemma 3.2(iii) sends $[p] \in V(pAp)$ to $[p] \in V(ApA)$. Also observe that, by [20, Theorem 1] (see also [5, Proposition 3.9]), every purely infinite simple C*-algebra has real rank zero, whence Lemma 3.2(iii) applies to Cuntz algebras.

We shall also use the fact that $V(-)$ is a functor from C*-algebras to abelian monoids that preserves finite direct sums and inductive (direct) limits [4, 5.2.3–5.2.4].

4. Distributive Subgroups with Respect to a Lattice Homomorphism

DEFINITION 4.1. For lattices D and M and a lattice homomorphism $\varphi \colon D \to M$, we say that an element a of M is *distributive with respect to φ,* if the map $D \to M$, $u \mapsto a \wedge \varphi(u)$ is a lattice homomorphism.

Observe that, as φ is a lattice homomorphism, it suffices to verify the condition

$$a \wedge \varphi(x \vee y) \leq (a \wedge \varphi(x)) \vee (a \wedge \varphi(y)), \qquad \text{for all } x, y \in D. \tag{4.1}$$

In particular, if D is finite, the unit of D is the join of all join-irreducible elements of D, so, if a is distributive with respect to φ, we get $a \wedge \varphi(1) = \bigvee (a \wedge \varphi(p) \mid p \in J(D))$. Observe that a is distributive with respect to φ iff $a \wedge \varphi(1)$ is distributive with respect to φ. We will use the following characterization of distributive elements.

LEMMA 4.2. *Let D be a finite distributive lattice, let M be a modular lattice, let $\varphi\colon D \to M$ be a lattice homomorphism, and let $a \leq \varphi(1)$ in M. Then a is distributive with respect to φ iff there exists a family $\bigl(a_p \mid p \in \mathrm{J}(D)\bigr)$ of elements of M that satisfies the following conditions:*

(i) $a_p \leq \varphi(p)$, *for all* $p \in \mathrm{J}(D)$.
(ii) $p \leq q$ *implies that* $a_p \leq a_q$, *for all* $p, q \in \mathrm{J}(D)$.
(iii) $a_p \wedge \varphi(p_*) = \bigvee \bigl(a_q \mid q \in \mathrm{J}(p_*)\bigr)$, *for all* $p \in \mathrm{J}(D)$.
(iv) $a = \bigvee \bigl(a_p \mid p \in \mathrm{J}(D)\bigr)$.

Proof. Suppose first that a is distributive with respect to φ and $a \leq \varphi(1)$, and put $a_p = a \wedge \varphi(p)$, for all $p \in \mathrm{J}(D)$. Observe that (i) and (ii) are trivially satisfied. For (iii), we compute

$$
\begin{aligned}
a_p \wedge \varphi(p_*) &= a \wedge \varphi(p_*) && \text{(by the definition of } a_p) \\
&= \bigvee (a \wedge \varphi(q) \mid q \in \mathrm{J}(p_*)) && \text{(because } a \text{ is distributive with respect to } \varphi) \\
&= \bigvee \bigl(a_q \mid q \in \mathrm{J}(p_*)\bigr).
\end{aligned}
$$

For (iv), we compute

$$
\begin{aligned}
a &= a \wedge \varphi(1) && \text{(because } a \leq \varphi(1)) \\
&= \bigvee (a \wedge \varphi(p) \mid p \in \mathrm{J}(D)) && \text{(because } a \text{ is distributive with respect to } \varphi) \\
&= \bigvee \bigl(a_p \mid p \in \mathrm{J}(D)\bigr).
\end{aligned}
$$

Conversely, let $\bigl(a_p \mid p \in \mathrm{J}(D)\bigr)$ satisfy (i)–(iv) above and set $a_u = \bigvee \bigl(a_p \mid p \in \mathrm{J}(u)\bigr)$, for all $u \in D$. So $a_u \leq \varphi(u)$, for all $u \in D$. Furthermore, as D is distributive, every join-irreducible element of D is join-prime, thus the map $\psi\colon D \to M$, $u \mapsto a_u$ is a join homomorphism. It also follows from condition (iv) that $a = a_1$.

We claim that the equality $a_v \wedge \varphi(u) = a_u$ holds for all $u \leq v$ in D. As D is finite, an easy induction proof reduces the problem to the case where $u \prec v$. Let p be a minimal element of D with the property that $p \leq v$ and $p \nleq u$. So p is join-irreducible, $p \wedge u = p_*$, and $p \vee u = v$. We compute

$$
\begin{aligned}
a_v \wedge \varphi(u) &= (a_u \vee a_p) \wedge \varphi(u) && \text{(because } \psi \text{ is a join homomorphism)} \\
&= a_u \vee (a_p \wedge \varphi(u)) && \text{(because } M \text{ is modular and } a_u \leq \varphi(u)) \\
&= a_u \vee (a_p \wedge \varphi(p) \wedge \varphi(u)) && \text{(because } a_p \leq \varphi(p)) \\
&= a_u \vee (a_p \wedge \varphi(p \wedge u)) && \text{(because } \varphi \text{ is a lattice homomorphism)} \\
&= a_u \vee (a_p \wedge \varphi(p_*)) && \text{(because } p \wedge u = p_*) \\
&= a_u \vee a_{p_*} && \text{(by condition (iii))} \\
&= a_u && \text{(because } p_* \leq u),
\end{aligned}
$$

which completes the proof of the claim. Taking $v = 1$ in this claim yields that $\psi(u) = a_u = a \wedge \varphi(u)$, for all $u \in D$. In particular, ψ is a meet homomorphism. □

THEOREM 4.3. *Let G be an abelian group, let D be a finite distributive lattice, let $\varphi\colon D \to \operatorname{Sub} G$, $u \mapsto G_u$ be a lattice homomorphism. Then every finitely generated subgroup of G_1 is contained in some finitely generated subgroup of G_1 that is distributive with respect to φ.*

Proof. We argue by induction on $|\operatorname{J}(D)|$. Denote by $\mathcal{H}(\varphi)$ the set of all families $\vec{A} = (A_p \mid p \in \operatorname{J}(D))$ of finitely generated subgroups of G such that $A_p \leqslant G_p$ and $p \leq q$ implies that $A_p \leqslant A_q$, for all $p, q \in \operatorname{J}(D)$. We put $A_u = \sum (A_p \mid p \in \operatorname{J}_D(u))$, for all $u \in D$. Further, we set $N(\vec{A}) = \{p \in \operatorname{J}(D) \mid A_p \cap G_{p_*} = A_{p_*}\}$. For $\vec{A}, \vec{B} \in \mathcal{H}(\varphi)$, let $\vec{A} \leqslant \vec{B}$ hold, if $A_p \leqslant B_p$ for all $p \in \operatorname{J}(D)$.

CLAIM. *For all $\vec{A} \in \mathcal{H}(\varphi)$ and all $p \in \operatorname{J}(D)$, there exists $\vec{B} \in \mathcal{H}(\varphi)$ such that $\vec{A} \leqslant \vec{B}$ and $N(\vec{A}) \cup \{p\} \subseteq N(\vec{B})$.*

Proof of Claim. As $A_p \cap G_{p_*} \leqslant G_{p_*} = \sum (G_q \mid q \in \operatorname{J}_D(p_*))$ and $A_p \cap G_{p_*}$ is finitely generated (because A_p is), there are finitely generated subgroups $H_q \leqslant G_q$, for $q \in \operatorname{J}_D(p_*)$, such that

$$A_p \cap G_{p_*} \leqslant \sum (H_q \mid q \in \operatorname{J}_D(p_*)). \tag{4.2}$$

As the interval $D' = \{x \in D \mid x \leq p_*\}$ is a sublattice of D with fewer join-irreducible elements (because $\operatorname{J}(D')$ is contained in $\operatorname{J}(D) \setminus \{p\}$), it follows from the induction hypothesis that there exists a finitely generated subgroup C of G_{p_*} that is distributive with respect to $\varphi\restriction_{D'}$, and that contains $\sum (H_q \mid q \in \operatorname{J}_D(p_*))$. By using (4.2) and the definition of C, we get

$$A_{p_*} \leqslant A_p \cap G_{p_*} \leqslant C \leqslant G_{p_*}. \tag{4.3}$$

We put

$$C_u = C \cap G_{p_* \wedge u}, \qquad \text{for all } u \in D. \tag{4.4}$$

By using (4.3), we obtain easily that

$$A_{p_* \wedge u} \leqslant C_u \leqslant G_{p_* \wedge u}, \qquad \text{for all } u \in D. \tag{4.5}$$

As C is distributive with respect to $\varphi\restriction_{D'}$ and D is distributive, the assignment $u \mapsto C_u$ defines a lattice homomorphism from D to $\operatorname{Sub} C$.

We prove that the family $\vec{B} = (B_q \mid q \in \operatorname{J}(D))$, where

$$B_q = A_q + C_q, \qquad \text{for all } q \in \operatorname{J}(D)$$

(where C_q is defined *via* (4.4)), is as required for the claim. First observe that $\vec{B}$ obviously belongs to $\mathcal{H}(\varphi)$. As both maps $u \mapsto A_u$ and $u \mapsto C_u$ are join homomorphisms from D to $\operatorname{Sub} G$, we obtain

$$B_u = A_u + C_u, \qquad \text{for all } u \in D. \tag{4.6}$$

For all $u \geq p_*$ in D, it follows from (4.3) and (4.4) that $C_u = C \cap G_{p_*} = C$, and so $B_u = A_u + C$. In particular, $B_p = A_p + C$ and (using (4.3)) $B_{p_*} = A_{p_*} + C = C$, thus

$$\begin{aligned}
B_p \cap G_{p_*} &= (A_p + C) \cap G_{p_*} \\
&= (A_p \cap G_{p_*}) + C \quad &\text{(using (4.3) and the modularity of Sub } G) \\
&= C \quad &\text{(using (4.3))},
\end{aligned}$$

so $B_p \cap G_{p_*} = B_{p_*}$, that is, $p \in N(\vec{B})$.

Now let $q \in N(\vec{A})$, we prove that $q \in N(\vec{B})$. If $q \leq p_*$, then, using (4.5) and (4.6), we get that $B_q = C_q$ and $B_{q_*} = C_{q_*}$, so $B_q \cap G_{q_*} = B_{q_*}$ (because $\vec{C}$ belongs to $\mathcal{H}(\varphi\restriction_{D'})$), that is, $q \in N(\vec{B})$. Suppose that $q \nleq p_*$. From $p_* \wedge q < q$ it follows that $p_* \wedge q \leq q_*$, and so $p_* \wedge q = p_* \wedge q_*$, thus (see (4.4)) $C_q = C_{q_*}$. It follows that

$$\begin{aligned}
B_q \cap G_{q_*} &= (A_q + C_{q_*}) \cap G_{q_*} \quad &\text{(because } C_q = C_{q_*}) \\
&= (A_q \cap G_{q_*}) + C_{q_*} \quad &\text{(using (4.5) and the modularity of Sub } G) \\
&= A_{q_*} + C_{q_*} \quad &\text{(because } q \in N(\vec{A})) \\
&= B_{q_*} \quad &\text{(using (4.6))},
\end{aligned}$$

and so $q \in N(\vec{B})$. $\hspace{2cm}$ □ Claim.

Now let A be a finitely generated subgroup of G_1. As $A \leqslant \sum (G_p \mid p \in \mathrm{J}(D))$ and A is finitely generated, there exists $\vec{A} = (A_p \mid p \in \mathrm{J}(D))$ in $\mathcal{H}(\varphi)$ such that $A \leqslant \sum (A_p \mid p \in \mathrm{J}(D))$. By applying the Claim above at most $|\mathrm{J}(D)|$ times, starting with $\vec{A}$, we obtain $\vec{B} = (B_p \mid p \in \mathrm{J}(D))$ in $\mathcal{H}(\varphi)$ such that $\vec{A} \leqslant \vec{B}$ and $N(\vec{B}) = \mathrm{J}(D)$. The latter condition means that $B_p \cap G_{p_*} = B_{p_*}$ for all $p \in \mathrm{J}(D)$. Hence, by applying Lemma 4.2 (with $M = \mathrm{Sub}\, G$), we obtain that the subgroup $B = \sum (B_p \mid p \in \mathrm{J}(D))$ is distributive with respect to φ. Furthermore, B is finitely generated (because all the B_ps are). Finally,

$$A \leqslant \sum (A_p \mid p \in \mathrm{J}(D)) \leqslant \sum (B_p \mid p \in \mathrm{J}(D)) = B,$$

so the subgroup B is as required. $\hspace{6cm}$ □

DEFINITION 4.4. For a lattice D and an abelian group G, a map $\varphi \colon D \to \mathrm{Sub}\, G$ satisfies the *purity condition*, if $u \leq v$ implies that $\varphi(u) \leqslant_{\mathrm{pure}} \varphi(v)$, for all $u \leq v$ in D.

We shall avoid the terminology 'pure homomorphism', as it conflicts with another one frequently used in lattice theory and universal algebra.

Remark 4.5. In case G is *torsion-free*, $A \leqslant_{\mathrm{pure}} B$ implies that $A \cap C \leqslant_{\mathrm{pure}} B \cap C$, for all subgroups A, B, and C of G (this condition is well-known to fail, as a rule, in the non torsion-free case). In particular, in the context of Theorem 4.3, if the original map $\varphi \colon u \mapsto G_u$ satisfies the purity condition, then so does the map $u \mapsto B \cap G_u$.

The lattice-theory oriented reader will observe that the proof of Theorem 4.3 depends only on a few lattice-theoretical properties of $\mathrm{Sub}\, G$. In order to state the corresponding lattice-theoretical generalization of Theorem 4.3, we need the following classical definitions. An element a in a complete lattice L is *compact*, if for every $X \subseteq L$, if $a \leq \bigvee X$, then there exists a finite subset Y of X such that $a \leq \bigvee Y$. We say that L is *compactly nœtherian*, if it is complete, every element of L is a supremum of compact elements, and every subelement of a compact element of L is compact. For example, for an abelian group G, the subgroup lattice $\mathrm{Sub}\, G$ is a compactly nœtherian modular lattice, in which the compact elements are exactly the finitely generated subgroups of G. Now we can state the announced generalization of Theorem 4.3. The proof is, *mutatis mutandis*, the same as the one of Theorem 4.3.

THEOREM 4.3$'$. *Let D be a finite distributive lattice, let M be a compactly nœtherian modular lattice, and let $\varphi \colon D \to M$ be a lattice homomorphism. Then every compact element of M below $\varphi(1)$ lies below some compact element $b \leq \varphi(1)$ of M such that the map $D \to M$, $u \mapsto b \wedge \varphi(u)$ defines a lattice homomorphism.*

5. Pure Approximations of Lattice Homomorphisms Satisfying the Purity Condition

DEFINITION 5.1. For a lattice D and an abelian group G, we say that a lattice homomorphism $\varphi \colon D \to \mathrm{Sub}\, G$ satisfying the purity condition is *purely finitely approximated*, if for every finitely generated subgroup H of G, there exists a lattice homomorphism $\psi \colon D \to \mathrm{Sub}\, G$ satisfying the purity condition such that $\psi(u)$ is finitely generated and $H \cap \varphi(u) \leqslant \psi(u) \leqslant \varphi(u)$, for all $u \in D$.

For an abelian group G and a positive integer m, we put $G[m] = \{x \in G \mid mx = 0\}$. We also put $T(G) = \bigcup (G[m] \mid m \in \mathbb{N})$, the *torsion subgroup* of G. The following lemma will make it possible to reduce the proof of Theorem 5.4 to the torsion case and the torsion-free case.

LEMMA 5.2. *Let D be a lattice, let G be an abelian group, and let $\varphi \colon D \to \mathrm{Sub}\, G$, $u \mapsto G_u$ be a lattice homomorphism satisfying the purity condition.*

Denote by $\pi\colon G \twoheadrightarrow G/T(G)$ *the canonical projection. Then each of the following maps is a lattice homomorphism satisfying the purity condition.*

(i) *The map* $\varphi[m]\colon D \to \operatorname{Sub} G[m]$, $u \mapsto G_u[m]$.
(ii) *The map* $T(\varphi)\colon D \to \operatorname{Sub} T(G)$, $u \mapsto T(G_u)$.
(iii) *The map* $\overline{\varphi}\colon D \to \operatorname{Sub}(G/T(G))$, $u \mapsto \pi G_u$.

Proof. (i) (see the proof of Proposition 3.4 in [11]). It is obvious that $\varphi[m]$ is a meet homomorphism. Let $u, v \in D$ and let $z \in G_{u \vee v}[m]$. As $z \in G_{u \vee v} = G_u + G_v$, there are $x \in G_u$ and $y \in G_v$ such that $z = x + y$. As $0 = mz = mx + my$, we get $mx = -my$, so $mx \in G_u \cap G_v = G_{u \wedge v}$. As $G_{u \wedge v} \leqslant_{\mathrm{pure}} G_u$, there exists $t \in G_{u \wedge v}$ such that $mx = mt$. As $z = (x - t) + (y + t)$ with $x - t \in G_u[m]$ and $y + t \in G_v[m]$, we get $G_{u \vee v}[m] = G_u[m] + G_v[m]$. Therefore, $\varphi[m]$ is a lattice homomorphism.

Let $u \leq v$ in D, let $x \in G_v[m]$, and let $n \in \mathbb{N}$ such that $nx \in G_u[m]$. Let d be the greatest common divisor of m and n. There are integers k and l such that $km + ln = d$, so, from $nx \in G_u[m]$ and $mx = 0$ it follows that $dx \in G_u[m]$, thus $dx \in G_u$. As $G_u \leqslant_{\mathrm{pure}} G_v$, there exists $y \in G_u$ such that $dx = dy$. As d divides m, we get $my = mx = 0$, so $y \in G_u[m]$. As d divides n, we get $nx = ny$. Therefore, $G_u[m] \leqslant_{\mathrm{pure}} G_v[m]$, thus completing the proof of (i).

As $T(G)$ is the directed union of all $G[m]$s, (ii) follows immediately from (i).

(iii). It is obvious that $\overline{\varphi}$ is a join homomorphism. Let $u, v \in D$ and let $\overline{x} \in \pi G_u \cap \pi G_v$, say $\overline{x} = \pi(x)$ for some $x \in G$. There are $a, b \in T(G)$ such that $x - a \in G_u$ and $x - b \in G_v$. Pick $m \in \mathbb{N}$ such that $ma = mb = 0$. We obtain that $mx \in G_u \cap G_v = G_{u \wedge v}$, hence, as φ satisfies the purity condition, $mx = my$ for some $y \in G_{u \wedge v}$, thus $x - y \in T(G)$, and so $\overline{x} = \pi(y) \in \pi G_{u \wedge v}$. Therefore, $\overline{\varphi}$ is a lattice homomorphism.

Let $u \leq v$ in D, let $\overline{x} \in \pi G_v$ and $m \in \mathbb{N}$ such that $m\overline{x} \in \pi G_u$. Writing $\overline{x} = \pi(x)$ for $x \in G_v$, we obtain that $mx \in G_u + T(G)$, and so $nmx \in G_u$ for some $n \in \mathbb{N}$, hence, as φ satisfies the purity condition, $nmx = nmy$ for some $y \in G_u$, so $x - y \in T(G)$, and so $\overline{x} = \pi(y) \in \pi G_u$. Therefore, $\overline{\varphi}$ satisfies the purity condition. $\square$

The statements of Lemma 5.3 and Theorem 5.4 relate the concepts introduced in Definitions 4.4 (purity condition) and 5.1 (purely finitely approximated). The following result deals with the torsion case, and it is implicit in [11].

LEMMA 5.3. *Let D be a finite distributive lattice and let G be an abelian torsion group. Then every lattice homomorphism $\varphi\colon D \to \operatorname{Sub} G$ satisfying the purity condition is purely finitely approximated.*

Proof. Write $\varphi(u) = G_u$, for all $u \in D$, and let H be a finitely generated subgroup of G. Pick a positive integer m such that all elements of H have

order dividing m, and put $H_u = H \cap G_u$, for all $u \in D$. As $H \leqslant G_1[m]$ and by Lemma 5.2, this reduces the problem to the case where $mG = \{0\}$.

Now we argue as in the proof of [11, Theorem 6.1]. For all $p \in \mathrm{J}(D)$, since $G_{p_*} \leqslant_{\mathrm{pure}} G_p$ and $mG_p = \{0\}$, it follows from Kulikov's Theorem (see [10, Theorem 27.5]) that $G_p = G_{p_*} \oplus K_p$ for some subgroup K_p of G_p. Hence, [11, Lemma 5.2] yields that

$$G_u = G_0 \oplus \bigoplus \big(K_p \mid p \in \mathrm{J}(u) \big), \quad \text{for all } u \in D. \tag{5.1}$$

As $H_u \leqslant G_u$ and H_u is finitely generated, for all $u \in D$, there are finitely generated subgroups $G_0' \leqslant G_0$ and $K_p' \leqslant K_p$, for $p \in \mathrm{J}(D)$, such that, putting

$$G_u' = G_0' \oplus \bigoplus \big(K_p' \mid p \in \mathrm{J}(u) \big),$$

we get $H_u \leqslant G_u'$, for all $u \in D$. The map $u \mapsto G_u'$ is the desired approximation. □

Now we remove the torsion assumption from Lemma 5.3.

THEOREM 5.4 (Pure approximation theorem). *Let D be a finite distributive lattice and let G be an abelian group. Then every lattice homomorphism from D to $\mathrm{Sub}\,G$ satisfying the purity condition is purely finitely approximated.*

Proof. Let $\varphi \colon D \to \mathrm{Sub}\,G$, $u \mapsto G_u$ be a lattice homomorphism satisfying the purity condition. Denote by $\pi \colon G \twoheadrightarrow G/T(G)$ be the canonical projection and let $\overline{\varphi} \colon D \to \mathrm{Sub}(G/T(G))$, $u \mapsto \pi G_u$.

Now let A be a finitely generated subgroup of G. Without loss of generality we may take $A \leqslant G_1$. By applying Theorem 4.3 to the group $G/T(G)$, the homomorphism $\overline{\varphi}$, and the subgroup πA, we obtain a finitely generated subgroup $\overline{H}$ of πG_1 containing πA such that the map $\overline{\psi} \colon D \to \mathrm{Sub}(G/T(G))$, $u \mapsto \overline{H}_u = \overline{H} \cap \pi G_u$ is a lattice homomorphism. As $G/T(G)$ is torsion-free and $\overline{\varphi}$ satisfies the purity condition (see Lemma 5.2(iii)), the map $\overline{\psi}$ is a lattice homomorphism satisfying the purity condition (see Remark 4.5). As $\overline{H}$ is a finitely generated subgroup of the torsion-free abelian group $G/T(G)$, it is free abelian (of finite rank). Denote by $\varepsilon \colon \overline{H} \hookrightarrow G/T(G)$ the inclusion map.

CLAIM. *There exists a group embedding $\alpha \colon \overline{H} \hookrightarrow G$ such that $\pi \circ \alpha = \varepsilon$ and $\alpha \overline{H}_u \leqslant G_u$ for all $u \in D$.*

Proof of Claim. For each $p \in \mathrm{J}(D)$, as $\overline{H}_{p_*} \leqslant_{\mathrm{pure}} \overline{H}_p$ and $\overline{H}$ is finitely generated, there exists $\overline{K}_p \leqslant \overline{H}_p$ such that $\overline{H}_p = \overline{H}_{p_*} \oplus \overline{K}_p$. Hence it follows from [11, Lemma 5.2] that

$$\overline{H}_u = \overline{H}_0 \oplus \bigoplus \big(\overline{K}_p \mid p \in \mathrm{J}(u) \big), \quad \text{for all } u \in D. \tag{5.2}$$

For all $p \in \mathrm{J}(D)$, as $\overline{K}_p \leqslant \overline{H}$ and $\overline{H}$ is free abelian (of finite rank), $\overline{K}_p$ is free abelian (of finite rank), thus projective. Hence, as $\overline{K}_p \leqslant \overline{H}_p \leqslant \pi G_p$ and denoting by $\pi_p \colon G_p \twoheadrightarrow \pi G_p$ the restriction of π and by $\varepsilon_p \colon \overline{K}_p \hookrightarrow \pi G_p$ the restriction of ε, we obtain a group homomorphism $\alpha_p \colon \overline{K}_p \to G_p$ such that $\pi_p \circ \alpha_p = \varepsilon_p$. Similarly, denoting by $\pi_0 \colon G_0 \twoheadrightarrow \pi G_0$ the restriction of π and by $\varepsilon_0 \colon \overline{H}_0 \hookrightarrow \pi G_0$ the restriction of ε, we obtain a group homomorphism $\alpha_0 \colon \overline{H}_0 \to G_0$ such that $\pi_0 \circ \alpha_0 = \varepsilon_0$. Applying (5.2) to $u = 1$, we get $\overline{H} = \overline{H}_0 \oplus \bigoplus \left(\overline{K}_p \mid p \in \mathrm{J}(D) \right)$, so we can define a group homomorphism $\alpha \colon \overline{H} \to G$ by the rule

$$\alpha \left(x_0 + \sum_{p \in \mathrm{J}(D)} x_p \right) = \alpha_0(x_0) + \sum_{p \in \mathrm{J}(D)} \alpha_p(x_p), \tag{5.3}$$

for all $x_0 \in \overline{H}_0$ and all $\left(x_p \mid p \in \mathrm{J}(D) \right) \in \prod \left(\overline{K}_p \mid p \in \mathrm{J}(D) \right)$. From $\pi_p \circ \alpha_p = \varepsilon_p$ for all $p \in \mathrm{J}(D) \cup \{0\}$ it follows that $\pi \circ \alpha = \varepsilon$. As ε is an embedding, so is α. Finally, let $u \in D$ and let $x \in \overline{H}_u$. It follows from (5.2) that x can be decomposed as

$$x = x_0 + \sum_{p \in \mathrm{J}(u)} x_p, \quad \text{where } x_0 \in \overline{H}_0 \text{ and } x_p \in \overline{K}_p \text{ for all } p \in \mathrm{J}(u).$$

As $\alpha_p(x_p) \in G_p$ for all $p \in \mathrm{J}(u) \cup \{0\}$, it follows from (5.3) that $\alpha(x) \in G_u$.
$\hfill \square$ Claim.

Put $H = \alpha \overline{H}$. As ε is an embedding and $\pi \circ \alpha = \varepsilon$, we obtain

$$H \cap T(G) = \{0\}. \tag{5.4}$$

As α is an embedding, the map $\psi \colon D \to \operatorname{Sub} H$, $u \mapsto H_u = \alpha \overline{H}_u$ is a lattice homomorphism satisfying the purity condition (because $\overline{\psi} \colon u \mapsto \overline{H}_u$ has these properties). For $u \in D$, we observe that

$$\pi(A \cap G_u) \leqslant \overline{H} \cap \pi G_u = \overline{H}_u = \pi H_u,$$

thus for all $x \in A \cap G_u$, there exists $y \in H_u$ such that $x - y \in T(G)$. As $y \in G_u$ (because $y \in H_u = \alpha \overline{H}_u \leqslant G_u$) and $x \in G_u$, we obtain that $x - y$ belongs to $T(G) \cap G_u = T(G_u)$, and so $x \in T(G_u) + H_u$. Hence, using (5.4), we have proved that $A \cap G_u \leqslant T(G_u) \oplus H_u$. As $A \cap G_u$ is finitely generated, there exists a finitely generated subgroup B_u of $T(G_u)$ such that $A \cap G_u \leqslant B_u \oplus H_u$.

It follows from Lemma 5.2(ii) that the map $T(\varphi) \colon D \to \operatorname{Sub} T(G)$, $u \mapsto T(G_u)$ is a lattice homomorphism satisfying the purity condition. Hence, applying Lemma 5.3 to this morphism and the sum of all B_us, we obtain a lattice homomorphism $D \to \operatorname{Sub} T(G)$, $u \mapsto G'_u$, satisfying the purity condition and with G'_1 finitely generated, such that $B_u \leqslant G'_u \leqslant T(G_u)$ for all $u \in D$. So $A \cap G_u \leqslant G'_u \oplus H_u$, for all $u \in D$. It follows from (5.4) that

the map $D \to \operatorname{Sub} G$, $u \mapsto G'_u \oplus H_u$ is a lattice homomorphism satisfying the purity condition, so it is as desired.   □

As an immediate corollary, we get a lattice-theoretical characterization of purity for embeddings of abelian groups, similar to the one mentioned in [12].

COROLLARY 5.5. *A subgroup A of an abelian group B is a pure subgroup iff for any finitely generated $H \leqslant B$, there are finitely generated $A' \leqslant A$ and $B' \leqslant B$ such that $A \cap H \leqslant A'$, $H \leqslant B'$, and $A' \leqslant^{\oplus} B'$.*

Proof. That the given condition implies purity is an easy exercise (take H monogenic). Conversely, suppose that $A \leqslant_{\text{pure}} B$ and let $H \leqslant B$ be a finitely generated subgroup. Denote by $D = \{0, 1\}$ the two-element chain and by $\varphi \colon D \to \operatorname{Sub} B$ the homomorphism sending 0 to A and 1 to B. As φ satisfies the purity condition and by Theorem 5.4, there exists a homomorphism $\psi \colon D \to \operatorname{Sub} B$ with finitely generated values such that $H \cap \varphi(u) \leqslant \psi(u) \leqslant \varphi(u)$, for all $u \in D$. Put $A' = \psi(0)$ and $B' = \psi(1)$.   □

6. Regular Refinement Monoids; the Classes $\overline{\mathcal{B}}$, $\overline{\mathcal{L}}$, and $\overline{\mathcal{R}}$

We shall use the notation and terminology of [11] concerning (abelian) monoids and semilattices of groups. In particular, every abelian monoid M is endowed with a partial preordering $\leq$ defined by $x \leq y$ iff there exists z such that $x + z = y$. We say that M is *conical*, if $x + y = 0$ implies that $x = y = 0$, for all $x, y \in M$. We say that M is *regular*, if $2x \leq x$, for all $x \in M$, and we say that M is a *semilattice of groups*, if M is a disjoint union of groups (i.e., subsemigroups each of which happens to be a group). We say that M is a *refinement monoid*, if for all $a_0, a_1, b_0, b_1 \in M$, if $a_0 + a_1 = b_0 + b_1$, then there are $c_{i,j} \in M$, for $i, j < 2$, such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$, for all $i < 2$. We put $\Lambda(M) = \{x \in M \mid 2x = x\}$. A *semilattice* is an abelian, idempotent monoid. It is *distributive*, if it is a refinement monoid.

We recall the following characterization of regular abelian monoids, see [13, Corollary IV.2.2], also [11, Lemma 2.1].

LEMMA 6.1. *An abelian monoid M is regular iff it is a semilattice of groups.*

A regular abelian monoid M is the disjoint union of all subgroups

$$G_M[e] = \{x \in M \mid e \leq x \leq e\}, \qquad \text{for } e \in \Lambda(M).$$

For $a \leq b$ in $\Lambda(M)$, the assignment $j_{a,b} \colon x \mapsto x + b$ defines the *natural map* from $G_M[a]$ to $G_M[b]$. It is a group homomorphism. If $j_{a,b}$ is an embedding for all $a \leq b$ in M, then we shall say that M satisfies the

embedding condition, denoted by (emb). If the range of $j_{a,b}$ is a pure subgroup of $G_M[b]$ for all $a \leq b$ in $\Lambda(M)$, then we shall say that M satisfies the *purity condition*, denoted by (pur).

The following definition introduces classes $\overline{\mathcal{B}}$, $\overline{\mathcal{L}}$, and $\overline{\mathcal{R}}$ of abelian monoids, which properly contain, respectively, the classes $\mathcal{B}$, $\mathcal{L}$, and $\mathcal{R}_{\mathrm{ep}}$ considered in [11].

Notation 6.2. Denote by $\overline{\mathcal{B}}$ the class of all finite direct sums of abelian monoids of the form $(\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}$, where $n \in \mathbb{N}$, or $\mathbb{Z} \sqcup \{0\}$ (where $\sqcup$ denotes disjoint union). Further, denote by $\overline{\mathcal{L}}$ the class of all direct limits of monoids from $\overline{\mathcal{B}}$, and by $\overline{\mathcal{R}}$ the class of all regular refinement monoids satisfying the conditions (emb) and (pur).

As $\overline{\mathcal{B}}$ is a class of finitely generated abelian monoids, closed under finite direct sums, the following lemma is an easy consequence of Proposition 3.1 and Section 4 in [11].

LEMMA 6.3. *The class $\overline{\mathcal{L}}$ is closed under direct limits, finite direct sums, and retracts, and contains as a member $A \sqcup \{0\}$, for any finitely generated abelian group A. Furthermore, $\overline{\mathcal{L}}$ is contained in $\overline{\mathcal{R}}$.*

LEMMA 6.4. *Any finitely generated monoid in $\overline{\mathcal{R}}$ belongs to $\overline{\mathcal{L}}$.*

Proof. Similar to the proof of [11, Proposition 5.3], the key point being this time that every pure subgroup of a finitely generated abelian group is a direct summand. $\square$

LEMMA 6.5. *Each monoid M in $\overline{\mathcal{R}}$ is the directed union of those finitely generated submonoids of M that belong to $\overline{\mathcal{R}}$.*

Proof. We argue as in the proof of [11, Theorem 6.1]. We must prove that any finite subset X of M is contained in some finitely generated submonoid of M lying in $\overline{\mathcal{R}}$. For convenience, we assume that $0 \in X$. Furthermore, by [11, Theorem 3.3], we may assume that

$$M = \bigsqcup_{e \in \Lambda} (\{e\} \times G_e) \subseteq \Lambda \times G,$$

for some distributive semilattice Λ and some abelian group G with subgroups G_e satisfying the following conditions:

  (i) $G = \bigcup_{e \in \Lambda} G_e$.
 (ii) $a \leq b$ implies that $G_a \leqslant_{\mathrm{pure}} G_b$, for all $a, b \in \Lambda$.
(iii) $G_0 = \{0\}$.
 (iv) $G_a + G_b = G_{a+b}$, for all $a, b \in \Lambda$.
  (v) $G_a \cap G_b = \bigcup_{e \leq a,b \text{ in } \Lambda} G_e$, for all $a, b \in \Lambda$.

Now we denote by $\operatorname{Id}\Lambda$ the lattice of all *ideals* of Λ, that is, those non-empty subsets A of Λ such that $x + y \in A$ iff $x, y \in A$, for all $x, y \in \Lambda$. As Λ is a distributive semilattice, $\operatorname{Id}\Lambda$ is a distributive lattice.

Now we set $G_A = \bigcup_{e \in A} G_e$, for all $A \in \operatorname{Id}\Lambda$. Observe that the union defining G_A is directed, and that $G_{[0,e]} = G_e$, for all $e \in \Lambda$. Hence the map $\operatorname{Id}\Lambda \to \operatorname{Sub}G$, $A \mapsto G_A$ is a lattice homomorphism satisfying the purity condition, and sending $\{0\}$ to $\{0\}$.

Write any $x \in X$ in the form $x = (e_x, g_x) \in M$. Denote by $\boldsymbol{D}$ the sublattice of $\operatorname{Id}\Lambda$ generated by $\{[0, e_x] \mid x \in X\}$ and by K the (finitely generated) subgroup of G generated by $\{g_x \mid x \in X\}$. As $\operatorname{Id}\Lambda$ is distributive, $\boldsymbol{D}$ is finite. Moreover, the ideal $\{0\}$ belongs to $\boldsymbol{D}$ since $0 \in X$. By Theorem 5.4, there exists a lattice homomorphism $\boldsymbol{D} \to \operatorname{Sub}G$, $A \mapsto G'_A$ satisfying the purity condition such that G'_A is finitely generated and $G_A \cap K \leqslant G'_A \leqslant G_A$, for all $A \in \boldsymbol{D}$. For each $P \in \operatorname{J}(\boldsymbol{D})$, as $G'_{P_*} \leqslant_{\text{pure}} G'_P$ and G'_P is finitely generated, there exists $H_P \leqslant G'_P$ such that $G'_P = G'_{P_*} \oplus H_P$. As $G'_{\{0\}} = \{0\}$, we obtain, using [11, Lemma 5.2], that

$$G'_A = \bigoplus (H_P \mid P \in \operatorname{J}_{\boldsymbol{D}}(A)), \qquad \text{for all } A \in \boldsymbol{D}.$$

Using the observations that X is finite and that for each $x \in X$, the element $[0, e_x]$ is the join of all join-irreducible elements of $\boldsymbol{D}$ below it, we obtain, as in the proof of [11, Theorem 6.1], elements $u_P \in P$, for $P \in \operatorname{J}(\boldsymbol{D})$, such that

$$e_x = \bigvee (u_P \mid P \in \operatorname{J}_{\boldsymbol{D}}([0, e_x])), \qquad \text{for all } x \in X.$$

Since each G'_P is a finitely generated subgroup of the directed union $G_P = \bigcup_{e \in P} G_e$, we may enlarge the elements u_P to ensure that $G'_P \leqslant G_{u_P}$, for all $P \in \operatorname{J}(\boldsymbol{D})$. Finally, denoting by $P^\dagger$ the largest element of $\boldsymbol{D}$ such that $P \nsubseteq P^\dagger$ (its existence is ensured by P being a join-irreducible element of the finite distributive lattice $\boldsymbol{D}$), we may enlarge u_P further to ensure that $u_P \in P \setminus P^\dagger$. We define a map $\varphi \colon \boldsymbol{D} \to \Lambda$ by the rule

$$\varphi(A) = \bigvee (u_P \mid P \in \operatorname{J}_{\boldsymbol{D}}(A)), \qquad \text{for all } A \in \boldsymbol{D}.$$

Now we conclude the proof as in the final section of the proof of [11, Theorem 6.1]. The construction of φ ensures that it is a semilattice embedding from $\boldsymbol{D}$ into Λ, that $\varphi(A) \in A$ for all $A \in \boldsymbol{D}$, and that $\varphi([0, e_x]) = e_x$ for all $x \in X$. Further, we set

$$N = \bigsqcup_{A \in \boldsymbol{D}} (\{\varphi(A)\} \times G'_A) \subseteq \Lambda \times G.$$

As $A \mapsto G'_A$ is a zero-preserving lattice homomorphism satisfying the purity condition, it follows from [11, Theorem 3.3] that N belongs to $\overline{\mathcal{R}}$. As all groups G'_A are finitely generated, N is finitely generated. As

$$G'_A = \sum_{P \in \mathbf{J}_{\mathbf{D}}(A)} G'_P \leqslant \sum_{P \in \mathbf{J}_{\mathbf{D}}(A)} G_{u_P} \leqslant \sum_{P \in \mathbf{J}_{\mathbf{D}}(A)} G_{\varphi(P)} = G_{\varphi(A)},$$

for all $A \in \mathbf{D}$, we see that N is contained in M. Finally, for each $x \in X$, as g_x belongs to $G_{e_x} = G_{[0,e_x]}$ and $g_x \in K$, we get $g_x \in G_{[0,e_x]} \cap K \subseteq G'_{[0,e_x]}$, so $x = (e_x, g_x) \in N$. Therefore, X is contained in N. $\qquad\square$

By using Lemmas 6.3, 6.4, and 6.5, we obtain our main monoid-theoretical result.

THEOREM 6.6. *The direct limits of finite direct sums of abelian monoids of the form $(\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}$, where $n \in \mathbb{N}$, or $\mathbb{Z} \sqcup \{0\}$, are exactly the regular conical refinement monoids satisfying* (emb) *and* (pur). *That is, $\overline{\mathcal{L}} = \overline{\mathcal{R}}$.*

An obvious adaptation of [11, Corollary 6.6] give us the following result.

COROLLARY 6.7. *Let (M, u) be an abelian monoid with order-unit. Then (M, u) is a direct limit of finite direct sums of pairs of the form $((\mathbb{Z}/n\mathbb{Z}) \sqcup \{0\}, \overline{m})$ and $(\mathbb{Z} \sqcup \{0\}, m)$ if and only if M is a regular conical refinement monoid satisfying* (emb) *and* (pur).

7. Lifting Monoid Maps by C*-Algebra Maps

DEFINITION 7.1. A *Cuntz algebra* is an algebra of the form $\mathcal{O}_n$, where $2 \leq n \leq \infty$. A *special Cuntz limit* is a C*-inductive limit of a sequence of finite direct sums of Cuntz algebras. An *extended Cuntz limit* is a C*-inductive limit of a sequence of finite direct sums of full matrix algebras over Cuntz algebras $\mathcal{O}_n$ for $2 \leq n < \infty$ and nonzero corners of $\mathcal{O}_\infty$ by projections.

The basic K-theoretic information concerning the Cuntz algebras $\mathcal{O}_n$, where $2 \leq n < \infty$, is usually summarized in the statements $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_1(\mathcal{O}_n) = 0$ [7, Theorems 3.7–3.8]. Also, $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$ [7, Corollary 3.11]. However, Cuntz also showed that the Murray–von Neumann equivalence classes of nonzero projections in a purely infinite simple C*-algebra A form a subgroup of $V(A)$ that maps isomorphically onto $K_0(A)$ under the natural map $V(A) \to K_0(A)$ [7, p. 188]. Moreover [3, Proposition 2.1, Corollary 2.2], $V(A) \cong \{0\} \sqcup K_0(A)$. It follows that $V(\mathcal{O}_n) \setminus \{0\}$ is a group isomorphic to $K_0(\mathcal{O}_n)$, that is, $V(\mathcal{O}_n) \cong (\mathbb{Z}/(n-1)\mathbb{Z}) \sqcup \{0\}$. It is routine to check that the corresponding isomorphism sends $[1_{\mathcal{O}_n}]$ to the coset $\overline{1}$ in $\mathbb{Z}/(n-1)\mathbb{Z}$, and thus we get

$$\left(V(\mathrm{M}_m(\mathcal{O}_n)), [1_{\mathrm{M}_m(\mathcal{O}_n)}] \right) \cong \left((\mathbb{Z}/(n-1)\mathbb{Z}) \sqcup \{0\}, \overline{m} \right), \tag{7.1}$$

for all $m \geq 1$ and $n \geq 2$.

Remark 7.2. Because of (7.1), for every $n \geq 2$, $k \geq 1$, and for every nonzero projection $p \in \mathrm{M}_k(\mathcal{O}_n)$, there exists $l \in \{1, 2, \ldots, n-1\}$ such that $p\mathrm{M}_k(\mathcal{O}_n)p \cong \mathrm{M}_l(\mathcal{O}_n)$. To see this, observe that as $p \neq 0$, there exists $l \in \{1, 2, \ldots, n-1\}$ such that $[p] = \bar{l} \in \mathbb{Z}/(n-1)\mathbb{Z}$. Denote by $\mathbb{K}$ the C*-algebra of compact operators over an infinite-dimensional, separable Hilbert space $\mathcal{H}$, set $I_t = \sum_{i=1}^{t} e_{i,i} \in \mathbb{K}$ for any $t \geq 1$, and set $E_l = 1 \otimes I_l \in \mathcal{O}_n \otimes \mathbb{K}$. From

$$\mathrm{M}_k(\mathcal{O}_n) \otimes \mathbb{K} \cong \mathcal{O}_n \otimes \mathrm{M}_k(\mathbb{C}) \otimes \mathbb{K} \cong \mathcal{O}_n \otimes \mathrm{M}_k(\mathbb{K}),$$

and $\mathrm{M}_k(\mathbb{K}) \cong \mathbb{K}$ (see [19, Proposition 1.10.2]), we obtain a natural isomorphism $\mathrm{M}_k(\mathcal{O}_n) \otimes \mathbb{K} \cong \mathcal{O}_n \otimes \mathbb{K}$. Let the projection $q \in \mathcal{O}_n \otimes \mathbb{K}$ correspond to $p \otimes I_1 \in \mathrm{M}_k(\mathcal{O}_n) \otimes \mathbb{K}$ under this isomorphism (whence $[q] = \bar{l} \in \mathbb{Z}/(n-1)\mathbb{Z}$). It follows that $q \sim E_l$ in $\mathcal{O}_n \otimes \mathbb{K}$, whence

$$p\mathrm{M}_k(\mathcal{O}_n)p \cong (p \otimes I_1)(\mathrm{M}_k(\mathcal{O}_n) \otimes \mathbb{K})(p \otimes I_1) \cong q(\mathcal{O}_n \otimes \mathbb{K})q \cong E_l(\mathcal{O}_n \otimes \mathbb{K})E_l \cong \mathrm{M}_l(\mathcal{O}_n),$$

as desired.

Analogously [7, Section 3], we get $V(\mathcal{O}_\infty) \cong \mathbb{Z} \sqcup \{0\}$, *via* an isomorphism sending $[1_{\mathcal{O}_\infty}]$ to 1 in $\mathbb{Z}$, and

$$\left(V(\mathrm{M}_m(\mathcal{O}_\infty)), [1_{\mathrm{M}_m(\mathcal{O}_\infty)}]\right) \cong \left(\mathbb{Z} \sqcup \{0\}, m\right), \qquad \text{for all } m \in \mathbb{N}. \tag{7.2}$$

We also need to consider the case of the pairs $(\mathbb{Z}, -m)$, with $m \in \mathbb{Z}^+$, which cannot be represented by any pair $(K_0(\mathrm{M}_n(\mathcal{O}_\infty)), [1_{\mathrm{M}_n(\mathcal{O}_\infty)}])$. We can solve this problem by using corners by projections of $\mathcal{O}_\infty$. Throughout Sections 7 and 8, we shall use the projections of $\mathcal{O}_\infty$ defined as

$$p_n = 1 - \sum_{i=1}^{n} s_i s_i^*, \quad \text{for all } n \geq 0 \quad (\text{so } p_0 = 1).$$

Observe that $[p_n] \in K_0(\mathcal{O}_\infty)$ corresponds to $-(n-1) \in \mathbb{Z}$. Hence

$$\left(V(p_n \mathcal{O}_\infty p_n), [p_n]\right) \cong \left(\mathbb{Z} \sqcup \{0\}, -(n-1)\right). \tag{7.3}$$

Remark 7.3. Because of (7.2) and (7.3), we get the following facts:
(i) For every $k \geq 1$ there exists a projection $p \in \mathcal{O}_\infty$ such that $p\mathcal{O}_\infty p \cong \mathrm{M}_k(\mathcal{O}_\infty)$. To see this, notice that for every $k \geq 1$ there exists a projection $p \in \mathcal{O}_\infty$ such that $[p] = k \in \mathbb{Z}$. So, as in Remark 7.2, $p \otimes I_1 \sim 1 \otimes I_k$ in $\mathcal{O}_\infty \otimes \mathbb{K}$, and hence

$$p\mathcal{O}_\infty p \cong (p \otimes I_1)(\mathcal{O}_\infty \otimes \mathbb{K})(p \otimes I_1) \cong (1 \otimes I_k)(\mathcal{O}_\infty \otimes \mathbb{K})(1 \otimes I_k) \cong \mathrm{M}_k(\mathcal{O}_\infty).$$

(ii) For every projection $p \in \mathcal{O}_\infty$ and every $k \in \mathbb{Z}^+$, one of the following cases occurs (by arguments similar to those in part (i)):

(a) If $\big(V(p\mathcal{O}_\infty p), [p]\big) \cong \big(\mathbb{Z} \sqcup \{0\}, k\big)$, then $p\mathcal{O}_\infty p \cong M_k(\mathcal{O}_\infty)$;

(b) If $\big(V(p\mathcal{O}_\infty p), [p]\big) \cong \big(\mathbb{Z} \sqcup \{0\}, -k\big)$, then $p\mathcal{O}_\infty p \cong M_k(p_2\mathcal{O}_\infty p_2)$;

(c) If $\big(V(p\mathcal{O}_\infty p), [p]\big) \cong \big(\mathbb{Z} \sqcup \{0\}, 0_\mathbb{Z}\big)$, then $p\mathcal{O}_\infty p \cong p_1\mathcal{O}_\infty p_1$.

Thus, in order to represent $\mathbb{Z} \sqcup \{0\}$ by corners of $\mathcal{O}_\infty$, we can restrict our arguments to those corners generated by 1, p_1, and p_2. Furthermore, by [15, Theorem 3.5(1)], the isomorphism

$$\tau : (V(\mathcal{O}_\infty), [1]) \rightarrow (V(p_2\mathcal{O}_\infty p_2), [p_2]), \tag{7.4}$$

is induced by a unital C*-algebra isomorphism

$$\psi : \mathcal{O}_\infty \rightarrow p_2\mathcal{O}_\infty p_2.$$

The remaining basic facts that we shall need are contained in the following lemmas.

LEMMA 7.4. *Let B be a C*-algebra and let $q \in B$ a projection. Then any normalized monoid homomorphism*

$$\alpha : (V(\mathcal{O}_\infty), [1]) \rightarrow (V(B), [q]),$$

is induced by a C*-algebra homomorphism $\phi : \mathcal{O}_\infty \rightarrow B$ that sends 1 to q. That is, $V(\phi) = \alpha$.

Proof. Set $a = [1]$ and $b_n = [p_n]$, for all $n \geq 1$. Observe that $a = [s_1 s_1^*] = a + b_1$ and $b_n = a + b_{n+1}$. As $[q] = \alpha([1]) = [q] + \alpha(b_1)$ and by Lemma 3.1, there exists a projection $q_1 \in qBq$ such that $q_1 \sim q$ and $[q - q_1] = \alpha(b_1)$. Suppose that we have constructed pairwise orthogonal projections $q_1, \ldots, q_n \in qBq$ such that $q_l \sim q$ and $\alpha(b_l) = \big[q - \sum_{i=1}^l q_i\big]$ for $1 \leq l \leq n$. As

$$\left[q - \sum_{i=1}^n q_i\right] = \alpha(b_n) = [q] + \alpha(b_{n+1}),$$

there exists a projection $q_{n+1} \in \big(q - \sum_{i=1}^n q_i\big)B\big(q - \sum_{i=1}^n q_i\big)$ such that $q_{n+1} \sim q$ and $\big[q - \sum_{i=1}^{n+1} q_i\big] = \alpha(b_{n+1})$. Thus we have constructed, by induction, a sequence $(q_i \mid i \geq 1)$ of pairwise orthogonal projections in qBq such that $q_n \sim q$ while $\alpha(b_n) = \big[q - \sum_{i=1}^n q_i\big]$ for all $n \geq 1$. Hence there exists a sequence $(t_i \mid i \geq 1)$ of elements of qBq with $t_i^* t_j = q\delta_{i,j}$ and $t_i t_i^* = q_i$ for all positive integers i, j. There exists a unique C*-algebra homomorphism $\phi : \mathcal{O}_\infty \rightarrow B$ such that $\phi(1) = q$ while $\phi(s_i) = t_i$ for all $i \geq 1$. As

$$V(\phi)([1]) = [q] = \alpha([1]),$$

$$V(\phi)([p_n]) = \left[q - \sum_{i=1}^n q_i\right] = \alpha([p_n]), \qquad \text{for all } n \geq 1,$$

and $\{[1]\} \cup \{[p_n] \mid n \geq 1\}$ generates the monoid $V(\mathcal{O}_\infty)$, we get $V(\phi) = \alpha$. $\quad\square$

LEMMA 7.5. *Let A be a finite direct sum of full matrix algebras over Cuntz algebras, let B a C*-algebra, and let $q \in B$ be a projection. Then any normalized monoid homomorphism*

$$\alpha \colon (V(A), [1_A]) \to (V(B), [q]),$$

is induced by a C*-algebra homomorphism $\phi \colon A \to B$ that sends 1_A to q. That is, $V(\phi) = \alpha$.

Proof. Write $A = \bigoplus_{j=1}^{r} M_{k_j}(\mathcal{O}_{n_j}) \oplus \bigoplus_{i=1}^{s} M_{l_i}(\mathcal{O}_\infty)$ for some $k_j, n_j, l_i \in \mathbb{N}$. Let p be the central projection of A corresponding to the unit of $\bigoplus_{j=1}^{r} M_{k_j}(\mathcal{O}_{n_j})$, and let $q_1, \ldots, q_s$ be the orthogonal central projections in A corresponding to $M_{l_1}(\mathcal{O}_\infty), \ldots, M_{l_s}(\mathcal{O}_\infty)$, respectively. Thus p and $q_1, \ldots, q_s$ are orthogonal central projections summing up to 1_A.

Each q_i is an orthogonal sum of pairwise equivalent projections $g_1^{(i)}, \ldots, g_{l_i}^{(i)}$ such that $g_1^{(i)} A g_1^{(i)} \cong \mathcal{O}_\infty$. In $V(A)$, we get the equation

$$[p] + \sum_{i=1}^{s} l_i [g_1^{(i)}] = [p] + \sum_{i=1}^{s} [q_i] = [1_A],$$

whence $\alpha([p]) + \sum_{i=1}^{s} l_i \alpha([g_1^{(i)}]) = [q]$ in $V(B)$. By Lemma 3.1, q is an orthogonal sum of projections $\overline{p}, \overline{q}_1, \ldots, \overline{q}_s$ of B such that $\alpha([p]) = [\overline{p}]$, $\alpha([q_i]) = [\overline{q}_i]$, and each $\overline{q}_i$ is an orthogonal sum of pairwise equivalent projections $h_1^{(i)}, \ldots, h_{l_i}^{(i)}$ such that $\alpha([g_1^{(i)}]) = [h_1^{(i)}]$.

As $pAp \cong \bigoplus_{j=1}^{r} M_{k_j}(\mathcal{O}_{n_j})$, it follows from [11, Lemma 7.1] that the restriction of α to $V(pAp)$ is induced by a C*-algebra homomorphism $\phi' \colon pAp \to B$ such that $\phi'(p) = \overline{p}$. Furthermore, as $g_1^{(i)} A g_1^{(i)} \cong \mathcal{O}_\infty$, the restriction of α to $V(g_1^{(i)} A g_1^{(i)})$ defines a normalized monoid morphism $\alpha_i \colon (V(\mathcal{O}_\infty), [1]) \to (V(B), [h_1^{(i)}])$. By Lemma 7.4, there exists a C*-algebra homomorphism $\psi_i \colon g_1^{(i)} A g_1^{(i)} \to B$ inducing α_i such that $\psi_i(g_1^{(i)}) = h_1^{(i)}$. As $q_i A q_i \cong M_{l_i}(g_1^{(i)} A g_1^{(i)})$ and $\overline{q}_i B \overline{q}_i \cong M_{l_i}(h_1^{(i)} B h_1^{(i)})$, the map ψ_i extends to a C*-algebra homomorphism $\phi_i \colon q_i A q_i \to \overline{q}_i B \overline{q}_i$ that induces the restriction of α to $V(q_i A q_i)$. As the equivalence classes of projections from pAp and from $g_1^{(i)} A g_1^{(i)}$, for $1 \leq i \leq s$, generate $V(A)$, the C*-algebra homomorphism

$$\phi = \phi' \oplus \bigoplus_{i=1}^{s} \phi_i \colon A \to \overline{p} B \overline{p} \oplus \bigoplus_{i=1}^{s} \overline{q}_i B \overline{q}_i = qBq \subseteq B,$$

induces α.   □

THEOREM 7.6. *An abelian monoid M is isomorphic to $V(A)$ for some special (resp., extended) Cuntz limit A if and only if*

(a) *M is a countable, regular conical refinement monoid.*
(b) *For all idempotents $e \leq f$ in M, the homomorphism $G_M[e] \to G_M[f]$ given by $x \mapsto x + f$ is injective, and $G_M[e] + f$ is a pure subgroup of $G_M[f]$.*

Proof. ($\Longrightarrow$): By (7.1) and (7.2) and since the $V(-)$ functor preserves direct limits and finite direct sums, the present implication follows from the easy direction of Theorem 6.6.

($\Longleftarrow$): Since M is countable, Theorem 6.6 implies that M is the direct limit of a sequence of the form

$$M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \xrightarrow{\alpha_3} \cdots$$

where each M_i is a finite direct sum of monoids $\mathbb{Z} \sqcup \{0\}$ and $(\mathbb{Z}/n_{ij}\mathbb{Z}) \sqcup \{0\}$ for some $2 \leq n_{ij} < \infty$. Hence, denoting by A_i the direct sum of the corresponding Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_{n_{ij}+1}$, there is an isomorphism $h_i \colon V(A_i) \to M_i$. Each of the homomorphisms

$$h_{i+1}^{-1}\alpha_i h_i \colon V(A_i) \longrightarrow V(A_{i+1})$$

sends $[1_{A_i}]$ to the class of a projection in A_{i+1}, and so, by Lemma 7.5, $h_{i+1}^{-1}\alpha_i h_i$ is induced by a C*-algebra homomorphism $\phi_i \colon A_i \to A_{i+1}$. Therefore $M \cong V(A)$, where A is the C*-inductive limit of the sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots \qquad\qquad \square$$

A structural description of the monoids appearing in Theorem 7.6 is easily obtained with the help of [11, Theorem 3.3], as follows.

COROLLARY 7.7. *Let M be an abelian monoid. Then $M \cong V(A)$ for some special (resp., extended) Cuntz limit A if and only if*

$$M \cong \bigsqcup_{e \in \Lambda} (\{e\} \times G_e) \subseteq \Lambda \times G,$$

where

(a) *Λ is a countable distributive semilattice.*
(b) *G is a countable abelian group.*
(c) *G_e is a pure subgroup of G for all $e \in \Lambda$.*
(d) *$G_0 = \{0\}$ and $\bigcup_{e \in \Lambda} G_e = G$.*
(e) *$G_e + G_f = G_{e+f}$ and $G_e \cap G_f = \bigcup_{g \in \Lambda, \, g \leq e, f} G_g$ for all $e, \, f \in \Lambda$.*

COROLLARY 7.8. *For every extended Cuntz limit A, there exists a special Cuntz limit B such that $V(A) \cong V(B)$.*

8. Algebras with Order-Unit

In the present section, we establish unital versions of Theorem 7.6 and Corollary 7.7. For this, we need suitable adaptations of Lemma 7.4 to the corners of $\mathcal{O}_\infty$ generated by the projections p_1 and p_2.

LEMMA 8.1. *Let B be a C*-algebra, and $q \in B$ a projection. Then any normalized monoid homomorphism*

$$\alpha \colon (V(p_2 \mathcal{O}_\infty p_2), [p_2]) \to (V(B), [q]),$$

is induced by a C*-algebra homomorphism $\phi \colon p_2 \mathcal{O}_\infty p_2 \to B$ that sends p_2 to q. That is, $V(\phi) = \alpha$.

   Proof. As noticed in (7.4), the normalized monoid isomorphism

$$\tau \colon (V(\mathcal{O}_\infty), [1]) \to (V(p_2 \mathcal{O}_\infty p_2)),$$

is induced by a unital C*-algebra isomorphism $\psi \colon \mathcal{O}_\infty \to p_2 \mathcal{O}_\infty p_2$, that is, $\tau = V(\psi)$. Thus, $\beta = \alpha\tau \colon (V(\mathcal{O}_\infty), [1]) \to (V(B), [q])$ is a normalized monoid morphism. By Lemma 7.4, there exists a C*-algebra homomorphism $\varphi \colon \mathcal{O}_\infty \to B$ sending 1 to q, such that $V(\varphi) = \beta = \alpha\tau$. Since ψ is an isomorphism and $V(-)$ is a functor, we get $\tau^{-1} = V(\psi^{-1})$, and thus $\alpha = V(\varphi\psi^{-1})$, where $\varphi\psi^{-1} \colon p_2 \mathcal{O}_\infty p_2 \to B$ is a C*-algebra homomorphism that sends p_2 to q. Thus, $\phi = \varphi\psi^{-1}$ is the desired morphism.   □

   In the case of the corner $p_1 \mathcal{O}_\infty p_1$, as $[p_1]$ is idempotent, we need to restrict the target algebras in order to preserve the 'lifting' property.

LEMMA 8.2. *Let B be a finite direct sum of full matrix algebras over Cuntz algebras $\mathcal{O}_n$ (for $2 \le n < \infty$) and $p\mathcal{O}_\infty p$ (for any nonzero projection $p \in \mathcal{O}_\infty$), and let $q \in B$ be a projection. Then any normalized monoid homomorphism*

$$\alpha \colon (V(p_1 \mathcal{O}_\infty p_1), [p_1]) \to (V(B), [q]),$$

is induced by a C*-algebra homomorphism $\phi \colon p_1 \mathcal{O}_\infty p_1 \to B$ that sends p_1 to q. That is, $V(\phi) = \alpha$.

   Proof. By Remark 7.3, we can write $B = \bigoplus_{l=1}^{r+s} B_l$, where $B_j = \mathrm{M}_{k_j}(\mathcal{O}_{n_j})$ for $1 \le j \le r$ (with $k_j, n_j \in \mathbb{N}$) and $B_{r+i} = p_{l_i} \mathcal{O}_\infty p_{l_i}$ for $1 \le i \le s$ (with $l_i \in \mathbb{Z}^+$). There exist pairwise orthogonal projections $q_i \in B_i$ such that $q = \sum_{i=1}^{r+s} q_i$. Since the functor $V(-)$ preserves finite direct sums, we can reduce the problem to the case where B is either $\mathrm{M}_{k_j}(\mathcal{O}_{n_j})$ or $p_{l_i} \mathcal{O}_\infty p_{l_i}$ with $l_i \in \mathbb{Z}^+$, by composing with the canonical projections $\pi_i \colon B \twoheadrightarrow B_i$. We get $\alpha_i = V(\pi_i)\alpha \colon V(p_1 \mathcal{O}_\infty p_1) \to V(B_i)$ with $\alpha_i([p_1]) = [q_i]$. If $q_i = 0$, then α_i is induced by the zero homomorphism $\phi_i \colon p_1 \mathcal{O}_\infty p_1 \to B_i$; so suppose that

$q_i \neq 0$. Since $[p_1]$ is an order-unit of $V(p_1 \mathcal{O}_\infty p_1)$, the image of α_i is contained in the order-ideal of $V(B_i)$ generated by $[q_i]$, whence α_i restricts to a normalized monoid morphism

$$\alpha_i : (V(p_1 \mathcal{O}_\infty p_1), [p_1]) \to (V(q_i B_i q_i), [q_i]).$$

Since B_i is a purely infinite simple C*-algebra, so is $q_i B_i q_i$.

Since $V(p_1 \mathcal{O}_\infty p_1) \setminus \{0\}$ is a group containing $[p_1]$, $[q_i] \neq 0$ in $V(q_i B_i q_i)$, and $V(q_i B_i q_i)$ is a conical monoid, it follows from [3, Corollary 2.2] that

$$\beta_i = \alpha_i \restriction_{V(p_1 \mathcal{O}_\infty p_1) \setminus \{0\}} : K_0(p_1 \mathcal{O}_\infty p_1) \to K_0(q_i B_i q_i),$$

is a group homomorphism such that $\beta_i([p_1]) = [q_i]$. By Remark 7.2, for each $i \geq 1$, $q_i B_i q_i$ is isomorphic to either $M_k(\mathcal{O}_n)$ (for some $k \geq 1$) or $p \mathcal{O}_\infty p$ (for some projection $p \in \mathcal{O}_\infty$). Thus, by [15, Lemma 3.7], there exists a unital C*-algebra homomorphism $\phi_i : p_1 \mathcal{O}_\infty p_1 \to q_i B_i q_i$ such that $K_0(\phi_i) = \beta_i$, and thus $V(\phi_i) = \alpha_i$. The map

$$\phi = \bigoplus_{i=1}^{r+s} \phi_i : p_1 \mathcal{O}_\infty p_1 \to q B q \subseteq B,$$

satisfies the desired properties. $\square$

Thus, we get the following version of Lemma 7.5.

LEMMA 8.3. *Let A, B be finite direct sums of full matrix algebras over either Cuntz algebras $\mathcal{O}_n$ $(2 \leq n < \infty)$ or $p \mathcal{O}_\infty p$ (for projections $p \in \mathcal{O}_\infty$). Then any normalized monoid homomorphism*

$$\alpha : (V(A), [1_A]) \to (V(B), [1_B]),$$

is induced by a C-algebra homomorphism $\phi : A \to B$ that sends 1_A to 1_B. That is, $V(\phi) = \alpha$.*

Outline of proof. By arguing as in the proof of Lemma 7.5 and using Remark 7.3 together with (7.3), we reduce the problem to the case where A is either $\mathcal{O}_\infty$, or $p_1 \mathcal{O}_\infty p_1$, or $p_2 \mathcal{O}_\infty p_2$. The first case is covered by Lemma 7.4, the second case by Lemma 8.2, and the third case by Lemma 8.1. $\square$

THEOREM 8.4. *Let (M, u) be an abelian monoid with order-unit. Then $(M, u) \cong (V(A), [1_A])$ for some unital extended Cuntz limit A if and only if*

(a) *M is a countable, regular conical refinement monoid.*
(b) *For all idempotents $e \leq f$ in M, the homomorphism $G_M[e] \to G_M[f]$ given by $x \mapsto x + f$ is injective, and $G_M[e] + f$ is a pure subgroup of $G_M[f]$.*

Proof. ($\Longrightarrow$): Theorem 7.6.

($\Longleftarrow$): Corollary 6.7 implies that (M, u) is the direct limit of a sequence of the form

$$(M_1, u_1) \xrightarrow{\alpha_1} (M_2, u_2) \xrightarrow{\alpha_2} (M_3, u_3) \xrightarrow{\alpha_3} \cdots$$

where each (M_i, u_i) is a finite direct sum of pairs $((\mathbb{Z}/n_{ij}\mathbb{Z}) \sqcup \{0\}, \overline{m}_{ij})$ and $(\mathbb{Z} \sqcup \{0\}, k_i)$ for some $n_{ij}, m_{ij} \in \mathbb{N}$, $k_i \in \mathbb{Z}$. In view of (7.1)–(7.3), there exist isomorphisms $h_i : (V(A_i), [1_{A_i}]) \to (M_i, u_i)$ where A_i is a direct sum of matrix algebras of the form either $M_{m_{ij}}(\mathcal{O}_{n_{ij}+1})$ or $p_{k_i} \mathcal{O}_\infty p_{k_i}$, with p_{k_i} being suitable projections. By Lemma 8.3, each of the normalized homomorphisms

$$h_{i+1}^{-1} \alpha_i h_i : (V(A_i), [1_{A_i}]) \longrightarrow (V(A_{i+1}), [1_{A_{i+1}}]),$$

is induced by a unital C*-algebra homomorphism $\phi_i : A_i \to A_{i+1}$. Therefore $(M, u) \cong (V(A), [1_A])$ where A is the C*-inductive limit of the sequence

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots \qquad \qquad \square$$

COROLLARY 8.5. *Let (M, u) be an abelian monoid with order-unit. Then $(M, u) \cong (V(A), [1_A])$ for some unital extended Cuntz limit A if and only if*

$$(M, u) \cong \left(\bigsqcup_{e \in \Lambda} (\{e\} \times G_e), (1, u_1) \right) \subseteq (\Lambda \times G_1, (1, u_1)),$$

where

(a) Λ *is a countable distributive semilattice with maximum element* 1.
(b) G_1 *is a countable abelian group.*
(c) G_e *is a pure subgroup of G_1 for all $e \in \Lambda$, and $G_0 = \{0\}$.*
(d) $G_e + G_f = G_{e+f}$ *and* $G_e \cap G_f = \bigcup_{g \in \Lambda, g \leq e, f} G_g$ *for all e, $f \in \Lambda$.*
(e) $u_1 \in G_1$.

Proof. ($\Longrightarrow$): By Corollary 7.7, M is isomorphic to a monoid of the form

$$M' = \bigsqcup_{e \in \Lambda} (\{e\} \times G_e) \subseteq \Lambda \times G,$$

for some countable distributive semilattice Λ and some countable abelian group G with subgroups G_e satisfying the conditions of that corollary. As M has an order-unit, Λ has a largest element. Conditions (a)–(e) are now all satisfied.

($\Longleftarrow$): With the help of [11, Theorem 3.3], it is clear that M satisfies conditions (a) and (b) of Theorem 8.4. $\qquad \square$

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