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Genus-zero Whitham hierarchies in conformal-map dynamics ☆

Luis Martínez-Alonso [a,*], Elena Medina [b]

[a] Departamento de Física Teórica II, Universidad Complutense, E-28040 Madrid, Spain
[b] Departamento de Matemáticas, Universidad de Cádiz, E-11510 Puerto Real, Cádiz, Spain

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Abstract

A scheme for solving quasiclassical string equations is developed to prove that genus-zero Whitham hierarchies describe the deformations of planar domains determined by rational conformal maps. This property is applied in normal matrix models to show that deformations of simply-connected supports of eigenvalues under changes of coupling constants are governed by genus-zero Whitham hierarchies.

MSC: 58B20

Keywords: Whitham hierarchy; Conformal maps; Normal matrix models

1. Introduction

Conformal mapping methods have been effectively applied in the analysis of interfacial free-boundary problems involving planar domains [1]. They have provided many exact solutions [2–4] which stimulated the research on possible underlying integrable structures. Thus, Wiegmann and Zabrodin discovered [5,6] that deformations of simply-connected domains with respect to changes of their exterior harmonic moments, treated as independent variables, are described by the dispersionless Toda hierarchy. They also formulated an algebro-geometric analysis [7,8] of the deformations of multiply-connected domains in terms of Whitham equations for Abelian differentials.

Recent research has shown [9,10] that many exact solutions of Laplacian growth models correspond to a special type of domains called *algebraic* or *quadrature domains*. In the simply-connected case the complement of a quadrature domain D is the image of the exterior of the unit disk under a conformal map given by a rational function

$$z(w) = r\,w + \sum_{n=0}^{N_0} \frac{u_{0,n}}{w^n} + \sum_{s=1}^{k} \sum_{n=1}^{N_s} \frac{u_{s,n}}{(w-a_s)^n}, \tag{1}$$

where the coefficient r is a positive number and the k poles $a_s \neq 0$ lie inside the unit circle. In this work we prove that the deformations of rational conformal maps under changes of the parameters $(r, u_{0,n}, u_{s,m}, a_s)$ such that $z(\bar{a}_s^{-1})$ are kept constant, turn out to be described by a solution of the genus-zero Whitham hierarchy $\mathrm{W}(n)$ with $n = 2k+2$ punctures [11]. It should be noted that according to a recent general result by Takasaki [12] the Whitham hierarchy $\mathrm{W}(n)$ is the quasiclassical limit of the n-component KP hierarchy.

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* Corresponding author.
E-mail address: luism@fis.ucm.es (L. Martínez-Alonso).

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Our analysis is based on solving a system of quasiclassical string equations which leads to the characterization of the conformal map (1) as a function $z(w, \boldsymbol{t})$ of the Whitham times $\boldsymbol{t}$. For $k = 0$ our result agrees with [5] (see also [13]) since W(2) is the dispersionless Toda hierarchy. However, for $k \geqslant 1$ the analysis of [5] does not apply because the Schwarz function of the boundary of D has poles outside D, consequently there are infinite exterior harmonic moments of D different from zero and, since $z(w)$ depends on a finite number of parameters only, these harmonic moments are not independent variables.

Deformations of quadrature domains naturally arise in the analysis of partition functions of $N \times N$ normal matrix models [14]

$$Z_N = \int e^{\frac{1}{\hbar} W(M, M^\dagger)}\, \mathrm{d}M\, \mathrm{d}M^\dagger, \tag{2}$$

with *quasiharmonic* potentials $W(z, \bar{z}) := -z\bar{z} + V(z) + \overline{V(z)}$ of the form

$$V(z) = \sum_{n=1}^{N_0+1} z^n t_{0,n} + \sum_{s=1}^{k} \left(-t_{s,0} \log(z - \beta_s) + \sum_{n=1}^{N_s - 1} \frac{t_{s,n}}{(z - \beta_s)^n} \right). \tag{3}$$

Integrating over eigenvalues and ignoring normalization factors, the partition function reduces to

$$Z_N = \int \prod_{i>j} |z_i - z_j|^2 e^{\frac{1}{\hbar} \sum_j W(z_j, \bar{z}_j)} \prod_j \mathrm{d}^2 z_j. \tag{4}$$

In the large N limit ($N \to \infty, \hbar N$ fixed) the eigenvalues densely occupy a bounded quadrature domain D in the complex plane (*the support of eigenvalues*). As a consequence of our analysis we prove that for simply-connected supports of eigenvalues the corresponding rational conformal map $z = z(w, \boldsymbol{t})$ as a function of the coupling constants $\boldsymbol{t}$ of the partition function represents a solution of the Whitham hierarchy W$(2k + 2)$.

2. String equations in Whitham hierarchies

The elements of the phase space for a genus-zero Whitham hierarchy W$(M + 1)$ are characterized by $M + 1$ *punctures* q_α ($\alpha = 0, \ldots, M$), where $q_0 := \infty$, of the extended complex p-plane and an associated set of local coordinates of the form

$$z_0 = p + \sum_{n=1}^{\infty} \frac{c_{0,n}}{p^n}, \qquad z_i = \frac{d_i}{p - q_i} + \sum_{n=0}^{\infty} d_{i,n} (p - q_i)^n, \quad i = 1, \ldots, M. \tag{5}$$

In what follows Greek and Latin suffixes will label indices of the sets $\{0, \ldots, M\}$ and $\{1, \ldots, M\}$, respectively. We will henceforth suppose that there exist positively oriented closed curves Γ_μ in the complex planes of the variables z_μ such that each function $z_\mu(p)$ determines a conformal map of the right-exterior of a circle $\gamma_\mu := z_\mu^{-1}(\Gamma_\mu)$ on the exterior of Γ_μ (we will assume that the circle γ_0 encircles all the γ_i) (see Fig. 1).

The flows of the Whitham hierarchy can be formulated as the following infinite system of quasiclassical Lax equations

$$\frac{\partial z_\alpha}{\partial t_{\mu,n}} = \{\Omega_{\mu n}, z_\alpha\}, \tag{6}$$

associated to the series of time parameters $\{t_{0,n}\colon n \geqslant 1;\ t_{i,n}\colon i = 1, \ldots, M, n \geqslant 0\}$. Here the Poisson bracket is defined as $\{F, G\} := \partial_p F \partial_x G - \partial_x F \partial_p G$, $x := t_{01}$ and the Hamiltonian functions are

$$\Omega_{\mu n} := \left(z_\mu^n\right)_{(\mu,+)}, \quad (n \geqslant 1), \qquad \Omega_{i0} := -\log(p - q_i), \tag{7}$$

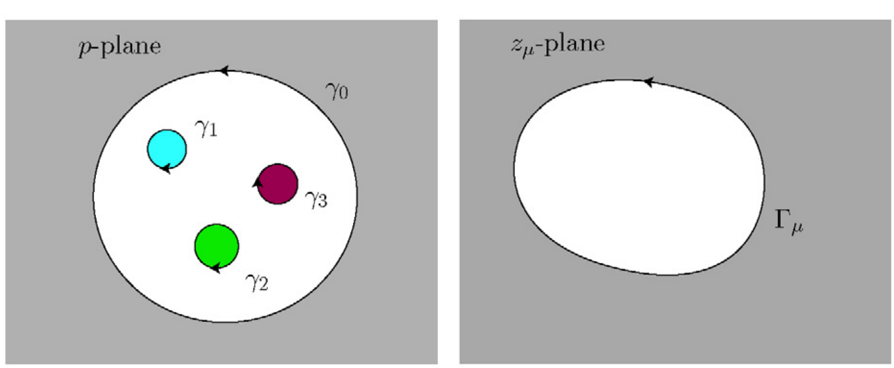

Fig. 1. Right-exteriors of γ_μ and Γ_μ.

where $(\cdot)_{(i,+)}$ and $(\cdot)_{(0,+)}$ stand for the projectors on the subspaces generated by $\{(p-q_i)^{-n}\}_{n=1}^{\infty}$ and $\{p^n\}_{n=0}^{\infty}$ in the corresponding spaces of Laurent series. These hierarchies [12] are the dispersionless limits of the multi-component KP hierarchies. In particular, for $M=0$ and $M=1$ they represent the dispersionless versions of the KP and Toda hierarchies, respectively.

In our analysis we will use an extended Lax formalism with Orlov functions

$$m_\alpha(z_\alpha,\boldsymbol{t})=\sum_{n=1}^{\infty} n t_{\alpha,n} z_\alpha^{n-1}+\frac{t_{\alpha,0}}{z_\alpha}+\sum_{n\geqslant 2}\frac{v_{\alpha n}}{z_\alpha^n}, \tag{8}$$

which verify the same Lax equations (6) as the variables z_α, and such that $\{z_\alpha, m_\alpha\}=1$. The parameter $t_{0,0}$ in (8) is defined by

$$t_{0,0}:=-\sum_{i=1}^{M} t_{i,0}. \tag{9}$$

The Whitham hierarchy can be formulated as the following system of equations

$$\mathrm{d}z_\alpha\wedge\mathrm{d}m_\alpha=\mathrm{d}\omega,\quad \forall\alpha, \tag{10}$$

where ω is the one-form

$$\omega:=\sum_{\mu,n}\Omega_{\mu n}\,\mathrm{d}t_{\mu,n}. \tag{11}$$

To see how to get from the system (10) to the Whitham hierarchy, note that by identifying the coefficients of $\mathrm{d}p\wedge\mathrm{d}t_{\mu n}$ and $\mathrm{d}x\wedge\mathrm{d}t_{\mu n}$ in (10) we obtain

$$\frac{\partial z_\alpha}{\partial p}\frac{\partial m_\alpha}{\partial t_{\mu n}}-\frac{\partial m_\alpha}{\partial p}\frac{\partial z_\alpha}{\partial t_{\mu n}}=\frac{\partial \Omega_{\mu n}}{\partial p},\qquad \frac{\partial z_\alpha}{\partial x}\frac{\partial m_\alpha}{\partial t_{\mu n}}-\frac{\partial m_\alpha}{\partial x}\frac{\partial z_\alpha}{\partial t_{\mu n}}=\frac{\partial \Omega_{\mu n}}{\partial x} \tag{12}$$

and, in particular, since $\Omega_{01}=p$, for $(\mu,n)=(0,1)$, the system (12) implies $\{z_\alpha,m_\alpha\}=1$. Thus, using this fact and solving (12) for $\partial_{t_{\mu n}}z_\alpha$ and $\partial_{t_{\mu n}}m_\alpha$, the Lax equations for (z_α,m_α) follow.

A natural form of characterizing solutions of Whitham hierarchies is provided by systems of string equations

$$\begin{cases} P_i(z_i,m_i)=P_0(z_0,m_0),\\ Q_i(z_i,m_i)=Q_0(z_0,m_0),\end{cases}\quad i=1,2,\dots,M, \tag{13}$$

where $\{P_\alpha,Q_\alpha\}_{\alpha=0}^{M}$ satisfy $\{P_\alpha(p,x),Q_\alpha(p,x)\}=1$. Given a solution $(z_\alpha(p,\boldsymbol{t}),m_\alpha(p,\boldsymbol{t}))$ of a system (13), if we denote

$$\mathcal{P}_\alpha(p,\boldsymbol{t}):=P_\alpha\big(z_\alpha(p,\boldsymbol{t}),m_\alpha(p,\boldsymbol{t})\big),\qquad \mathcal{Q}_\alpha(p,\boldsymbol{t}):=Q_\alpha\big(z_\alpha(p,\boldsymbol{t}),m_\alpha(p,\boldsymbol{t})\big),$$

it is clear that

$$\mathrm{d}\mathcal{P}_\alpha\wedge\mathrm{d}\mathcal{Q}_\alpha=\mathrm{d}\mathcal{P}_\beta\wedge\mathrm{d}\mathcal{Q}_\beta,\quad\forall\alpha,\beta. \tag{14}$$

On the other hand $\{P_\alpha(p,x),Q_\alpha(p,x)\}=\{z_\alpha,m_\alpha\}=1$, so that solutions of a system of string equations verify

$$\mathrm{d}\mathcal{P}_\alpha\wedge\mathrm{d}\mathcal{Q}_\alpha=\mathrm{d}z_\beta\wedge\mathrm{d}m_\beta,\quad\forall\alpha,\beta. \tag{15}$$

Theorem. *Let $(z_\alpha(p,\boldsymbol{t}),m_\alpha(p,\boldsymbol{t}))$ be a solution of (13) which admits expansions of the form (5), (8), (9) and such that the coefficients of the two-forms (15) are meromorphic functions of the complex variable p with finite poles at $\{q_1,\dots,q_M\}$ only. Then $(z_\alpha(p,\boldsymbol{t}),m_\alpha(p,\boldsymbol{t}))$ is a solution of the Whitham hierarchy.*

Proof. In view of the hypothesis of the theorem the coefficients of the two-forms (15) with respect to the basis

$$\{\mathrm{d}p\wedge\mathrm{d}t_{\alpha,n},\ \mathrm{d}t_{\alpha,n}\wedge\mathrm{d}t_{\beta,m}\}$$

are determined by their principal parts at q_μ $(\mu=0,\dots,M)$, so that by taking (15) into account we may write

$$\mathrm{d}z_\alpha\wedge\mathrm{d}m_\alpha=\sum_{\mu=0}^{M}(\mathrm{d}z_\mu\wedge\mathrm{d}m_\mu)_{(\mu,+)},\quad\forall\alpha.$$

Moreover the terms in these decompositions can be found by using the expansions (8) of the functions m_μ as follows

$$\mathrm{d}z_\mu\wedge\mathrm{d}m_\mu=\mathrm{d}z_\mu\wedge\left(\sum_{n=1}^{\infty}nz_\mu^{n-1}\,\mathrm{d}t_{\mu,n}+\frac{\mathrm{d}t_{\mu,0}}{z_\mu}+\sum_{n\geqslant2}\frac{\mathrm{d}v_{\mu n}}{z_\mu^n}\right)=\mathrm{d}\left(\sum_{n=1}^{\infty}z_\mu^n\,\mathrm{d}t_{\mu,n}+\log z_\mu\,\mathrm{d}t_{\mu,0}-\sum_{n\geqslant2}\frac{1}{n-1}\frac{\mathrm{d}v_{\mu n}}{z_\mu^{n-1}}\right),$$

so that

$$(\mathrm{d}z_\mu \wedge \mathrm{d}m_\mu)_{(\mu,+)} = \mathrm{d}\left(\sum_{n=1}^{\infty}\big(z_\mu^n\big)_{(\mu,+)}\,\mathrm{d}t_{\mu,n} - (1-\delta_{\mu 0})\log(p-q_\mu)\,\mathrm{d}t_{\mu,0}\right) = \mathrm{d}\left(\sum_n \Omega_{\mu n}\,\mathrm{d}t_{\mu,n}\right).$$

Thus we find

$$\mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha = \mathrm{d}\omega = \mathrm{d}\left(\sum_{\mu,n}\Omega_{\mu n}\,\mathrm{d}t_{\mu,n}\right), \quad \forall\alpha$$

and, consequently, this proves that the functions $(z_\alpha(p,\boldsymbol{t}), m_\alpha(p,\boldsymbol{t}))$ determine a solution of the Whitham hierarchy. □

3. Integrable dynamics of quadrature domains

Let us consider a rational conformal map $z(w)$ of the form (1) with k poles a_s inside the unit circle and define

$$\tilde z(w) := \overline{z\big(\bar w^{-1}\big)} = \frac{r}{w} + \sum_{n=0}^{N_0}\bar u_{0,n}w^n + \sum_{s=1}^{k}\sum_{n=1}^{N_s}\frac{\bar u_{s,n}\,w^n}{(1-w\bar a_s)^n}. \tag{16}$$

In order to establish the connection between the deformations of the conformal map $z(w)$ and the genus-zero Whitham hierarchies we introduce the change of variable

$$p = Rw := rw + u_{0,0},$$

where r and $u_{0,0}$ are the first coefficients of $z(w)$ in (1). As a function of the new variable p the conformal map is normalized at infinity

$$z(p) = p + \mathcal{O}(1/p), \quad p\to\infty.$$

Moreover, if we define

$$q_0 := \infty, \qquad q_s := Rb_s, \qquad q_{s+k} := Ra_s, \qquad q_{2k+1} := u_{0,0},$$

it is clear that $z(p)$ and $\tilde z(p)$ become rational functions of p with poles at (q_0, q_s, q_{2k+1}) and (q_0, q_{s+k}, q_{2k+1}) respectively.

We are going to prove that deformations of $z(w)$ with respect to the coefficients

$$\boldsymbol{u} := (r, u_{0,n}, u_{s,m}, a_s),$$

such that $\beta_s = z(w)|_{b_s}$ are kept constant, are described by the Whitham hierarchy $\mathrm{W}(2k+2)$. To this end we introduce Whitham variables (z_α, m_α) on the $2k+2$ punctures q_α of the p-plane

$$\begin{cases} z_0 = z, \quad m_0 = \tilde z \quad (\text{near } q_0=\infty),\\ z_s = \frac{1}{z-\beta_s}, \quad m_s = -(z-\beta_s)^2\tilde z \quad (\text{near } q_s),\\ z_{s+k} = \frac{1}{\tilde z-\bar\beta_s}, \quad m_{s+k} = (\tilde z-\bar\beta_s)^2 z \quad (\text{near } q_{s+k}),\\ z_{2k+1} = \tilde z, \quad m_{2k+1} = -z \quad (\text{near } q_{2k+1}). \end{cases} \tag{17}$$

It is clear that these variables are rational functions of p with possible poles at the punctures q_α only. Moreover, they satisfy the system of string equations

$$\begin{cases} \frac{1}{z_s} + \beta_s = m_{s+k}z_{s+k}^2 = -m_{2k+1} = z_0,\\ -m_s z_s^2 = \frac{1}{z_{s+k}} + \bar\beta_s = z_{2k+1} = m_0. \end{cases} \tag{18}$$

Obviously the functions z_α are of the form (5). On the other hand due to (1) and (16) it follows that the functions m_α defined in (17) verify expansions of the form (8)

$$m_0 = \tilde z = \sum_{n=1}^{N_0+1} n t_{0,n} z_0^{n-1} + \frac{t_{0,0}}{z_0} + \cdots, \quad w\to\infty,$$

$$m_s = -z_s^{-2}\tilde z = \sum_{n=1}^{N_s-1} n t_{s,n} z_s^{n-1} + \frac{t_{s,0}}{z_s} + \cdots, \quad w\to b_s,$$

$$m_{s+k} = z_{s+k}^{-2} z = \sum_{n=1}^{N_s-1} n t_{s+k,n} z_{s+k}^{n-1} + \frac{t_{s+k,0}}{z_{s+k}} + \cdots, \quad w\to a_s,$$

$$m_{2k+1} = -z = \sum_{n=1}^{N_0+1} n t_{2k+1,n} z_{2k+1}^{n-1} + \frac{t_{2k+1,0}}{z_{2k+1}} + \cdots, \quad w \to 0, \tag{19}$$

where the time parameters $\boldsymbol{t} := (t_{\alpha,n})$ are rational functions in $\boldsymbol{u}$

$$t_{\alpha n} = Q_{\alpha,n}(\boldsymbol{u}). \tag{20}$$

The functions $Q_{\alpha,n}(\boldsymbol{u})$ satisfy certain constraints which can be characterized by considering the map

$$C : f \mapsto Cf, \qquad Cf(w) := \overline{f(\bar{w}^{-1})}. \tag{21}$$

Observe that in terms of the variable p

$$Cf(p) = \overline{f(\mathcal{I}p)}, \tag{22}$$

where $\mathcal{I}p = r^2/(\overline{p - u_{0,0}}) + u_{0,0}$ is the inversion with respect to the circle $|p - u_{0,0}|^2 = r^2$. From (17) it is clear that

$$z_{2k+1} = Cz_0, \qquad m_{2k+1} = -Cm_0, \qquad z_{s+k} = Cz_s, \qquad m_{s+k} = -Cm_s, \tag{23}$$

which implies

$$Q_{2k+1,n}(\boldsymbol{u}) = -\overline{Q_{0,n}(\boldsymbol{u})}, \qquad Q_{s+k,n} = -\overline{Q_{s,n}(\boldsymbol{u})}. \tag{24}$$

Furthermore, we can prove that

$$\sum_{\alpha} Q_{\alpha,0}(\boldsymbol{u}) = 0. \tag{25}$$

Indeed from (17) we deduce

$$m_0 \, \mathrm{d}z_0 = m_s \, \mathrm{d}z_s = \tilde{z} \, \mathrm{d}z, \qquad m_{2k+1} \, \mathrm{d}z_{2k+1} = m_{s+k} \, \mathrm{d}z_{s+k} = -z \, \mathrm{d}\tilde{z}.$$

Hence

$$2\pi i \sum_{\alpha} Q_{\alpha,0} = \sum_{\alpha} \oint_{\Gamma_\alpha} m_\alpha \, \mathrm{d}z_\alpha = \sum_{\alpha} \oint_{\gamma_\alpha} \tilde{z} \, \partial_p z \, \mathrm{d}p = 0,$$

where we have taken into account that $\tilde{z}\,\partial_p z$ is a rational function of p with poles at the punctures q_α only, and the fact that

$$\sum_{\alpha} \gamma_\alpha \sim 0 \quad \text{in} \quad \mathbb{C} \setminus \{q_1, \ldots, q_{2k+1}\}.$$

Notice that due to (24) the constraint (25) can be rewritten as

$$\operatorname{Im} Q_{0,0} + \sum_{s} \operatorname{Im} Q_{s,0}(\boldsymbol{u}) = 0. \tag{26}$$

Under appropriate conditions one can determine $\boldsymbol{u}$ as a function of $(\boldsymbol{t}, \beta_s, \bar{\beta}_s)$. To this end we consider the system

$$\begin{cases} t_{\alpha,n} = Q_{\alpha,n}(\boldsymbol{u}), \\ \beta_s = z(b_s, \boldsymbol{u}), \end{cases} \tag{27}$$

where the time parameters

$$\boldsymbol{t} := (t_{\alpha,n}), \quad \alpha = 0, \ldots, 2k+1; \; n = 0, \ldots, \tilde{N}_\alpha; \qquad \tilde{N}_0 = \tilde{N}_{2k+1} = N_0 + 1, \qquad \tilde{N}_s = \tilde{N}_{s+k} = N_s - 1, \tag{28}$$

are assumed to satisfy

$$t_{2k+1,n} = -\bar{t}_{0,n}, \qquad t_{s+k,n} = -\bar{t}_{s,n}, \qquad \sum_{\alpha} t_{\alpha,0} = 0. \tag{29}$$

Firstly, we observe that $\boldsymbol{u}$ constitutes a set of

$$2\left(N_0 + \sum_{s} N_s + k\right) + 3, \tag{30}$$

real variables given by r and the real and imaginary parts of $(u_{0,n}, u_{s,m}, a_i)$. On the other hand, in view of (24) and (25) we may ignore the equations corresponding to $\alpha = s+k, 2k+1$ in (27). In this way the system (27) reduces to $2(N_0 + 2 + \sum_s N_s + k)$ real equations, but due to (26) one of them is a consequence of the others. Therefore, we are lead to a system of equal number of equations and unknowns which under appropriate conditions will determine $\boldsymbol{u}$, and consequently (z_α, m_α) as functions of $(\boldsymbol{t}, \beta_s)$.

In this way we have determined a rational solution $(z_\alpha(p), m_\alpha(p))$ of the system of string equations (18) which depends on $(\boldsymbol{t}, \beta_1, \dots, \beta_k)$ and satisfies the asymptotic conditions (5) and (8). Therefore, from the above theorem we conclude that this solution evolves with respect to $\boldsymbol{t}$ according to the Whitham hierarchy $\mathrm{W}(2k+2)$.

It is interesting to notice the following identity involving the times $t_{\alpha 0}$ and the area T of the domain D. Let Γ be the positively oriented boundary of D and take small positively oriented closed curves Γ_s' around the points β_s of the z-plane, then we have that

$$t_{0,0} = \frac{1}{2\pi i}\oint_{\Gamma_0} \tilde{z}\,\mathrm{d}z = \sum_s \frac{1}{2\pi i}\oint_{\Gamma_s'} \tilde{z}\,\mathrm{d}z + \frac{1}{2\pi i}\oint_{\Gamma} \tilde{z}\,\mathrm{d}z = \sum_s \frac{1}{2\pi i}\oint_{\Gamma_s} z_s^{-2}\tilde{z}\,\mathrm{d}z_s + \frac{1}{2\pi i}\oint_{\Gamma} \bar{z}\,\mathrm{d}z = -\sum_s t_{s,0} + \frac{1}{\pi}T. \tag{31}$$

4. Exact solutions

The above analysis can be used to generate solutions of $\mathrm{W}(2k+2)$ of the form (1). Let us consider the case in which only simple poles arise

$$z = rw + u_0 + \sum_{s=1}^{k} \frac{v_s}{w - a_s}, \qquad \tilde{z} = \frac{r}{w} + \bar{u}_0 + \sum_{s=1}^{k} \frac{\bar{v}_s w}{1 - \bar{a}_s w}.$$

By identifying the coefficients of z_0^0 and z_0^{-1} in the expansion of $m_0 = \tilde{z}$ as $z_0 = z \to \infty$ $(w \to \infty)$, we get

$$\bar{u}_0 - \sum_{s=1}^{k} \frac{\bar{v}_s}{\bar{a}_s} = t_{0,1}, \qquad r\left(r - \sum_{s=1}^{k} \frac{\bar{v}_s}{\bar{a}_s^2}\right) = t_{0,0}, \tag{32}$$

and identifying the coefficient of z_s^{-1} in the expansion of $m_s = -(z-\beta_s)^2\tilde{z}$ as $z_s \to \infty$ $(w \to b_s)$ yields

$$\frac{\bar{v}_s}{\bar{a}_s^2}\left(r - \sum_{s'=1}^{k} \frac{v_{s'}\bar{a}_s^2}{(1 - a_{s'}\bar{a}_s)^2}\right) = t_{s,0}. \tag{33}$$

Finally, the equations $z(w)|_{w=b_s} = \beta_s$ read

$$\frac{r}{\bar{a}_s} + u_0 + \sum_{s'=1}^{k} \frac{v_{s'}\bar{a}_s}{1 - \bar{a}_s a_{s'}} = \beta_s. \tag{34}$$

The system (32)–(34) determines (r, u_0, v_s, a_s) in terms of $(t_{0,0}, t_{0,1}, t_{s,0}, \beta_s)$.

Examples

A solution of $\mathrm{W}(4)$ is obtained by deforming the conformal map (aircraft wind)

$$z = rw + u_0 + \frac{v}{w - a}, \qquad \tilde{z} = \frac{r}{w} + \bar{u}_0 + \frac{\bar{v}w}{1 - \bar{a}w}.$$

The solution of the corresponding equations (32)–(34) is given by:

$$a = \frac{r^2 - At_{0,0}}{r(\bar{\beta} - t_{0,1})(1 - A)}, \qquad v = \frac{(r^2 - \bar{t}_{0,0})(r^2 - t_{0,0}A)^2}{r^3(\bar{\beta} - t_{0,1})^2(1 - A)^2},$$
$$u_0 = \frac{r^4 - r^2 t_{0,0}A + r^2\bar{t}_{0,1}(\bar{\beta} - t_{0,1})(1 - A) + |t_{0,0}|^2 A - r^2\bar{t}_{0,0}}{r^2(\bar{\beta} - t_{0,1})(1 - A)},$$

with

$$r^2 = \frac{2|t_{0,0}|^2 A^2}{|\bar{\beta} - t_{0,1}|^2(2A^3 - 3A^2 + A) + (t_{0,0} + t_{1,0})(A - 1) + (t_{0,0} + \bar{t}_{0,0})A},$$

where $A = |a|^2$ is implicitly defined by

$$\begin{aligned}
&2|\bar{\beta} - t_{0,1}|^4 A^5 + \big[2(t_{0,0} + t_{1,0}) - 2(t_{0,0} + \bar{t}_{0,0}) - 5|\bar{\beta} - t_{0,1}|^2\big]|\bar{\beta} - t_{0,1}|^2 A^4 \\
&\quad + 4\big[(t_{0,0} + \bar{t}_{0,0}) - (t_{0,0} + t_{1,0}) + |\bar{\beta} - t_{0,1}|^2\big]|\bar{\beta} - t_{0,1}|^2 A^3 \\
&\quad + \big[(t_{0,0} + t_{1,0})^2 - (t_{0,0} + \bar{t}_{0,0})^2 + 4|t_{0,0}|^2 - 2(t_{0,0} + \bar{t}_{0,0})|\bar{\beta} - t_{0,1}|^2 + 2(t_{0,0} + t_{1,0})|\bar{\beta} - t_{0,1}|^2 - |\bar{\beta} - t_{0,1}|^4\big]A^2 \\
&\quad - 2(t_{0,0} + t_{1,0})^2 A + (t_{0,0} + t_{1,0})^2 = 0.
\end{aligned}$$

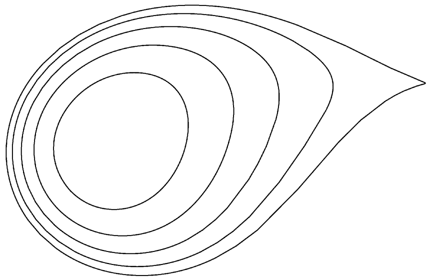

Fig. 2. Solution corresponding to $k=1$.

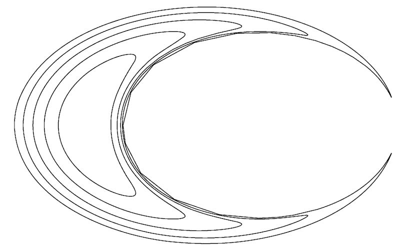

Fig. 3. Solution corresponding to $k=2$.

It can be proved that a reduction of the system (32)–(34) is obtained by setting

$$z=rw+u+\sum_{s=1}^{l}\left(\frac{v_s}{w-a_s}+\frac{\bar{v}_s}{w-\bar{a}_s}\right),\qquad \tilde{z}=\frac{r}{w}+u+\sum_{s=1}^{l}\left(\frac{\bar{v}_s w}{1-\bar{a}_s w}+\frac{v_s w}{1-a_s w}\right),\tag{35}$$

and by assuming

$$u, t_{0,0}, t_{0,1}\in\mathbb{R},\quad t_{s+l,0}=\bar{t}_{s,0},\quad \beta_{s+l}=\bar{\beta}_s.$$

In the simplest case $l=1$ the reduced system of implicit equations reads

$$u-\frac{v_1}{a_1}-\frac{\bar{v}_1}{\bar{a}_1}=t_{0,1},\qquad r\left(r-\frac{v_1}{{a_1}^2}-\frac{\bar{v}_1}{\bar{a}_1^2}\right)=t_{0,0},\qquad \bar{v}_1\left(\frac{r}{\bar{a}_1^2}-\frac{\bar{v}_1}{(1-\bar{a}_1^2)^2}-\frac{v_1}{(1-a_1\bar{a}_1)^2}\right)=t_{1,0},$$
$$\frac{r}{\bar{a}_1}+u+\frac{v_1\bar{a}_1}{1-a_1\bar{a}_1}+\frac{\bar{v}_1\bar{a}_1}{1-\bar{a}_1^2}=\beta_1,\tag{36}$$

and determines a solution of W(6).

Figs. 2 and 3 exhibit deformations of domains with respect to changes of the area T such that the Whitham times $t_{s,0}$ are kept constant. Observe that the evolution of the boundary develops cusp-like singularities [15].

5. Whitham times and coupling constants of normal matrix models

Let us assume that the support of eigenvalues of the normal matrix model (2) is a simply-connected domain D. For example this is the case if $k=1$ [8]. Then by applying the saddle point method to the large N limit of (4) it follows that

$$\bar{z}=V'(z)+\frac{1}{2\pi i}\iint_D\frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{z'-z},\quad z\in D.\tag{37}$$

Let us consider now the rational conformal map $z(w)$ associated to D, from (37) it follows that the function $\tilde{z}(w)$ can be extended as a meromorphic function of z outside D by

$$\tilde{z}(z)=V'(z)+\frac{1}{2\pi i}\iint_D\frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{z'-z}.\tag{38}$$

In fact $\tilde{z}(z)$ represents the Schwarz function of the boundary Γ of D. By using (3) and (17) one can rewrite (38) as

$$\tilde{z}(z)=\sum_{n=1}^{N_0+1}nz_0^{n-1}t_{0,n}-\sum_{s=1}^{k}\left(\sum_{n=1}^{N_s-1}nz_s^{n+1}t_{s,n}+z_s t_{s,0}\right)+\frac{1}{2\pi i}\iint_D\frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{z'-z}.\tag{39}$$

By comparing this identity with our definition (19) of the Whitham times and by taking into account (31), we conclude that the Whitham times coincide with the coupling constants of the normal matrix model.

Finally, we notice that by using the strategy deployed in the proof of the Theorem of Section 2 and by taking (39) into account, it follows that the deformations with respect to the parameters β_s and $\bar{\beta}_s$ are described by the flows

$$\frac{\partial z_\alpha}{\partial\beta_s}=t_{s,0}\frac{\partial z_\alpha}{\partial t_{s,1}}+\sum_{n=1}^{N_s-2}nt_{s,n}\frac{\partial z_\alpha}{\partial t_{s,n+1}}+(N_s-1)t_{s,N_s-1}\left\{\left(z_s^{N_s}\right)_{(s,+)},z_\alpha\right\},\tag{40}$$

and

$$\frac{\partial z_\alpha}{\partial \bar{\beta}_s} = t_{s+k,0}\frac{\partial z_\alpha}{\partial t_{s+k,1}} + \sum_{n=1}^{N_s-2} n t_{s+k,n}\frac{\partial z_\alpha}{\partial t_{s+k,n+1}} + (N_s-1)t_{s+k,N_s-1}\big\{\big(z_{s+k}^{N_s}\big)_{(s+k,+)}, z_\alpha\big\}. \tag{41}$$

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