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Unconditionally Cauchy series and Cesàro summability

A. Aizpuru *, A. Gutiérrez-Dávila, A. Sala

Departamento de Matemáticas, Universidad de Cádiz, Apdo. 40, 11510-Puerto Real (Cádiz), Spain

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Abstract

In this paper we obtain new characterizations of weakly unconditionally Cauchy series and unconditionally convergent series through Cesàro summability. We study new spaces associated to a series in a Banach space; as a consequence, new characterizations of complete and barrelled normed spaces are proved.

Keywords: (Weakly) unconditionally Cauchy series; Cesàro-convergence; Weak (and weak-*) Cesàro-convergence; Orlicz–Pettis theorem; Completeness and barrelledness

1. Introduction

Let X be a normed space. For any given series $\sum_i x_i$ in X, let us consider the set $S(\sum_i x_i)$ (respectively $S_{\mathrm{w}}(\sum_i x_i)$) of sequences $(a_i)_i \in l_\infty$ such that $\sum_i a_i x_i$ converges (respectively converges for the weak topology). The set $S(\sum_i x_i)$ (respectively $S_{\mathrm{w}}(\sum_i x_i)$), endowed with the sup norm, will be called the space of convergence (respectively weak convergence) associated to the series $\sum_i x_i$. The space X is complete if and only if for every weakly unconditionally Cauchy (wuc) series $\sum_i x_i$ in X the space $S(\sum_i x_i)$ is complete [7]. This result remains valid if the space $S(\sum_i x_i)$ is replaced by $S_{\mathrm{w}}(\sum_i x_i)$ [7]. Let us observe that these characterizations of the completeness of a normed space X using the spaces of convergence (and weak convergence) associated to weakly unconditionally Cauchy series (wuc, [3,4,6]) in X. Furthermore, a normed

* Corresponding author.
E-mail addresses: antonio.aizpuru@uca.es (A. Aizpuru), antonio.gutierrez@uca.es (A. Gutiérrez-Dávila), antonio.sala@uca.es (A. Sala).

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space X is barrelled if and only if each series $\sum_i x_i^*$ in X^* such that $S_{w^*}(\sum_i x_i^*) = l_\infty$ (i.e., its space of weak* convergence is l_∞) is wuc [7].

In this paper we extend these results to the model of Cesàro-convergence. If $\sum_i x_i$ is a series in a topological vector space X, then the series is Cesàro-convergent to x_0 [5] if $\lim_n \frac{1}{n}(S_1 + \cdots + S_n) = x_0$, where $S_n = \sum_{i=1}^n x_i$ for $n \in \mathbb{N}$. In this case we will write C-$\sum_i x_i = x_0$. It is clear that if $\sum_i x_i = x_0$ then C-$\sum_i x_i = x_0$.

Section 2 (respectively 3 and 4) deals with spaces of Cesàro-convergence (respectively weak Cesàro-convergence and weak* Cesàro-convergence). It is shown that a series in a Banach space is wuc if and only if its space of Cesàro-convergence (or weak Cesàro-convergence) is complete. Furthermore, if this equivalence is true for each series in a normed space X, then X must be a Banach space. In this way, we obtain new characterizations of completeness of a normed space through the properties of the space of Cesàro-convergence (or weak Cesàro-convergence) associated to weak series. We also prove a characterization of barrelledness (see Section 4), which is similar to the aforementioned one, but we consider the weak* Cesàro-convergence instead of the weak* convergence.

In the last section of this paper we obtain two new versions of the classical Orlicz–Pettis theorem by using the Cesàro-sums of subseries in the weak topology. Recently [9–12], etc.) some theorems of Orlicz–Pettis type have been obtained. Many of them use supremum properties of natural families (i.e., a subfamily of $P(\mathbb{N})$ which contains the finite subsets). Swartz [9] proved a generalization of the Orlicz–Pettis theorem dealing with FQ σ-families (see Section 5). Also Wu Junde and Lu Shijie [12] obtained an interesting improvement in the framework of topological vector spaces. They used the *signed-weak gliding hump property* (S-WGHP), which can be translated to families of $P(\mathbb{N})$ as a supremum property, similar to property FQ. This generalization considers a dual pair $[X, Y]$ and replaces the subseries-$\sigma(X, Y)$ convergence (or, equivalently, the m_0-multiplier-$\sigma(X, Y)$ convergence, where m_0 is the scalar-valued sequence space which satisfies that for each $(t_i)_i \in m_0$ the set $\{t_i\colon i \in \mathbb{N}\}$ is finite) by the λ-multiplier-$\sigma(X, Y)$ convergence, where λ is a scalar-valued sequences space which has the S-WGHP and contains c_{00}. Wu Junde and Lu Shijie prove that these assumptions imply the λ-multiplier-$\tau(X, Y)$ convergence for a series, where $\tau(X, Y)$ is the Mackey topology.

It is also interesting the improvement that the same authors have proved for Abelian topological groups [11].

Also, the separation properties of natural families (see the definition of property S_1 in Section 5) have led to new theorems of Orlicz–Pettis type [2].

By considering natural families with the property S_1 and FQ σ-families, we extend these results to Cesàro summability.

2. The space of Cesàro-convergence

Let $\sum_i x_i$ be a series in a Banach space X and let

$$S_C\left(\sum_i x_i\right) = \left\{(a_i)_i \in l_\infty\colon \sum_i a_i x_i \text{ is Cesàro-convergent}\right\},$$

endowed with the sup norm. This space will be called the space of Cesàro-convergence associated to the series $\sum_i x_i$. The following theorem characterizes the completeness of the space $S_C(\sum_i x_i)$.

Theorem 2.1. *The following conditions are equivalent*:

(1) $\sum_i x_i$ *is a weakly unconditionally Cauchy series* (*wuc*).
(2) $S_C(\sum_i x_i)$ *is a complete space.*
(3) $c_0 \subseteq S_C$.

Proof. (1) $\Rightarrow$ (2). Let $\sum_i x_i$ be a wuc series and let

$$H = \sup\left\{ \left\| \sum_{i=1}^{n} a_i x_i \right\| : |a_i| \leqslant 1,\ 1 \leqslant i \leqslant n,\ n \in \mathbb{N} \right\}.$$

Let $(a^n)_n$ be a sequence in $S_C(\sum_i x_i)$ with $\lim_n a^n = a^0$ in l_∞. We will prove that $a^0 \in S_C(\sum_i x_i)$.

For any $m \in \mathbb{N}$ and $k \in \mathbb{N}$, let S_k^m be the kth partial sum $S_k^m = \sum_{i=1}^k a_i^m x_i$ and, similarly, let $S_k^0 = \sum_{i=1}^k a_i^0 x_i$. It is sufficient to prove that $(\frac{1}{n}\sum_{i=1}^n S_i^0)_n$ is a Cauchy sequence.

Let $m_0 \in \mathbb{N}$ be such that $\|a^0 - a^{m_0}\| < \frac{\varepsilon}{5H}$ and let n_0 be such that

$$\left\| \frac{1}{p}\sum_{i=1}^{p} S_i^{m_0} - \frac{1}{q}\sum_{i=1}^{q} S_i^{m_0} \right\| < \frac{\varepsilon}{5}$$

for $p, q \in \mathbb{N}$, $p > q \geqslant n_0$. It is clear that

$$\left\| \frac{1}{p}\sum_{i=1}^{p} S_i^{0} - \frac{1}{q}\sum_{i=1}^{q} S_i^{0} \right\| \leqslant \left\| \frac{1}{p}\sum_{i=1}^{p} \left(S_i^0 - S_i^{m_0}\right) \right\| + \left\| \frac{1}{q}\sum_{i=1}^{q} \left(S_i^0 - S_i^{m_0}\right) \right\| + \left\| \frac{1}{p}\sum_{i=1}^{p} S_i^{m_0} - \frac{1}{q}\sum_{i=1}^{q} S_i^{m_0} \right\|,$$

and that, for $j \in \mathbb{N}$,

$$\left\| \sum_{i=1}^{j} \frac{5H}{\varepsilon}\left(a_i^0 - a_i^{m_0}\right) x_i \right\| \leqslant H.$$

Hence,

$$\|S_j^0 - S_j^{m_0}\| = \left\| \sum_{i=1}^{j} \left(a_i^0 - a_i^{m_0}\right) x_i \right\| \leqslant \frac{\varepsilon}{5}.$$

Now, it is easy to check that $(\frac{1}{n}\sum_{i=1}^n S_i^0)_n$ is a Cauchy sequence.

It is obvious that (2) $\Rightarrow$ (3). Let us prove that (3) $\Rightarrow$ (1). If the series is not wuc, let $f \in X^*$ be such that $\sum_i |f(x_i)| = \infty$.

Inductively, we will construct a sequence $(\alpha_i)_i \in c_0$ such that $\sum_i \alpha_i f(x_i) = \infty$ and $\alpha_i f(x_i) \geqslant 0$; then, C-$\sum_i \alpha_i f(x_i) = \infty$, which is a contradiction.

Let m_1 be such that $\sum_{i=1}^{m_1} |f(x_i)| > 2 \cdot 2$. We define $\alpha_i = \frac{1}{2}$ if $f(x_i) \geqslant 0$ and $\alpha_i = -\frac{1}{2}$ if $f(x_i) < 0$ for $i \in \{1, 2, \ldots, m_1\}$. It is clear that $\sum_{i=1}^{m_1} \alpha_i f(x_i) > 2$ and $\alpha_i f(x_i) \geqslant 0$ for $i \in \{1, 2, \ldots, m_1\}$.

Let $m_2 > m_1$ be such that $\sum_{i=m_1+1}^{m_2} |f(x_i)| > 2^2 \cdot 2^2$. We define $\alpha_i = \frac{1}{2^2}$ if $f(x_i) \geqslant 0$ and $\alpha_i = -\frac{1}{2^2}$ if $f(x_i) < 0$ for $i \in \{m_1 + 1, \ldots, m_2\}$. Similarly, it can be deduced that $\sum_{i=m_1+1}^{m_2} \alpha_i f(x_i) > 2^2$ and $\alpha_i f(x_i) \geqslant 0$ for $i \in \{m_1 + 1, \ldots, m_2\}$.

Now, it is easy to conclude the inductive argument to obtain the sequence $(\alpha_i)_i$ with the properties given above. This completes the proof. □

Remark 2.2. The space of convergence.

Let $\sum_i x_i$ be a series in a normed space X and let

$$S\left(\sum_i x_i\right) = \left\{(a_i)_i \in l_\infty\colon \sum_i a_i x_i \text{ converges}\right\},$$

endowed with the sup norm. It is obvious that $S(\sum_i x_i)$ is a subspace of l_∞ and $S(\sum_i x_i) \subseteq S_{\mathrm{C}}(\sum_i x_i)$. If X is a Banach space, then $\sum_i x_i$ is wuc if and only if $S(\sum_i x_i)$ is complete [7]. Theorem 2.1 gives us a similar characterization by considering Cesàro-convergence.

Theorem 2.1 and Remark 2.2 allow us to obtain the following theorem.

Corollary 2.3. *Let X be a Banach space and let $\sum_i x_i$ be a series in X. The following properties are equivalent*:

(1) $\sum_i x_i$ *is wuc.*
(2) $S(\sum_i x_i)$ *is a complete space.*
(3) $c_0 \subseteq S(\sum_i x_i)$.
(4) $S_{\mathrm{C}}(\sum_i x_i)$ *is a complete space.*
(5) $c_0 \subseteq S_{\mathrm{C}}(\sum_i x_i)$.
(6) $\sum |f(x_i)|$ *is Cesàro-convergent for every* $f \in X^*$.

Proof. The equivalence of properties (1), (2) and (3) can be found in [7]. The remaining equivalences are consequence of Theorem 2.1. □

Remark 2.4. Let X be a normed space and let $\sum_i x_i$ be a series in X. Let us consider the following two spaces:

$$S^{\mathrm{C}}\left(\sum_i x_i\right) = \left\{(a_i)_i \in l_\infty\colon \sum_i a_i x_i \text{ is a Cauchy series}\right\},$$

$$S^{\mathrm{C}}_{\mathrm{C}}\left(\sum_i x_i\right) = \left\{(a_i)_i \in l_\infty\colon \sum_i a_i x_i \text{ is a Cesàro Cauchy series}\right\}.$$

From the discussion given above, it can be deduced that

(A) $\sum_i x_i$ is wuc $\Leftrightarrow S^{\mathrm{C}}(\sum_i x_i)$ is complete $\Leftrightarrow c_0 \subseteq S^{\mathrm{C}}(\sum_i x_i)$.
(B) $\sum_i x_i$ is wuc $\Leftrightarrow S^{\mathrm{C}}_{\mathrm{C}}(\sum_i x_i)$ is complete $\Leftrightarrow c_0 \subseteq S^{\mathrm{C}}_{\mathrm{C}}(\sum_i x_i)$.

The following theorem characterizes the completeness of a normed space using the space $S_{\mathrm{C}}(\sum_i x_i)$.

Theorem 2.5. *Let X be a normed space. X is a Banach space if and only if $S_{\mathrm{C}}(\sum_i x_i)$ is complete for any wuc series $\sum_i x_i$.*

Proof. By Theorem 2.1, the condition is necessary. Conversely, if X is not a complete space, let $\sum_i x_i$ be a series in X such that $\|x_i\| < \frac{1}{i2^i}$, for $i \in \mathbb{N}$, and $\sum_i x_i = x^{**} \in X^{**} \setminus X$. It is clear that C-$\sum_i x_i = x^{**}$.

Let $\sum_i z_i$ be the series defined by $z_{2i-1} = ix_i$ and $z_{2i} = -ix_i$, for $i \in \mathbb{N}$. Clearly $\sum_i z_i$ is wuc. Let us consider the sequence $(a_i)_i \in c_0$ defined by $a_{2i-1} = \frac{1}{2i}$ and $a_{2i} = \frac{-1}{2i}$, for $i \in \mathbb{N}$. We have C-$\sum_i a_i z_i \in X^{**} \setminus X$ and so $S_{\mathrm{C}}(\sum_i z_i)$ is not a complete space. □

A series $\sum_i x_i$ in a normed space X is wuc if and only if the map $T : S(\sum_i x_i) \to X$ defined by $T((a_i)_i) = \sum_i a_i x_i$ is continuous [1]. We now give a similar characterization of wuc series.

Theorem 2.6. *Let $\sum_i x_i$ be a series in a normed space X. The series $\sum_i x_i$ is wuc if and only if the map $T : S_{\mathrm{C}}(\sum_i x_i) \to X$ defined by $T((a_i)_i) = \mathrm{C}\text{-}\sum_i a_i x_i$ is continuous.*

Proof. If the map T is continuous, we prove $\sum_i x_i$ is a wuc series. Since $c_{00} \subseteq S_{\mathrm{C}}(\sum_i x_i)$ and T is continuous, we have that

$$\sup\left\{\left\| \sum_{i=1}^{n} a_i x_i \right\| : n \in \mathbb{N} \mathrm{y} |a_i| \leqslant 1\right\} \leqslant \|T\|.$$

Hence $\sum_i x_i$ is a wuc series.

Conversely, let

$$H = \sup\left\{\left\| \sum_{i=1}^{n} a_i x_i \right\| : n \in \mathbb{N} \mathrm{y} |a_i| \leqslant 1\right\}$$

and let $(a_i)_i \in B_{S_{\mathrm{C}}(\sum_i x_i)}$. Write $S_j = \sum_{i \leqslant j} a_i x_i$ for $j \in \mathbb{N}$. It is clear that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} S_i \right\| = \left\| \frac{1}{n} \sum_{i=1}^{n} (n - i + 1) a_i x_i \right\| \leqslant H$$

for $n \in \mathbb{N}$. Therefore, T is continuous. □

3. The space of weak Cesàro-convergence

We now consider the space of weak Cesàro-convergence associated to a series $\sum_i x_i$ in a normed space X:

$$S_{\mathrm{wC}}\left(\sum_i x_i\right) = \left\{(a_i)_i \in l_\infty \colon \sum_i a_i x_i \text{ is w-Cesàro-convergent}\right\},$$

endowed with the sup norm.

Theorem 3.1. *Let $\sum_i x_i$ be a series in the Banach space X. The following statements are equivalent*:

(1) *$\sum_i x_i$ is wuc.*
(2) *$S_{\mathrm{wC}}(\sum_i x_i)$ is a complete space.*
(3) *$c_0 \subseteq S_{\mathrm{wC}}(\sum_i x_i)$.*

Proof. (1) ⇒ (2). Let $\sum_i x_i$ be a wuc series and let

$$H = \sup\left\{\left\| \sum_{i=1}^{n} a_i x_i \right\| : n \in \mathbb{N} \mathrm{y} |a_i| \leqslant 1\right\}.$$

Let $(a^k)_k$ be a sequence in $S_{\mathrm{wC}}(\sum_i x_i)$ with $\lim_k a^k = a^0$ in l_∞. We prove that $a^0 \in S_{\mathrm{wC}}(\sum_i x_i)$. Define $z_k = \mathrm{wC}\text{-}\sum_i a_i^k x_i$, where $a^k = (a_i^k)_i$ and $k \in \mathbb{N}$.

Let us check that $(z_k)_k$ is a Cauchy sequence. Let $\varepsilon > 0$ and let $k_0 \in \mathbb{N}$ be such that $\|a^k - a^0\| < \frac{\varepsilon}{5H}$ for $k \geqslant k_0$. Let us suppose $p, q \in \mathbb{N}$ satisfy $p, q \geqslant k_0$ and $f_0 \in S_{X^*}$ is such that $\|z_p - z_q\| = f_0(z_p - z_q) = \mathrm{C}\text{-}\sum_i (a_i^p - a_i^q) f_0(x_i)$. We can assume that $\|a^p - a^q\| \neq 0$ (if there exists a constant subsequence of $(a_k)_k$ then $a^0 \in S_{\mathrm{wC}}(\sum_i x_i)$), and it is obvious that

$$\frac{1}{\|a^p - a^q\|} \left\| \frac{1}{m} \sum_{i=1}^{m} (m-i+1)\big(a_i^p - a_i^q\big) x_i \right\| \leqslant H$$

for $m \in \mathbb{N}$. Therefore $\|z_p - z_q\| < \frac{2\varepsilon}{5}$ for $p, q \geqslant k_0$. This proves that $(z_k)_k$ is a Cauchy sequence.

Let $z_0 = \lim_k z_k$. We prove that $z_0 = \mathrm{wC}\text{-}\sum_i a_i^0 x_i$. Consider $k \in \mathbb{N}$ with $\|z_k - z_0\| < \frac{\varepsilon}{5}$ and $\|a^k - a^0\| < \frac{\varepsilon}{5H}$. Let $f \in B_{X^*}$ and $m_1 \in \mathbb{N}$ be such that

$$\left\| \frac{1}{m} \sum_{i=1}^{m} (m-i+1) a_i^k f(x_i) - f(z_k) \right\| < \frac{\varepsilon}{5}$$

for $m \geqslant m_1$. Then

$$\begin{aligned} &\left\| \frac{1}{m} \sum_{i=1}^{m} \big((m-i+1) a_i^0 f(x_i)\big) - f(z_0) \right\| \\ &\quad \leqslant \left\| \frac{1}{m} \sum_{i=1}^{m} (m-i+1)\big(a_i^0 - a_i^k\big) f(x_i) \right\| + \left\| \frac{1}{m} \sum_{i=1}^{m} \big((m-i+1) a_i^k f(x_i)\big) - f(z_k) \right\| \\ &\qquad + \big\| f(z_k) - f(z_0) \big\| < \frac{3\varepsilon}{5} \end{aligned}$$

for $m \geqslant m_1$.

It is obvious that (2) $\Rightarrow$ (3). An analysis similar to that in the proof of (3) $\Rightarrow$ (1) in Theorem 2.1 shows that (3) $\Rightarrow$ (1). □

We now consider the space $S_{\mathrm{wC}}(\sum_i x_i)$ and obtain a characterization similar to the one in Theorem 2.5.

Theorem 3.2. *A normed space X is a Banach space if and only if $S_{\mathrm{wC}}(\sum_i x_i)$ is complete for any wuc series $\sum_i x_i$ in X.*

Proof. The proof is similar to the one in Theorem 2.5: in the notation of this proof, we only need to observe that $(a_i)_i \notin S_{\mathrm{wC}}(\sum_i z_i)$ because $\mathrm{C}\text{-}\sum_i a_i f(z_i) = x^{**}(f)$. □

Remark 3.3. The space of weak convergence.

Let $\sum_i x_i$ be a series in a normed space X and let

$$S_{\mathrm{w}}\bigg(\sum_i x_i\bigg) = \bigg\{(a_i)_i \in l_\infty \colon \sum_i a_i x_i \text{ is w convergent}\bigg\},$$

endowed with the sup norm. It is clear that $S_{\mathrm{w}}(\sum_i x_i) \subseteq S_{\mathrm{wC}}(\sum_i x_i)$. As in the previous section, Theorems 3.1 and 3.2 have been obtained using the spaces $S_{\mathrm{w}}(\sum_i x_i)$ instead of $S_{\mathrm{wC}}(\sum_i x_i)$ [7].

4. The space of weak* Cesàro-convergence

In this section we study the convergence spaces associated to a series in the dual space $(X^*, \sigma(X^*, X))$.

Let X be a normed space and let $\sum_i f_i$ be a series in X^*. Let us consider

$$S_{\mathrm{w}^*\mathrm{C}}\left(\sum_i f_i\right) = \left\{(a_i)_i \in l_\infty\colon \sum_i a_i f_i \text{ is w}^*\text{-Cesàro-convergent}\right\},$$

endowed with the sup norm. This space will be called the space of weak* Cesàro-convergence associated to the series $\sum_i f_i$.

As in the previous sections, we can consider the space of weak* convergence

$$S_{\mathrm{w}^*}\left(\sum_i f_i\right) = \left\{(a_i)_i \in l_\infty\colon \sum_i a_i f_i \text{ is w}^* \text{ convergent}\right\}.$$

It is clear that

$$S_{\mathrm{w}^*}\left(\sum_i f_i\right) \subset S_{\mathrm{w}^*\mathrm{C}}\left(\sum_i f_i\right).$$

The following theorem is similar to [7, Theorem 4.1] but we consider the space $S_{\mathrm{w}^*\mathrm{C}}(\sum_i f_i)$.

Theorem 4.1. *Let X be a normed space and let $\sum_i f_i$ be a series in X^*. Let us consider the following statements*:

(1) $\sum_i f_i$ *is wuc.*
(2) $S_{\mathrm{w}^*\mathrm{C}}(\sum_i f_i) = l_\infty$.
(3) *If $x \in X$ and $M \subseteq \mathbb{N}$, then $\sum_{i \in M} f_i(x)$ is Cesàro-convergent.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *The space X is a barrelled space if and only if* (3) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2). If $(a_i)_i \in l_\infty$, then the series $\sum_i a_i f_i$ is w* convergent in X^*. Therefore the w*-Cesàro sum $\sum_i a_i f_i$ exists.

It is easy to check that (2) $\Rightarrow$ (3).

We now prove the last statement. Let us suppose that X is a barrelled normed space. Define

$$E = \left\{\sum_{i=1}^{n} a_i f_i\colon n \in \mathbb{N} \text{ and } |a_i| \leqslant 1\right\}.$$

In order to prove (3) $\Rightarrow$ (1), it is sufficient to show that E is pointwise bounded. On the contrary, let us suppose that there exists $x_0 \in X$ such that the series $\sum_i |f_i(x_0)|$ diverges. Write $M = \{i \in \mathbb{N}\colon f_i(x_0) \geqslant 0\}$ and $R = \{i \in \mathbb{N}\colon f_i(x_0) < 0\}$. With this notation, either $\sum_{i \in M} f_i(x_0)$ or $\sum_{i \in R}(-f_i)(x_0)$ is divergent, which contradicts (3).

Assume that statements (1) and (3) are equivalent for a normed space X. Next we prove that X is a barrelled space. Suppose, contrary to our claim, that there exists a pointwise bounded set $M \subseteq X^*$ which is not bounded. Let $f_i \in M$ be such that $\|f_i\| > 2^i \cdot 2^i$ for $i \in \mathbb{N}$. Define $g_i = \frac{1}{2^i} f_i$ for $i \in \mathbb{N}$, it is obvious that the series $\sum_i g_i$ satisfies (3) and also $\|g_i\| > 2^i$, which is impossible. □

5. Some versions of Orlicz–Pettis theorem for Cesàro-convergence

Theorem 5.1. *Let X be a Banach space. A series $\sum_i x_i$ in X is unconditionally convergent (uco) if and only if $\sum_i x_i$ is subseries weakly Cesàro-convergent.*

Proof. It is clear that every unconditionally convergent series is subseries weakly Cesàro-convergent.

Conversely, let $\sum_i x_i$ be a series in X such that $\sum_{i\in M} x_i$ is weakly Cesàro-convergent for every $M \subseteq \mathbb{N}$. We first prove that $\sum_{i\in M} \varepsilon_i x_i$ is a weakly Cesàro-convergent series for every $M \subseteq \mathbb{N}$ and every sequence $(\varepsilon_i)_i$ in $\{-1, 1\}$. Define $\alpha_i = 1$ if $\varepsilon_i = 1$ and $\alpha_i = 0$ if $\varepsilon_i \neq 1$ for $i \in M$. Let $\beta_i = -1$ if $\varepsilon_i = -1$ and $\beta_i = 0$ if $\varepsilon_i \neq -1$ for $i \in \mathbb{N}$.

It is obvious that wC-$\sum_{i\in M} \varepsilon_i x_i =$ wC-$\sum_{i\in M} \alpha_i x_i +$ wC-$\sum_{i\in M} \beta_i x_i$ and so $\sum_{i\in M} \varepsilon_i x_i$ is weakly Cesàro-convergent.

Let us prove that $\sum_i x_i$ is wuc. On the contrary, consider $f \in X^*$ with $\sum_i |f(x_i)| = \infty$ and let $\varepsilon_i = 1$ if $f(x_i) \geqslant 0$ and $\varepsilon_i = -1$ if $f(x_i) < 0$ for $i \in \mathbb{N}$. It is clear that $\sum_i \varepsilon_i f(x_i) = \infty$, which is a contradiction.

We are now in a position to show $\sum_i x_i$ is subseries weakly convergent. Let $M \subseteq \mathbb{N}$ and let $x_0 =$ wC-$\sum_{i\in M} x_i$. Since $\sum_i x_i$ is wuc, the series $\sum_{i\in M} f(x_i)$ is convergent for $f \in X^*$ and $f(x_0) =$ C-$\sum_{i\in M} f(x_i)$. Hence $x_0 =$ w-$\sum_{i\in M} x_i$. The classical Orlicz–Pettis theorem allows us to conclude the proof. □

From the previous result we can easily conclude the following one.

Corollary 5.2. *Let $\sum_i x_i$ be a series in the Banach space X. The following statements are equivalent:*

(1) *$\sum_i x_i$ is uco.*
(2) *$S_{\mathrm{C}}(\sum_i x_i) = l_\infty$.*
(3) *$S_{\mathrm{wC}}(\sum_i x_i) = l_\infty$.*

Definition 5.3. Let $\mathcal{F}$ be a natural family and let $\phi_0(\mathbb{N})$ be the family of finite subsets of $\mathbb{N}$. It is said that $\mathcal{F}$ is an FQ σ-family [8] if for every sequence $(A_i)_i$ of mutually disjoint sets in $\phi_0(\mathbb{N})$ there exists an infinite set $M \subseteq \mathbb{N}$ such that $A = \bigcup_{i\in M} A_i \in \mathcal{F}$.

Definition 5.4. Let $\mathcal{F}$ be a natural family. A series $\sum_i x_i$ in a normed space X is $\mathcal{F}$-convergent (respectively $\mathcal{F}$-weakly convergent) if $\sum_{i\in A} x_i$ is convergent (respectively weakly convergent) for every $A \in \mathcal{F}$ [2].

Let $\mathcal{F}$ be an FQ σ-family. Swartz [9] proved that each $\mathcal{F}$-convergent series in a Banach space is uco. We next extend this result to Cesàro-convergence.

Theorem 5.5. *A series $\sum_i x_i$ in the Banach space X is uco if and only if there exists an FQ σ-family $\mathcal{F}$ such that $\sum_{i\in A} x_i$ is w-Cesàro-convergent for every $A \in \mathcal{F}$.*

Proof. Let $\sum_i x_i$ be a series such that there exists wC-$\sum_{i\in A} x_i$ for every $A \in \mathcal{F}$, where $\mathcal{F}$ denotes an FQ σ-family. As in the proof of Theorem 5.1, it is sufficient to show that $\sum_i x_i$ is wuc (let us observe that every $\mathcal{F}$-weakly convergent series is uco).

If $\sum_i x_i$ is not wuc, there exists $f \in X^*$ with $\sum_i |f(x_i)| = +\infty$. With the notation $P = \{i\colon f(x_i) \geqslant 0\}$ and $Q = \{i\colon f(x_i) < 0\}$, we can assume that $\sum_{i\in P} f(x_i) = +\infty$. We can inductively construct a disjoint sequence $(A_n)_n$ in $\phi_0(\mathbb{N}) \cap P$ such that $\sup A_n < \inf A_{n+1}$ and $\sum_{i\in A_n} f(x_i) > n$ for each $n \in \mathbb{N}$.

Consider $M \subseteq \mathbb{N}$ with $A = \bigcup_{n\in M} A_n \in \mathcal{F}$. It is clear that $\sum_{i\in A} f(x_i) = +\infty$, which contradicts $\sum_{i\in A} f(x_i)$ is Cesàro-convergent. □

We now introduce a separation property which has allowed us to improve the mentioned Swartz's result concerning FQ σ-families [2].

Definition 5.6. It is said that a natural family $\mathcal{F}$ has property S_1 [2] if for every pair $[(A_i)_i, (B_i)_i]$ of disjoint sequences of mutually disjoint sets in $\phi_0(\mathbb{N})$ there exist an infinite set $M \subseteq \mathbb{N}$ and $B \in \mathcal{F}$ such that $A_i \subseteq B$ and $B_i \subseteq B^{\mathrm{C}}$ for $i \in M$.

It is obvious that each FQ σ-family has the property S_1. However there exists natural families with property S_1 which are not FQ σ-families [2].

The following generalization of the Orlicz–Pettis theorem is proved in [2]. Let $\sum_i x_i$ be a series in a Banach space X and let $\mathcal{F}$ be a natural family with the property S_1. Then, $\sum_i x_i$ is uco if and only if $\sum_i x_i$ is $\mathcal{F}$-weakly convergent.

The property S_1 and the weak Cesàro-convergence allow us to characterize the unconditional convergence of a series in a Banach space. An analysis similar to that in the proofs of Theorems 5.1 and 5.5 shows the main step of this argument is to prove that each series $\sum_i x_i$ with $\sum_{i\in A} x_i$ weakly Cesàro-convergent for $A \in \mathcal{F}$ is wuc.

Lemma 5.7. *Let X be a Banach space. A series $\sum_i x_i$ in X is wuc if and only if there exists a natural family $\mathcal{F}$ with the property S_1 such that the partial sums $\sum_{i\in A} x_i$ are bounded for each $A \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a natural family with the property S_1 and let $\sum_i x_i$ be a series such that the partial sums $\sum_{i\in A} x_i$ are bounded for each $A \in \mathcal{F}$. The sequence $(\sum_{i\in A\cap[1,n]} x_i)_n$ is bounded for each $A \in \mathcal{F}$. Suppose, contrary to our claim, that the series $\sum_i x_i$ is not wuc. There exists $f \in X^*$ with $\sum_i |f(x_i)| = +\infty$. Let $P = \{i\colon f(x_i) \geqslant 0\}$ and $Q = \{i\colon f(x_i) < 0\}$. We can assume that $\sum_{i\in P} f(x_i) = +\infty$. As in the proof of Theorem 5.1, let $(A_n)_n$ be a disjoint sequence in $\phi_0(\mathbb{N}) \cap P$ such that $\sup A_n < \inf A_{n+1}$ and $\sum_{i\in A_n} f(x_i) > n$ for $n \in \mathbb{N}$. Let $p_n = \inf A_n$, $q_n = \sup A_n$ and $B_n = [p_n, q_n] \setminus A_n$ for $n \in \mathbb{N}$. The pair of sequences $[(A_n)_n, (B_n)_n]$ allow us to obtain $A \in \mathcal{F}$ and an infinite set $M \subseteq \mathbb{N}$ such that $A_n \subseteq A$ and $B_n \subseteq A^{\mathrm{C}}$ for $n \in M$ (see Definition 5.6). Write $S_j = \sum_{i\in A,\, i\leqslant j} f(x_i)$. It is clear that the sequence $(S_{q_n} - S_{p_n - 1})_n$ is not bounded, which contradicts our hypothesis. □

From the previous result we deduce the following characterization of unconditional convergence.

Corollary 5.8. *Let $\sum_i x_i$ be a series in a Banach space X. The following statements are equivalent*:

(1) *$\sum_i x_i$ is uco.*
(2) *There exists a natural family $\mathcal{F}$ with the property S_1 such that the series $\sum_{i\in A} x_i$ is weakly Cesàro-convergent and its partial sums are bounded for every $A \in \mathcal{F}$.*

Let us observe that there exists Cesàro-convergent series such that the sequence of its partial sums is not bounded.

Remark 5.9. Let $\sum_i x_i$ be a wuc series in a normed space X. It is clear that $\sum_i a_i x_i$ is wuc for each $(a_i)_i \in l_\infty$. Therefore if $(a_i)_i \in S_{\mathrm{C}}(\sum_i x_i)$ then $\sum_i a_i x_i$ is weakly Cauchy and Cesàro-convergent, and so this series is weakly convergent. From this we have $S_{\mathrm{C}}(\sum_i x_i) \subseteq S_{\mathrm{w}}(\sum_i x_i)$, but we do not know what conditions allow us to obtain the equality of both spaces.

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