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Linear structure of sets of divergent sequences and series

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Abstract

We show that there exist infinite dimensional spaces of series, every non-zero element of which, enjoys certain pathological property. Some of these properties consist on being (i) conditional convergent, (ii) divergent, or (iii) being a subspace of l_∞ of divergent series. We also show that the space $l_1^\omega(X)$ of all weakly unconditionally Cauchy series in X has an infinite dimensional vector space of non-weakly convergent series, and that the set of unconditionally convergent series on X contains a vector space E, of infinite dimension, so that if $x \in E \setminus \{0\}$ then $\sum_i \|x_i\| = \infty$.

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1. Introduction

This paper contributes to the search for large vector spaces of functions on $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) which have special properties. Given such a property, we say that the subset M of functions on $\mathbb{K}$ which satisfy it is *spaceable* if $M \cup \{0\}$ contains a *closed* infinite dimensional subspace. The set M will be called *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space. We will say that the set

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M is μ-lineable if it contains a vector space of dimension μ. Also, we denote $\lambda(M)$ the maximum dimension of such a vector space [1].

One of the first results in this area was proved by Fonf et al. [5], who showed that the set of *nowhere differentiable functions* on [0, 1] is spaceable in $\mathscr{C}[0,1]$. Moreover, Rodríguez-Piazza proved [7] that the X in [5] can be chosen to be isometrically isomorphic to any separable Banach space. In this work, we continue the search for *pathological* vector spaces. Particularly, we study *large* vector spaces of sequences and series enjoying several *special* properties. There are many examples of scalar series in $\mathbb{K}$ which are conditionally convergent, unconditionally convergent, and examples of unconditionally convergent vector series $\sum_i x_i$ in a Banach space X so that $\sum_i \|x_i\| = \infty$ [4]. Here we study the lineability of sets of scalar and vector series enjoying, amongst others, these *special* properties.

2. Sequences, scalar series, and vector series

A family $\{A_\alpha : \alpha \in I\}$ of infinite subsets of $\mathbb{N}$ is called *almost disjoint* if $A_\alpha \cap A_\beta$ is finite whenever $\alpha, \beta \in I$ and $\alpha \neq \beta$. The usual procedure to generate such a family is the following. Let us denote I the irrationals in [0, 1] and $\{q_n : n \in \mathbb{N}\}$ denotes $[0,1] \cap \mathbb{Q}$. For every $\alpha \in I$ we choose a subsequence $(q_{n_k})_k$ of $\{q_n : n \in \mathbb{N}\}$ so that $\lim_{k\to\infty} q_{n_k} = \alpha$ and we define $A_\alpha = \{n_k : k \in \mathbb{N}\}$. By construction we obtain that $\{A_\alpha : \alpha \in I\}$ is an almost disjoint family of subsets of $\mathbb{N}$. In this paper, we use this notion on several occasions.

If V denotes the set of conditionally convergent series then, clearly, $V \cup \{0\}$ is not a vector space in $CS(\mathbb{K})$, the set of convergent series. The following theorem shows that the set of conditionally convergent series is c-lineable.

Theorem 2.1. *$CS(\mathbb{K})$ contains a vector space E verifying the following properties*:

(i) *Every $x \in E \setminus \{0\}$ is a conditionally convergent series.*
(ii) $\dim(E) = c$.
(iii) $\operatorname{span}\{E \cup c_{00}\}$ *is an algebra and its elements are either elements of c_{00} or conditionally convergent series.*

Proof. Let us fix any conditionally convergent series $\sum_i a_i$ such that $a_i \neq 0$ for every $i \in \mathbb{N}$. Consider the family $(A_\alpha)_{\alpha\in I}$ of almost disjoint subsets of $\mathbb{N}$. We have that $\operatorname{card}(I) = c$. For every $\alpha \in I$ we define x^α given by $x_i^\alpha = a_n$ if $i = n$th element of A_α and $x_i^\alpha = 0$ otherwise. For every $\alpha \in I$ we have that the series $\sum_i x_i^\alpha$ has a conditionally convergent subseries and, therefore, it is conditionally convergent. Let $E = \operatorname{span}\{x^\alpha : \alpha \in I\}$. We have that $\{x^\alpha : \alpha \in I\}$ is a linearly independent family, and so $\dim(E) = c$.

Suppose that $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{K} \setminus \{0\}$ and $\{\alpha_1, \dots, \alpha_n\} \subset I$. We now see that $z = \lambda_1 x^{\alpha_1} + \cdots + \lambda_n x^{\alpha_n}$ is a conditionally convergent series. Indeed, it is easy to check that there exists $A \subset A_{\alpha_1}$, infinite, so that $A_{\alpha_1} \setminus A$ is finite, and verifying that $A \cap (A_{\alpha_2} \cup \cdots \cup A_{\alpha_n}) = \varnothing$. Then $\sum_{i\in A} z_i = \sum_{i\in A} \lambda_i x_i^\alpha$ is conditionally convergent and, therefore, so is z.

To check that $\operatorname{span}(E \cup c_{00})$ is an algebra it suffices to notice that

$$(\lambda_1 x^{\alpha_1} + \cdots + \lambda_n x^{\alpha_n})(\mu_1 x^{\beta_1} + \cdots + \mu_m x^{\beta_m}) \in c_{00}$$

if $\{\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m\} \subset \mathbb{K} \setminus \{0\}$ and $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\} \subset I$. $\square$

This same technique can be also used to prove the following results in a similar way:

Theorem 2.2. *There exists a vector space* $E \subset BS(\mathbb{K})$ (*the set of all series with bounded partial sums*) *such that*:

(i) *Every* $x \in E \setminus \{0\}$ *is a divergent series.*
(ii) $\dim(E) = c$ *and* E *is non-separable.*
(iii) $\operatorname{span}\{E \cup c_{00}\}$ *is an algebra and every element of it is either a divergent series or is an element of* c_{00}.

Theorem 2.3. *There exists a vector space* $E \subset l_\infty$ *such that*:

(i) $\dim(E) = c$.
(ii) *Every* $x \in E \setminus \{0\}$ *is a divergent sequence.*
(iii) $E \oplus c_0$ *is an algebra.*
(iv) *Every element in* $\mathrm{cl}(E) + c_0$ *is either a divergent sequence or a sequence in* c_0.

If X is a Banach space and $\sum_i x_i$ is a series in X, we say that $\sum_i x_i$ is *unconditionally convergent* (UC) if, for every permutation π of $\mathbb{N}$, we have that $\sum_{i=1}^{\infty} x_{\pi(i)}$ converges. We say that $\sum_i x_i$ is *weakly unconditionally Cauchy* (WUC) if $\sum_{i=1}^{\infty} |f(x_i)| < \infty$ for every $f \in X^*$, the dual space of X. It is also known that [2,3,6] if X is a Banach space, then there exists a WUC series in X which is convergent but which is not unconditionally convergent if and only if X has a copy of c_0. It is a well known fact that every infinite dimensional Banach space has a series $\sum_i x_i$ which is unconditionally convergent and so that $\sum_i \|x_i\| = \infty$ [4]. The technique from the proof of Theorem 2.1 lead to the following final results:

Theorem 2.4. *There exists a vector space* $E \subset l_1^{\omega}(c_0)$ (*the space of all weakly unconditionally Cauchy series in* c_0) *verifying*:

(i) $\dim(E) = c$.
(ii) *If* $x \in E \setminus \{0\}$ *then* $\sum_i x_i$ *is not weakly convergent.*

Theorem 2.5. *Let X be an infinite dimensional Banach space. Then there exists a vector subspace* E *of* $\mathrm{UC}(X)$ *such that* $\dim(E) = c$, *and if* $x \in E \setminus \{0\}$ *then* $\sum_i \|x_i\| = \infty$.

To conclude, notice that from Theorem 2.4 it follows that X has a copy of c_0 if and only if there exists a vector subspace E of $l_1^{\omega}(X)$ with $\dim(E) = c$, so that every non-zero element of E is a non-weakly convergent series.

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