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Multiple solutions for the Schwarzian Korteweg–de Vries equation in (2 + 1) dimensions ☆

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Abstract

In this paper we find new families of solutions for the (2 + 1)-dimensional integrable Schwarzian Korteweg–de Vries equation, that depend up to two arbitrary functions and a solution of a Riemann wave equation. Some of these solutions exhibit a rich dynamic, with a wide variety of qualitative behavior and structures that are exponentially localized. We have also found several families of overturning and intertwining solutions for the equation, that correspond to the nonconstant solutions of Riemann equations.

1. Introduction

The Lax representation [12] is one of the main tools in the theory of integrable systems: this representation let the construction of special solutions of the corresponding nonlinear system by means of a pair (L, T) of linear problems (Lax pairs). If this pair possesses a nontrivial spectral parameter then the nonlinear system is called Lax integrable (or inverse scattering transformation integrable). Many of these Lax integrable models have interesting properties: an infinite number of symmetries and/or conservation laws, multi-soliton solutions, etc. However, although the physical world has three spatial dimensions, the most known models are (1 + 1) dimensional. Therefore, in the last decade, many efforts have been made to obtain higher-dimensional integrable systems from well-known (1 + 1) integrable systems. A typical way, but not the unique, of constructing higher dimensional integrable systems is to modify the Lax pair (L, T) of the basic equation, by adding dependence on more spatial variables, in such way that the new pair $(\hat{L}, \hat{T})$ verifies the Lax equation $[\hat{L}, \hat{T}] = 0$. The new corresponding system usually has solutions that have their counterpart in the basic equation but does also have a great variety of new solutions. Many new properties can be seen to be exhibited by such solutions which are not displayed by the corresponding basic (1 + 1) dimensional system.

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The basic Korteweg–de Vries (KdV) equation is integrable and it is the origin of many integrable equations. One of these extensions is based on a result of Lie, who proved that the Schwarzian of a function f is the unique elementary function of the derivatives Df of f, excluding f itself, which is invariant under Möbius transformations. An important property of the KdV equation is that it possesses infinitely many symmetries. This fact has been related with the invariance under Möbius transformation of the Schwarzian Korteweg–de Vries (SKdV) equation

$$-\frac{\phi_t}{\phi_x} = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{1}$$

where the right hand side of this equation is the Schwarzian derivative of ϕ [9]. This equation was introduced in [10,18] and has been studied in [4,8,20].

Eq. (1) has been the basis for several generalizations and extensions. One of them can be obtained by the method we have indicated above. The resulting equation is

$$W_t + \frac{1}{4}W_{xxz} - \frac{W_x W_{xz}}{2W} - \frac{W_{xx}W_z}{4W} + \frac{W_x^2 W_z}{2W^2} - \frac{W_x}{8}\left(\partial_x^{-1}\left(\frac{W_x^2}{W^2}\right)\right)_z = 0, \tag{2}$$

where we denote $\partial_x^{-1} f = \int f\, dx$. We will call this equation as the $(2+1)$ Schwarzian Korteweg–de Vries equation ($(2+1)$ SKdV equation). This model was first considered by Kudryashov and Pickering [11]. Toda and Yu [17] obtained Eq. (2) by using Lax pairs and proved that this equation passes the integrability Painlevé test in the sense of the Weiss–Tabor–Carnevale method; they also proved that (2) is invariant under Möbius transformation; therefore, if W is a nonnull solution of (2) then $1/W$ is also a solution of (2).

The Calogero–Bogoyavlenskii–Schiff (CBS) equation.

$$u_{xt} + u_x u_{xz} + \frac{1}{2}u_z u_{xx} + \frac{1}{4}u_{xxxz} = 0 \tag{3}$$

can directly be obtained from the KdV equation by the method of extending the corresponding Lax pair. The CBS equation is equivalent to the Ablowitz–Kaup–Newell–Segur (AKNS) equation

$$4h_{xt} + h_{xxxz} + 8h_{xz}h_x + 4h_z h_{xx} = 0. \tag{4}$$

Bogoyavlenskii [2] found overturning solutions for a CBS equivalent equation. These solutions include functions which are solutions of the well known Riemann wave equation $\Phi_t + \Phi\Phi_z = 0$, that is also called the inviscid Burgers equation and has a great interest in hydrodynamics and traffic flow. Solutions of the Riemann wave equation have overturning (whiplash) phenomena in the wave front for all nonconstant solutions, which become multiple valued [16]. Some aspects of this class of solutions has been studied by Whitham in [19]. Although overturning solutions do not frequently appear in the literature, they have been found for some nonlinear coupled systems and the breaking soliton equation [6,7].

By means of the transformations

$$W = \phi_x, \quad \phi = \exp(\psi), \quad \psi_x = u, \quad \psi_t = v, \tag{5}$$

the $(2+1)$ SKdV equation (2) can be transformed into the system

$$\begin{aligned} &4u^2 v_x - 4uu_x v + u^2 u_{xxz} - uu_{xx}u_z - 3uu_x u_{xz} + 3u_x^2 u_z - u^4 u_z = 0, \\ &u_t - v_x = 0. \end{aligned} \tag{6}$$

Through Miura transform

$$h_x = \frac{u_{xx}}{4u} - \frac{3u_x^2}{8u^2} - \frac{u^2}{8}, \quad h_z = -\frac{v}{u}, \tag{7}$$

system (6) is related to the Ablowitz–Kaup–Newell–Segur (AKNS) equation.

Some explicit solutions of (2) has been found [3,5,15]: travelling waves, kinks, solitons and multi-solitons, etc. In this paper, we focus our attention in obtaining new classes of solutions for (2). The overturning and intertwining phenomena have not been studied for the known solutions of $(2+1)$ SKdV equation. So, to widen the family of exact solutions, we have used the nonclassical method of symmetry reductions [1,13,14]. We have obtained and studied several families of new solutions for Eq. (2) that exhibit overturning phenomena. A class of our new solutions can be transformed into the solutions found by Bogoyavlenskii [2] for (3). Some of our solutions do also exhibit intertwining phenomena among the branches of solutions.

2. Nonclassical symmetries of the SKdV in (2 + 1)-dimensions

In order to obtain new solutions of Eq. (2), we use transformations (5) and we apply the nonclassical method of reduction to system (6).

We consider a one-parameter Lie group of infinitesimal transformations in (x,z,t,u,v) and the associated Lie algebra of infinitesimal symmetries of the form

$$G = X\partial_x + Z\partial_z + T\partial_t + U\partial_u + V\partial_v, \tag{8}$$

where X, Z, T, U and V depend on (x,z,t,u,v). We require that (8) leaves invariant system (6) and the invariant surface conditions

$$Xu_x + Zu_z + Tu_t - U = 0, \quad Xv_x + Zv_z + Tv_t - V = 0. \tag{9}$$

This yields to an overdetermined nonlinear system of equations for the infinitesimals X, Z, T, U and V. There are several cases to consider:

Case 1. $T \neq 0$. We can set $T = 1$ without loss of generality and we obtain

$$\begin{aligned} &X = \alpha(t)x + \beta(t), \quad Z = \eta(z,t), \quad T = 1,\\ &U = -\alpha u, \quad V = \frac{v\eta_t}{\eta} + u\left((\alpha x + \beta)\frac{\eta_t}{\eta} - (\alpha_t x + \beta_t)\right), \end{aligned} \tag{10}$$

where $\beta = \beta(t)$ is an arbitrary function and α and η must satisfy the following equations:

$$\eta_t + \eta\eta_z + 2\alpha\eta = 0, \quad \alpha\eta_z + \alpha_t + 2\alpha^2 = 0. \tag{11}$$

We can distinguish two subcases:

1.1. If $\alpha(t) \neq 0$ then system (11) implies that

$$\alpha = \frac{c_3 - c_1}{2(c_3 t + c_4)}, \quad \eta = \frac{c_1 z + c_2}{c_3 t + c_4}.$$

This reduction has been obtained by Lie classical symmetries [15] and gives known solutions.

1.2. If $\alpha(t) \equiv 0$ then the first equation in (11) becomes the well known Riemann wave equation $\eta_t + \eta\eta_z = 0$; the second equation becomes the identity $0 = 0$.

By solving the corresponding characteristic equation, we obtain the reductions

$$u = f(w,\eta), \quad v = \eta_t g(w,\eta) - \gamma_t f(w,\eta), \quad w = x - \gamma, \tag{12}$$

where $\gamma(t) = \int \beta(t)\,dt$ and the functions $f(w,\eta)$ and $g(w,\eta)$ satisfy the following system of partial differential equations in $(1+1)$ dimensions

$$\begin{aligned} &f^4 f_\eta - 3f_\eta f_w^2 + f(f_w(-4\eta g + 3f_{w\eta}) + f_\eta f_{ww}) + f^2(4\eta g_w - f_{ww\eta}) = 0,\\ &g_w - f_\eta = 0. \end{aligned} \tag{13}$$

We now obtain some exact solutions for the reduced system (13).

1.2.I. If we set $f = \sqrt{\epsilon\eta}h(w)$ and $g = k(w)/\sqrt{\epsilon\eta}$, system (13) becomes

$$h^4 - 8kh_w + 8hk_w = 0, \quad 2k_w - \epsilon h = 0, \tag{14}$$

where $\epsilon^2 = 1$. System (14) is equivalent to the ordinary differential equation (ODE)

$$\epsilon(k_w^2 - kk_{ww}) + k_w^4 = 0. \tag{15}$$

This autonomous equation admits solutions that are implicitly defined by a rather complicated equation. However, if $\epsilon = -1$, some particular solutions for Eq. (15) are of the form $k = \pm w + c_1$, so $h = \pm 2$. Consequently, Eq. (2) has solutions of the form

$$W = c(z,t)\rho(z,t)\exp(\pm 2\rho(z,t)(x + a(t))), \tag{16}$$

where $\rho = \sqrt{-\eta}$ satisfies the Riemann equation $\rho_t - \rho^2\rho_z = 0$ and $c(z,t)$ is an arbitrary smooth function.

1.2.II. If we set $f = \rho h(y)$, $g = w^2 k(y)$, $\epsilon^2 = 1$, $y = \sqrt{\epsilon\eta}w$ and $\rho = \sqrt{\epsilon\eta}$ then system (13) becomes

$$\begin{aligned} &h^5 + yh^4h_y - 3yh_y^3 + hh_y(3h_y + 4y(-2yk + h_{yy})) + h^2(-2h_{yy} + y(16k + 8yk_y - h_{yyy})) = 0,\\ &2y^2k_y + 4yk - \epsilon(h + yh_y) = 0. \end{aligned} \tag{17}$$

This system is equivalent to the ODE

$$4\epsilon h^3 + h^5 + yh^4 h_y - 3yh_y^3 + hh_y(-8c_1 + 3h_y + 4yh_{yy}) - h^2(2h_{yy} + yh_{yyy}) = 0. \tag{18}$$

By integrating this equation once with respect to y, Eq. (18) can be reduced to

$$\epsilon(2y^2h^2h_{yy} - 3y^2hh_y^2 - y^2h^5) - 4y^2h^3 + 2c_2h^3 + 4c_1(yhh_{yy} - yh_y^2 + hh_y - yh^4 - 4\epsilon yh^2) = 0. \tag{19}$$

If we set $c_1 = 0$, (19) has solutions in terms of Bessel functions.

Some particular solutions of (19) are

(i) for $\epsilon = 1$ and $c_1 = 0$, $h = \pm 2\csc(2y)$;
(ii) for $\epsilon = 1$ and $c_1 = \frac{\mp 1}{2}$, $h = \frac{\pm\tanh(y)}{-1+y\coth(y)}$;
(iii) for $\epsilon = -1$ and $c_1 = 0$, $h = \pm 2(\sinh(2y))^{-1}$;
(iv) for $\epsilon = -1$ and $c_1 = \frac{\pm 1}{2}$, $h = \frac{\pm\tan^2(y)}{-y+\tan(y)}$.

Let us recall that if W is a solution of (2) then $1/W$ is a solution of (2). By considering the transformations (5) as well as the reductions (12), we obtain, from solutions (i) and (ii), the following solutions of (2)

$$W = c(z)\rho^{\pm 1}(z,t)\cos^{\mp 2}(\rho(z,t)(x + a(t))), \tag{20}$$

$$W = c(z)\rho^{\pm 1}(z,t)\sin^{\mp 2}(\rho(z,t)(x + a(t))), \tag{21}$$

$$W = c(z)\tanh^{\pm 2}(\rho(z,t)(x + a(t))), \tag{22}$$

$$W = c(z)\rho^{\pm 2}(z,t)(1 - \rho(z,t)(x + a(t))\coth(\rho(z,t)(x + a(t))))^{\mp 2}, \tag{23}$$

where a, c are arbitrary smooth functions and $\rho(z,t)$ satisfies the Riemann equation $\rho_t + \rho^2\rho_z = 0$.

Similarly, from solutions (iii) and (iv) of (19), we obtain the solutions

$$W = c(z)\rho^{\pm 1}(z,t)\cosh^{\mp 2}(\rho(z,t)(x + a(t))), \tag{24}$$

$$W = c(z)\rho^{\pm 1}(z,t)\sinh^{\mp 2}(\rho(z,t)(x + a(t))), \tag{25}$$

$$W = c(z)\tan^{\pm 2}(\rho(z,t)(x + a(t))), \tag{26}$$

$$W = c(z)\rho^{\pm 2}(z,t)(1 - \rho(z,t)(x + a(t))\cot(\rho(z,t)(x + a(t))))^{\mp 2}, \tag{27}$$

where a, c are arbitrary smooth functions and $\rho(z,t)$ satisfies the Riemann equation $\rho_t - \rho^2\rho_z = 0$.

1.2.III. If we set $f = h(y), g(w) = b_\eta(h(y) + \frac{c_1}{\eta})$, $y = w + b(\eta)$ and b arbitrary, (13) becomes

$$h_y(-h^4 + 3h_y^2 + 4h(c_1 - h_{yy})) + h^2h_{yyy} = 0. \tag{28}$$

This autonomous equation can be reduced to a second order ODE whose solutions can be expressed in terms of elliptic functions.

If $c_1 = 0$ some particular solutions are $h = \pm c_2\csc(c_2y)$, $h = \pm c_2(\sinh(c_2y))^{-1}$ and $h = \pm c_2\sec(c_2y)$, where c_2 is constant. From some of these solutions of (28), by setting $c_2 = 2$ and by considering the transformations (5), we obtain the following solutions of (2)

$$W = c(z)\sin^{\pm 2}(x + a(t) + b(\eta)), \tag{29}$$

$$W = c(z)\cosh^{\pm 2}(x + a(t) + b(\eta)), \tag{30}$$

$$W = c(z)\sinh^{\pm 2}(x + a(t) + b(\eta)), \tag{31}$$

where c, a, b are arbitrary smooth functions and $\eta(z,t)$ is a solution of the Riemann equation $\eta_t + \eta\eta_z = 0$.

Case 2. $T \equiv 0$. We can distinguish two subcases:

2.1. If $T \equiv 0$ and $Z \neq 0$ then, without loss of generality, we can set $Z = 1$.

By solving the determining equations, we find the following infinitesimals:

2.1.1

$$X = \frac{-x}{2z + c_1}, \quad Z = 1, \quad T = 0, \quad U = \frac{u}{2z + c_1}, \quad V = 0. \tag{32}$$

The similarity variables and solutions are given by

$$w = x^2(2z + c_1), \quad u = \sqrt{2z + c_1}\,f(w,t), \quad v = g(w,t). \tag{33}$$

These reductions have already been obtained in [15].

2.1.2

$$X = 0, \quad Z = 1, \quad T = 0, \quad U = \frac{u}{2z + c_1}, \quad V = \frac{v}{2z + c_1}. \tag{34}$$

The corresponding similarity solutions can be written as

$$u = 2\sqrt{2z + c_1} f(x,t), \quad v = 2\sqrt{2z + c_1} g(x,t). \tag{35}$$

This transformation reduces system (6) to system

$$\begin{aligned} &f^4 + g f_x - f g_x = 0, \\ &f_t - g_x = 0. \end{aligned} \tag{36}$$

Since $g = \frac{-f^4 + f f_t}{f_x}$, system (36) is reduced to the equation

$$(f^3 - f_t) f_{xx} - 4f^2 f_x^2 + f_x f_{xt} = 0. \tag{37}$$

We now obtain several exact solutions of Eq. (37), which lead to solutions of Eq. (2), by setting some specific ansatzs.

(i) Eq. (37) admits solutions of the form $f = \pm t^{-1}\sqrt{F(w) - t/2}$ where $w = x/\sqrt{t} + a(t)$ and F satisfies $5F_w^2 - 2FF_{ww} = 0$. This equation can easily be integrated and the corresponding exact solution of (2) is

$$W = c(z)\sqrt{t^{-1}(-1 + c_2(tx)^{-2/3})} \exp\left(t^{-3/2}\sqrt{2(2z + c_1)}(c_2 - (tx)^{2/3})^{3/2}\right), \tag{38}$$

where c_1 and c_2 are arbitrary constants.

(ii) A second solution of (37) has the form $f = F(w)$, where $w = x + a(t)$ and F satisfies the equation $F_{ww}F - 4F_w^2 = 0$. Clearly, this equation admits the solutions $F = k_2(3w + k_1)^{-1/3}$, where k_1 and k_2 are arbitrary constants. These solutions give us solutions of Eq. (2) of the form

$$W = \frac{c(z)\exp\left(3k_4\sqrt{2z + c_1}(2k_4^2 t + (x + a(t))^{2/3})\right)}{\sqrt[3]{x + a(t)}}, \tag{39}$$

where a, c are arbitrary smooth functions and c_1, k_4 are arbitrary constants.

(iii) Eq. (37) does also admit solutions of the form $f = F(w)(t + k_1)^{-1/2}$ where $w = (x + a(t))(t + k_1)^{-n}$ and F satisfies $F''(F + 2F^3) - F'^2(8F^2 + 2n + 1) = 0$. This equation admits solutions in terms of hypergeometrics functions and, for $n = \frac{3}{2}$, we again obtain solutions (39).

2.2. If $T \equiv 0$, $Z \equiv 0$ and $X \neq 0$ then we can set $X = 1$, without loss of generality. By solving the determining equations, we find the infinitesimals:

$$X = 1, \quad Z = 0, \quad T = 0, \quad U = 0, \quad V = h(z,t,u), \tag{40}$$

where h is an arbitrary function.

The similarity solutions $u = f(z,t)$, $v = xh(z,t) + g(z,t)$ allow us to reduce system (6) to the following system:

$$\begin{aligned} &h = f_t, \\ &4f_t - f^2 f_z = 0. \end{aligned} \tag{41}$$

By means of the transformations (5), we obtain the following solution of the Eq. (2):

$$W = c(z,t) f(z,t) \exp(x f(z,t)), \tag{42}$$

where c is an arbitrary function and f satisfies the Riemann equation $4f_t - f^2 f_z = 0$.

3. Solutions of Riemann equations

In Section 2, we have obtained several solutions of Eq. (2) that depend on the solutions of Riemann equations of the form

$$\eta_t + k\eta^n \eta_z = 0, \tag{43}$$

where $k \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$.

It must be observed that by means of the change of the dependent variable

$$\rho = k\eta^n, \tag{44}$$

Eq. (43) becomes

$$\rho_t + \rho\rho_z = 0. \tag{45}$$

Hence, if η is a solution of (43) then $\rho = k\eta^n$ is a solution of (45). Conversely, if n is odd and ρ is a solution of (45) then $\eta = \left(\frac{\rho}{k}\right)^{1/n}$ is a solution of (43). When n is even and $k > 0$ (resp. $k < 0$) and ρ is a positive (resp. negative) solution of (45) then $\eta = \left(\frac{\rho}{k}\right)^{1/n}$ is also a solution of (43).

Riemann equations of the form (43) have been widely studied for several reasons. From the mathematical point of view, Eq. (43) is a simple nonlinear equation that rises multiple-valued solutions. The dynamical system associated to Eq. (43) is

$$\begin{cases} \frac{dz}{ds} = k\eta^n, \\ \frac{dt}{ds} = 1, \\ \frac{d\eta}{ds} = 0. \end{cases} \tag{46}$$

If h is some smooth function defined in some open interval of $\mathbb{R}$ then the solution of system (46) that satisfies the initial condition $(z(0), t(0), \eta(0)) = (u, 0, h(u))$ is given by

$$(z(s), t(s), \eta(s)) = (kh(u)^n s + u, s, h(u)). \tag{47}$$

Most of the properties of solutions of (43) can be derived from (47). For instance, (47) implies that, as $t = s$ varies, the points of the graph of η move parallel to the z axis with constant speed $kh(u)^n = k\eta^n$. This means that any solution $\bar{\eta}$ of (43) satisfies

$$\bar{\eta}(z, t) = \bar{\eta}(z - k\bar{\eta}(z, t)^n t, 0). \tag{48}$$

Therefore, the solutions can become multiple-valued and the phenomenon of the overturning (whiplash) of the wave front occurs in all nonconstant solutions of our Riemann equations.

Eq. (47) can also be used to obtain a parametric representation of the solutions of (43). By eliminating s and u in (47), we obtain that the solution η of (43) that satisfies the initial condition $\eta(z, 0) = h(z)$ is implicitly given by

$$\eta = h(z - k\eta^n t). \tag{49}$$

In order to show some qualitative aspects of our solutions, we must choose specific initial conditions for (43). Clearly any initial condition of the form $\eta(z, 0) = C$, where C is a constant, gives a constant solution of (43); therefore, we will mainly consider nonconstant initial-conditions. Since the graph points of any solution η move parallel to the z axis with constant speed $kh(u)^n = k\eta^n$, the points that correspond to local maxima of $|h|$ move faster than the neighboring points; if h is not bounded there are points in the graph that move with an arbitrary large speed. This leads us to consider only functions h that are bounded. The simplest cases to consider are those where h is positive and attains an unique local maximum or h is monotonous. To fix our ideas we will consider the initial conditions

$$\eta(z, 0) = \frac{1}{z^2 + 1}, \tag{50}$$

$$\eta(z, 0) = 1 + \tanh(z). \tag{51}$$

With these simple initial conditions we can show most of the qualitative aspects of our solutions. The analysis of solutions that correspond to another initial conditions could be done by following the lines we now sketch. Of course, we could consider many other sets of initial conditions. If, for instance, $|\eta(z, 0)|$ attains two local maxima then there are intervals, of the variable t, where $\eta(z, t)$ is 3-valued and intervals where $\eta(z, t)$ is 5-valued. Similarly, if $|\eta(z, 0)|$ has an infinite number of local maxima then there are intervals where $\eta(z, t)$ is $3, 5, 7, \ldots$-valued.

For further reference, we will study the multiple-valuedness of the solutions in three different cases, appearing in Section 2. We first consider the Riemann equation

$$\eta_t - \eta^2\eta_z = 0. \tag{52}$$

We can choose any initial condition $\eta(z,0) = h(z)$ for Eq. (52); the corresponding solutions are implicitly given by (49), with $k = -1$ and $n = 2$.

We first consider the initial condition (50). The solutions of (50)–(52) are implicitly given by

$$G(z,t,\eta) \equiv \eta - \frac{1}{(\eta^2 t + z)^2 + 1} = 0. \tag{53}$$

The implicit function theorem cannot be applied, to obtain η as a function of (z,t), around the points (z,t,η) that satisfy the system $G(z,t,\eta) = 0$, $G_\eta(z,t,\eta) = 0$. This system is given by (53) and the equation

$$1 + \frac{4(\eta^2 t + z)\eta t}{[(\eta^2 t + z)^2 + 1]^2} = 0. \tag{54}$$

System (53) and (54) can be considered with unknowns z and η, as functions of t. It can be checked that for $|t| < t_0 = \frac{54}{25\sqrt{5}}$ this system has no solutions and that for $|t| > t_0$ the system has exactly two real solutions. These two values of z, as functions of t, are represented in Fig. 1(left). The parametric equations of the curve are given by

$$(t(s), z(s)) = \left(-\frac{(1+s^2)^3}{4s}, \frac{1}{4s} + \frac{5}{4}s\right).$$

It is clear that $z(t_0) = -\frac{\sqrt{5}}{2}$. The points P of the graph are bifurcation points. The branch containing P divides a small open neighborhood of P in two open regions; in one of them the problem (50)–(52) has only one solution and in the other region this problem has three solutions. In Fig. 1(right) we represent the combined graph of the solutions for several values of t, indicated in the figure.

If we now consider the initial condition (51) the results are similar; the bifurcation diagram that correspond to Fig. 1(left) does only contain a graph like the right part of that figure. In this case, the parametric equations of the curve are

$$(t(s), z(s)) = \left(\frac{\cosh^2 s}{2(1 + \tanh s)}, s - \frac{e^s}{2}\cosh s\right).$$

Now, in a neighborhood of $(\bar{z}, \bar{t})$ with $\bar{t} < t_0 = 27/64$ the problem (51) and (52) has an unique solution defined in $\mathbb{R}$ and when $\bar{t} > t_0$ there is an interval of z where the problem (51) and (52) has three branches of solutions. It is clear that $z_0 = z(t_0) = \frac{1}{4}(\ln 4 - 3)$.

Let us observe that equation $\eta_t + \eta^2\eta_z = 0$ is equivalent to Eq. (52) through the change $t \leftrightarrow -t$.

Our third case study will be the Riemann equation

$$\eta_t + \eta\eta_z = 0. \tag{55}$$

For this equation we can also consider any initial condition $\eta(z,0) = h(z)$. The corresponding solutions are implicitly given by equation $\eta = h(z - \eta t)$. We first consider, as before, initial condition (50). In this case the solutions of (50)–(55) are implicitly given by equation $\eta = \frac{1}{(z-\eta t)^2+1}$. It can be checked, as before, that for $|t| > t_1 = \frac{8}{3\sqrt{3}}$ Eq. (55) has three branches of solutions. The existence domain of these three branches and the graphs of the solutions of (55), for several values of t, are similar to those appearing in Fig. 1.

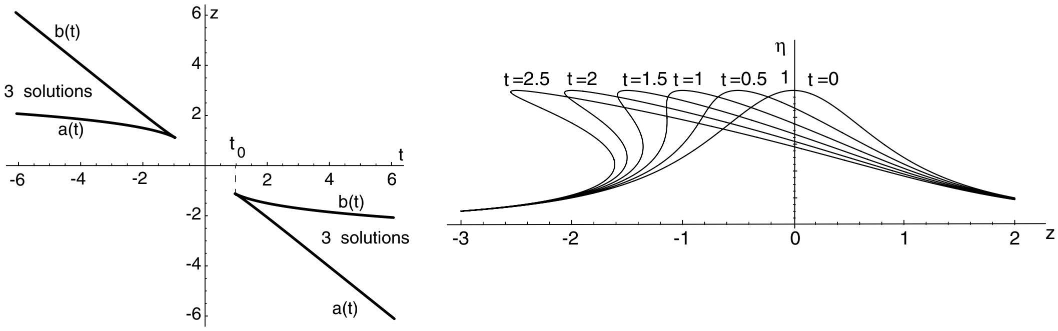

Fig. 1. Bifurcation points for the solutions of $G(z,t,\eta) \equiv \eta - \frac{1}{(\eta^2 t + z)^2 + 1} = 0$ (left). Combined graphs of the solutions of $G(z,t,\eta) = 0$ (right).

The analysis of the problem (51)–(55) is similar. Now, for $t < 1$ the problem (51)–(55) has an unique solution defined in $\mathbb{R}$; for $t > 1$ there is an interval of z where the problem (51)–(55) has three branches of solutions.

4. Multiple solutions

In this section we will study the behavior of some of the solutions of (2) we have found. We will only consider three types of solutions; the properties of the remaining solutions can be derived by following similar arguments.

(I) Solution of type (24):

$$W_1(x,z,t) = c(z)\rho(z,t)\cosh^{-2}(\rho(z,t)(x+a(t))), \tag{56}$$

where $\rho_t - \rho^2\rho_z = 0$. It must be observed that when $\rho =$ constant the solution (56) corresponds to the classic soliton. Clearly, ρ modulates the amplitude and the wave speed.

(II) Solution of type (22):

$$W_2(x,z,t) = c(z)\tanh^2(\rho(z,t)(x+a(t))), \tag{57}$$

where $\rho_t + \rho^2\rho_z = 0$. When $\rho =$ constant, this solution corresponds to a *kink–antikink*. The role of ρ in (57) is to modulate the speed of the wave.

(III) Solution of type (30):

$$W_3(x,z,t) = c(z)\cosh^{-2}(x+a(t)+b(\rho)), \tag{58}$$

where $\rho_t + \rho\rho_z = 0$. When $\rho =$ constant, this solution corresponds to a classic soliton. When $b(\rho)$ is not constant, $b(\rho)$ determines the displacement of the wave.

The behavior of these solutions will be studied when $c(z) \equiv 1$, $a(t) \equiv 0$ and $b(\rho) = \rho$. The initial condition we will consider for these three types of solutions is $\rho(z,0) = \frac{1}{z^2+1}$; later, we will comment some aspects of the solutions that corresponds to the initial condition $\rho(z,0) = 1 + \tanh(z)$. With these conditions, let us observe that, for $i = 1,2,3$, the dependence of W_i on (z,t) is through $\rho(z,t)$. Hence we can also write $W_1 = H_1(x,\rho) = \rho\cosh^{-2}(\rho x)$, $W_2 = H_2(x,\rho) = \tanh^2(\rho x)$ and $W_3 = H_3(x,\rho) = \cosh^{-2}(x+\rho)$. It is clear that W_1 and W_2 are even functions of the variable x.

I. For a fixed $x \in \mathbb{R}$ we have

$$\frac{\partial H_1}{\partial \rho}(x,\rho) = \cosh^{-2}(\rho x)[1 - 2\rho x\tanh(\rho x)].$$

We denote by y_0 the unique positive solution of equation $1 - 2y\tanh(y) = 0$. It is clear that $1 - 2y\tanh(y) > 0$ for $0 < y < y_0$ and that $1 - 2y\tanh(y) < 0$ for $y > y_0$. The sign of $\frac{\partial H_1}{\partial \rho}(x,\rho)$ and, therefore, the growing of H_1 with respect to ρ depends on the product $x\rho(z,t)$.

Let (z,t) be such that $t > t_0$ and $z \in]a(t), b(t)[$ (see Fig. 1). Then in a neighborhood of (z,t) the Riemann equation (52) has three solutions that will be denoted by ρ_i, $i \in \{1,2,3\}$, with $\rho_1(z,t) \leqslant \rho_2(z,t) \leqslant \rho_3(z,t)$. The corresponding solutions of Eq. (1) will be denoted by $W_1^{(i)}(x,z,t) = \rho_i(z,t)\cosh^{-2}(\rho_i(z,t)x)$, $i \in \{1,2,3\}$. Let us observe that $W_1^{(i)}$ are even functions of the variable x. It is clear that for any $x > 0$ we have that $x\rho_1(z,t) \leqslant x\rho_2(z,t) \leqslant x\rho_3(z,t)$. For $x > 0$ sufficiently small we have that $x\rho_3(z,t) \leqslant y_0$ and therefore $W_1^{(1)}(x,z,t) \leqslant W_1^{(2)}(x,z,t) \leqslant W_1^{(3)}(x,z,t)$. However, if $x > 0$ is such that $y_0 < x\rho_1(z,t)$ then the relative order among the three branches is reversed: we have $W_1^{(1)}(x,z,t) \geqslant W_1^{(2)}(x,z,t) \geqslant W_1^{(3)}(x,z,t)$. This change is produced through three intermediate steps:

- $W_1^{(1)}(x,z,t) < W_1^{(3)}(x,z,t) < W_1^{(2)}(x,z,t)$. This may occur when $x\rho_1(z,t) < x\rho_2(z,t) < y_0 < x\rho_3(z,t)$ and when $x\rho_1(z,t) < y_0 < x\rho_2(z,t) < x\rho_3(z,t)$.
- $W_1^{(3)}(x,z,t) < W_1^{(1)}(x,z,t) < W_1^{(2)}(x,z,t)$. This may occur when $x\rho_1(z,t) < x\rho_2(z,t) < y_0 < x\rho_3(z,t)$ and when $x\rho_1(z,t) < y_0 < x\rho_2(z,t) < x\rho_3(z,t)$.
- $W_1^{(3)}(x,z,t) < W_1^{(2)}(x,z,t) < W_1^{(1)}(x,z,t)$. This may occur when $x\rho_1(z,t) < y_0 < x\rho_2(z,t) < x\rho_3(z,t)$ and when $y_0 < x\rho_1(z,t) < x\rho_2(z,t) < x\rho_3(z,t)$.

Therefore, we have an intertwining effect among the branches of solutions. In Fig. 2 we represent former situation for a fixed (z,t) such that $t > t_0$ and $z \in]a(t), b(t)[$. In Fig. 3(left) we represent W_1 for $t = 0$. The right part of Fig. 3

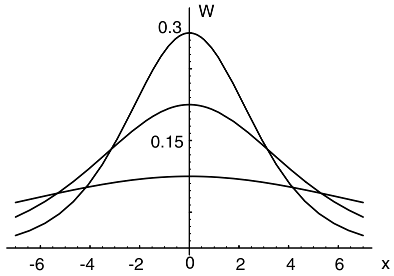

Fig. 2. Branches of the solutions of the form $W = \rho\cosh^{-2}(\rho x)$, with $\rho_t - \rho^2\rho_z = 0$ and $\rho(z, 0) = 1/(z^2 + 1)$, for a fixed (z, t) with $t > t_0$.

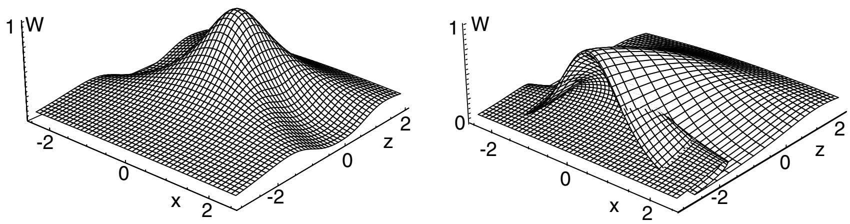

Fig. 3. Graphs of $W = \rho\cosh^{-2}(\rho x)$, with $\rho_t - \rho^2\rho_z = 0$ and $\rho(z,0) = 1/(z^2 + 1)$, for $t = 0$ and $t = 2.5$.

indicates, for $t = 2.5$, how the three branches of W_1 are assembled; the intertwining phenomenon can clearly be appreciated.

If we consider that ρ satisfies (51) and (52) then we obtain, for $t = 0$ and $t = 1$, the values of W_1 that are represented in Fig. 4. The overturning and intertwining phenomena do also appear. This is a bidimensional structure that is exponentially localized on a line. If, instead of initial condition (50), we consider an initial condition for (52) with several local maxima then the former intertwining phenomena appear several times for the corresponding solution.

II. We now analyze the solution given by $W_2(x, z, t) = c(z)\tanh^2(\rho(z, t)(x + a(t)))$, when $a(t) \equiv 0$, $c(z) \equiv 1$ and ρ is implicitly defined by equation $\rho(z, t) = \frac{1}{(\rho^2 t - z)^2 + 1}$. In this case the multiple-valuedness of W_2 does also appears. However let us check that the different branches of W do not mutually intersect; i.e. there is no intertwining among the branches of solutions. As for **I**, for a fixed (z, t) such that $t > t_0$ and $z \in]a(t), b(t)[$, we denote by $\rho_i(z, t)$, $i \in \{1, 2, 3\}$, the three solutions of the corresponding Riemann equation. We denote by $W_2^{(i)}$, $i \in \{1, 2, 3\}$, the corresponding branches of solutions of Eq. (2). Let us observe that $W_2^{(i)}$ are even functions of the variable x. For any $x > 0$ the auxiliary function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(\alpha) = \tanh^2(\alpha x)$ is increasing, because $g'(\alpha) = 2x\tanh(\alpha x)/\cosh(\alpha x) > 0$. Hence W_2 grows with ρ and

$$W_2^{(1)}(x, z, t) \leqslant W_2^{(2)}(x, z, t) \leqslant W_2^{(3)}(x, z, t).$$

Therefore the branches does not intertwine as x grows. In Fig. 5 we represent the graphs of W for $t = 0$ and $t = 2$. The overturning phenomenon can be appreciated in the right figure.

If we consider that ρ satisfies the equation $\rho_t + \rho^2\rho_z = 0$ and the initial condition (51) then we obtain, for $t = 0$ and $t = 1$, the values of W_2 that are represented in Fig. 6. The overturning phenomenon does also appear:

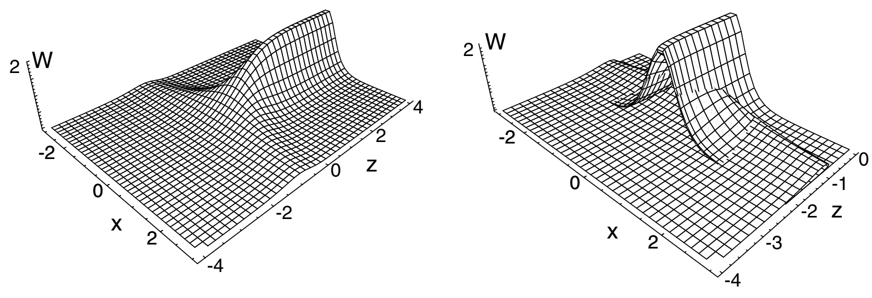

Fig. 4. Graphs of $W = \rho\cosh^{-2}(\rho x)$, with $\rho_t - \rho^2\rho_z = 0$ and $\rho(z,0) = 1 + \tanh(z)$, for $t = 0$ and $t = 1$.

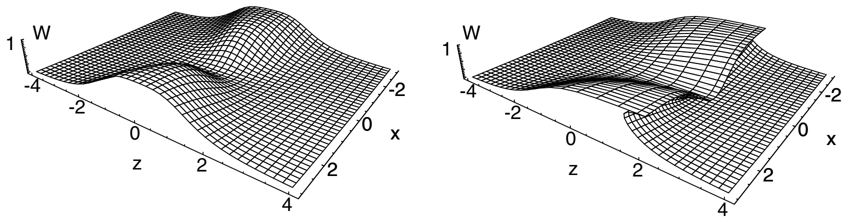

Fig. 5. Graphs of $W = \tanh^2(\rho x)$, with $\rho_t + \rho^2\rho_z = 0$ and $\rho(z,0) = 1/(z^2+1)$, for $t = 0$ and $t = 2$.

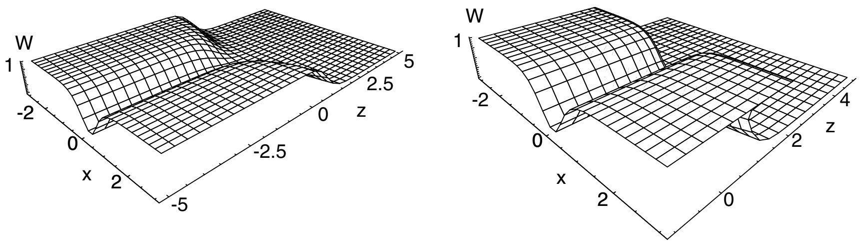

Fig. 6. Graphs of $W = \tanh^2(\rho x)$, with $\rho_t + \rho^2\rho_z = 0$ and $\rho(z,0) = 1 + \tanh(z)$, for $t = 0$ and $t = 1$.

III. We now consider solution $W_3(x,z,t) = c(z)\cosh^{-2}(x + a(t) + b(\rho))$, where $\rho_t + \rho\rho_z = 0$, $a(t) \equiv 0$, $c(z) \equiv 1$ and $b(\rho) = \rho$. In this case $W_3 = H_3(x,\rho) = \cosh^{-2}(x+\rho)$. As for **I** and **II**, if (z,t) is such that $t > t_1 = \frac{8}{3\sqrt{3}}$ and $z \in]a(t), b(t)[$, we denote by $\rho_i(z,t)$, $i = 1,2,3$, the three solutions of the corresponding Riemann equation. We denote by $W_3^{(i)}$, $i \in \{1,2,3\}$, the corresponding branches of solutions of Eq. (2). We observe that for a fixed $x \in \mathbb{R}$ we have $\frac{\partial H_3}{\partial \rho}(x,\rho) = (-2)\cosh^{-3}(x+\rho)\,\sinh(x+\rho)$. Hence $\frac{\partial H_3}{\partial \rho}(x,\rho)$ is positive (resp. negative) for $x + \rho < 0$ (resp. $x + \rho > 0$). Therefore, for a fixed (z,t), W_3 attains a local maximum at $x = -\rho(z,t)$. This fact implies, as in case **I**, that the relative order among the branches of W_3 can vary depending on the values of (x,z), and the intertwining phenomenon appears. This can be appreciated in Fig. 7, where we represent the graphs of W_3 for $t = 0$ and $t = 3$.

If we consider that ρ satisfies (51)–(55) then we obtain, for $t = 0$ and $t = 3$, the values of W_3 that are represented in Fig. 8. The overturning and intertwining phenomena do also appear.

We now comment some additional properties of the solutions we have found. We have chosen $c(z) \equiv 1$ in solutions (56)–(58). We could use nonconstant functions $c(z)$ that rise different behavior. As an example, we have that, for solutions of type (56) with ρ satisfying (50)–(52), the overturning phenomenon appears for $z \in]a(t), b(t)[$ (see the left part of Fig. 1). If we take the function $c = c(z)$ such that $c(z)$ is null for $z \notin [-z_0, z_0]$ the overturning and intertwining phenomena disappear.

We have also chosen $a(t) \equiv 0$ in solutions (56)–(58). Since these three solutions depends on $x + a(t)$, the effect of considering a nonnull function $a(t)$ is to obtain a displacement in x, as t varies.

It is easy to check that, for W of type (56),

$$\int_{-\infty}^{\infty} W(x,z,t)\,dx = \int_{-\infty}^{\infty} c(z)\rho(z,t)\cosh^{-2}(\rho(z,t)(x+a(t)))\,dx = 2c(z),$$

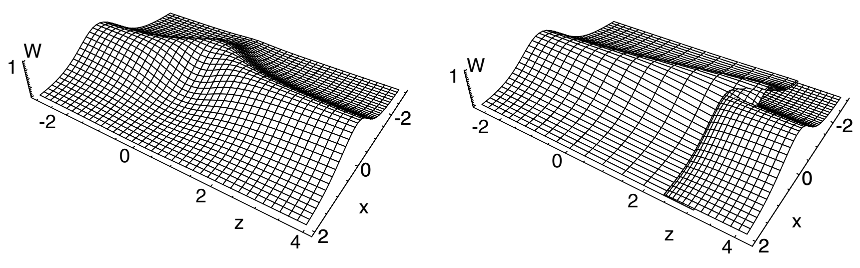

Fig. 7. Graphs of $W = \cosh^{-2}(x+\rho)$, with $\rho_t + \rho\rho_z = 0$ and $\rho(z,0) = 1/(z^2+1)$, for $t = 0$ and $t = 3$.

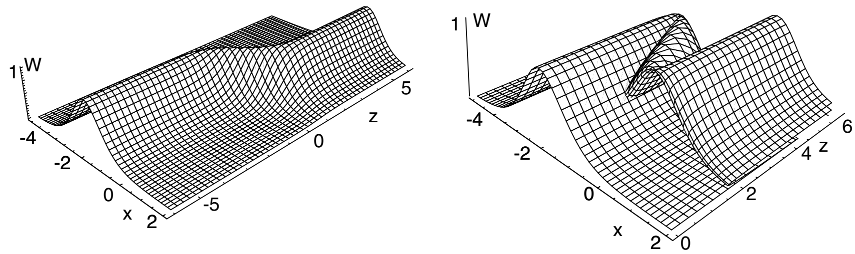

Fig. 8. Graphs of $W = \cosh^{-2}(x + \rho)$, with $\rho_t + \rho\rho_z = 0$ and $\rho(z,0) = 1 + \tanh(z)$, for $t = 0$ and $t = 3$.

i.e. $\int_{-\infty}^{\infty} W(x,z,t)\,\mathrm{d}x$ does not depend on t. Hence, if $\rho(z,t)$ is any solution of equation $\rho_t - \rho^2\rho_t = 0$ defined for $(z,t) \in (z_1,z_2) \times (t_1,t_2)$ then the corresponding solution given by (56) is such that

$$\int_{z_1}^{z_2} \int_{-\infty}^{\infty} W(x,z,t)\,\mathrm{d}x\,\mathrm{d}z = \int_{z_1}^{z_2} c(z)\,\mathrm{d}z \tag{59}$$

for $t \in (t_1,t_2)$. This proves that the mass determined by W on the band $\{(x,z) : (x,z) \in \mathbb{R} \times (z_1,z_2)\}$ is conserved for $t \in (t_1,t_2)$. The solutions given by (58) have the same property.

Finally, let us observe that the overturning solutions of type **II** are related with the overturning solutions found by Bogoyavlenskii [2] for Eq. (3): $u = -2\lambda\tanh(\lambda x)$ with $\lambda_t + 4\lambda^2\lambda_z = 0$, is a solution of (3) that corresponds to the solution $h = \rho(z,t)\tanh(\rho(z,t)x)$, with $\rho_t + \rho^2\rho_z = 0$, of the AKNS Eq. (4). By Miura transform (7), this last solution becomes $W = c(z)\tanh^{-2}(\rho x)$, which is a solution of (2) of type **II**.

Conversely, the solutions $W = c(z)\tanh^{\pm 2}(\rho x)$, $W = c(z)\rho(z,t)\cosh^{-2}(\rho x)$, $W = c(z)\cosh^{-2}(x + f(\rho(z,t)))$ of Eq. (2), with $\rho(z,t)$ satisfying the corresponding Riemann wave equation, give solutions for AKNS Eq. (4) like $h = \rho\tanh^{\mp 1}(\rho\ x) + d(t)$, $h = \frac{-x\rho^2}{2} + d(z,t)$, $h = \frac{-x}{2} + d(z,t)$, where d is an integration constant.

5. Conclusions

For the (2 + 1) SKdV equation we have found several families of solutions. Some of these families depend on two arbitrary functions and an arbitrary solution of a Riemann wave equation. Since the Riemann equations admit multiple-valued solutions, we obtain multiple-valued solutions for the (2 + 1) SKdV equation. This induces a great variety of exotic exponentially localized solutions; for many of them, the phenomena of overturning and/or intertwining among the branches of solutions do appear. These solutions include the corresponding to the overturning solutions found by Bogoyavlenskii for a related equation.

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