

Nonstable K -theory for Graph Algebras

P. Ara · M.A. Moreno · E. Pardo

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Abstract We compute the monoid $V(L_K(E))$ of isomorphism classes of finitely generated projective modules over certain graph algebras $L_K(E)$, and we show that this monoid satisfies the refinement property and separative cancellation. We also show that there is a natural isomorphism between the lattice of graded ideals of $L_K(E)$ and the lattice of order-ideals of $V(L_K(E))$. When K is the field \mathbb{C} of complex numbers, the algebra $L_{\mathbb{C}}(E)$ is a dense subalgebra of the graph C^* -algebra $C^*(E)$, and we show that the inclusion map induces an isomorphism between the corresponding monoids. As a consequence, the graph C^* -algebra of any row-finite graph turns out to satisfy the stable weak cancellation property.

Key words graph algebra · weak cancellation · separative cancellation · refinement monoid · nonstable K -theory · ideal lattice

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P. Ara (✉)
Departament de Matemàtiques, Universitat Autònoma de Barcelona,
08193 Bellaterra (Barcelona), Spain
e-mail: para@mat.uab.cat

M.A. Moreno · E. Pardo
Departamento de Matemáticas, Universidad de Cádiz, Apartado 40,
11510 Puerto Real (Cádiz), Spain

M.A. Moreno
e-mail: mariangeles.moreno@uca.es

E. Pardo
e-mail: enrique.pardo@uca.es

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1 Introduction

The Cuntz–Krieger algebras \mathcal{O}_A , introduced by Cuntz and Krieger [13] in 1980, constitute a prominent class of C^* -algebras. The algebras \mathcal{O}_A were originally associated to a finite matrix A with entries in $\{0, 1\}$, but it was quickly realized that they could also be viewed as the C^* -algebras of a finite directed graph [26]. These C^* -algebras, as well as those arising from various infinite graphs, have been the subject of much investigation (see e.g., [5, 14, 17, 18, 22]). Although Raeburn and Szymański [22] have computed the K_0 and K_1 -groups of a graph C^* -algebra $C^*(E)$ associated with any row-finite graph E , the actual structure of the monoid $V(C^*(E))$ of Murray–von Neumann equivalence classes of projections in matrix algebras over $C^*(E)$ seems to remain unnoticed. One of the goals of this paper is to fill this gap. Another major goal is to show some nice decomposition and cancellation properties of projections over graph C^* -algebras, which also hold for purely algebraic versions of them.

Any graph C^* -algebra $C^*(E)$ is the completion, in an appropriate norm, of a certain $*$ -subalgebra $L_{\mathbb{C}}(E)$, which is just the $*$ -subalgebra generated by the canonical projections and partial isometries that generate $C^*(E)$ as a C^* -algebra. We show that the natural inclusion $\psi: L_{\mathbb{C}}(E) \rightarrow C^*(E)$ induces a monoid isomorphism $V(\psi): V(L_{\mathbb{C}}(E)) \rightarrow V(C^*(E))$ (Theorem 7.1). In this algebraic vein, similar algebras $L_K(E)$ can be constructed over an arbitrary field K , and we show that the monoid $V(L_K(E))$ does not depend on the field K . The algebras $L_K(E)$ have been already considered recently by Abrams and Aranda Pino in [1], under the name of Leavitt path algebras. They provide a generalization of Leavitt algebras of type $(1, n)$, introduced by Leavitt [19] in 1962, just in the same way as graph C^* -algebras $C^*(E)$ provide a generalization of Cuntz algebras.

The decomposition properties of projections in (matrix algebras over) a C^* -algebra A are faithfully reflected in the structure of the monoid $V(A)$. This is an essential ingredient in the so-called nonstable K -theory for C^* -algebras; cf. [7]. A similar statement holds true in Ring Theory, where the monoid $V(R)$ is usually described in terms of the finitely generated projective R -modules; see for example [2] and [4]. For a C^* -algebra A , the two versions of $V(A)$, obtained by viewing A as a C^* -algebra or viewing A as a plain ring, agree, see Section 2. Moreover, important information about the lattice of ideals of a ring R is faithfully codified in the monoid $V(R)$; see for example [15, Theorem 2.1]. The subsets of $V(R)$ corresponding to ideals in R are the so-called order-ideals of $V(R)$, which are the submonoids S of $V(R)$ such that, for $x, y \in V(R)$, we have $x + y \in S$ if and only if $x \in S$ and $y \in S$. Then, [15, Theorem 2.1] asserts that the lattice of all order-ideals of $V(R)$ is isomorphic with the lattice of all *trace ideals* of R .

We consider an Abelian monoid M_E associated with a directed row-finite graph E , and we prove that this monoid is naturally isomorphic with the monoid of isomorphism classes of finitely generated projective modules over $L_K(E)$ (see Theorem 3.5). This uses the nice machinery developed by Bergman in [6] to compute the monoids $V(R)$ of algebras R obtained by means of some universal constructions. We also show that the monoid M_E is naturally isomorphic to $V(C^*(E))$ (Theorem 7.1), and indeed that the natural map $L_{\mathbb{C}}(E) \rightarrow C^*(E)$ induces an isomorphism

$V(L_{\mathbb{C}}(E)) \rightarrow V(C^*(E))$, but our proof of this fact is quite involved, basically because we do not have at our disposal a C^* -version of Bergman’s machinery. Rather, our proof uses the computation in [22] of $K_0(C^*(E))$, which implies that $K_0(C^*(E))$ agrees with the universal group of M_E . We then use techniques from nonstable K -theory to deduce the equality of the monoids $V(C^*(E))$ and M_E . As a consequence of this fact and of our monoid theoretic study of the monoid M_E , we get that $C^*(E)$ always has stable weak cancellation (Corollary 7.2) (equivalently, $C^*(E)$ is separative, see Proposition 2.1). An analogous result holds for all graph algebras $L_K(E)$ (Corollary 6.5). We remark that various stability results for wide classes of rings and C^* -algebras can be proved under the additional hypothesis of separativity; see for example [2, 3, 12, 21].

The most important tools from nonstable K -theory we use are the concepts of refinement and separative cancellation. These properties are faithfully reflected in monoid theoretic properties of the associated monoid $V(A)$. A substantial part of this paper is devoted to establish these properties for the monoid M_E , using just monoid theoretic techniques. Both concepts were defined and studied in [2]. The definitions will be recalled in Section 2.

We now summarize the contents of the rest of sections of the paper. Section 3 contains the definition of the (Leavitt) graph algebras $L_K(E)$ and of the monoid M_E associated with a row-finite graph E . The monoid M_E is isomorphic to F_E/\sim , where F_E is the free Abelian monoid on E^0 and \sim is a certain congruence on F_E . Our basic tool for the monoid theoretic study of M_E is a precise description of this congruence, which is given in Section 4, which also contains the proof of the refinement property of M_E . In Section 5, we shall establish an isomorphism between the lattice \mathcal{H} of saturated hereditary subsets of E^0 , the lattice of order-ideals of M_E , and the lattice of graded ideals of $L_K(E)$. This result parallels [5, Theorem 4.1], where an isomorphism between the lattice of saturated hereditary subsets of E^0 and the lattice of closed gauge-invariant ideals of the C^* -algebra $C^*(E)$ is obtained. The separativity property of the monoid M_E is obtained in Section 6. Finally we show in Section 7 that $V(C^*(E))$ is naturally isomorphic with M_E . This result, together with all properties we have obtained for M_E , enables us to conclude that $C^*(E)$ has stable weak cancellation.

2 Basic Concepts

Our references for K -theory for C^* -algebras are [8] and [24]. For algebraic K -theory, we refer the reader to [25]. For a unital ring R , let $M_{\infty}(R)$ be the directed union of $M_n(R)$ ($n \in \mathbb{N}$), where the transition maps $M_n(R) \rightarrow M_{n+1}(R)$ are given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. We define $V(R)$ to be the set of isomorphism classes (denoted $[P]$) of finitely generated projective left R -modules, and we endow $V(R)$ with the structure of a commutative monoid by imposing the operation

$$[P] + [Q] := [P \oplus Q]$$

for any isomorphism classes $[P]$ and $[Q]$. Equivalently [8, Chapter 3], $V(R)$ can be viewed as the set of equivalence classes $V(e)$ of idempotents e in $M_{\infty}(R)$ with the operation

$$V(e) + V(f) := V\left(\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right)$$

for idempotents $e, f \in M_\infty(R)$. The group $K_0(R)$ of a unital ring R is the universal group of $V(R)$. Recall that, as any universal group of an Abelian monoid, the group $K_0(R)$ has a standard structure of partially pre-ordered Abelian group. The set of positive elements in $K_0(R)$ is the image of $V(R)$ under the natural monoid homomorphism $V(R) \rightarrow K_0(R)$. Whenever A is a C^* -algebra, the monoid $V(A)$ agrees with the monoid of Murray–von Neumann equivalence classes of projections in $M_\infty(A)$; see [8, 4.6.2 and 4.6.4] or [24, Exercise 3.11]. It follows that the algebraic version of $K_0(A)$ coincides with the operator-theoretic one.

We now review some important decomposition and cancellation properties concerning finitely generated projective modules. In the context of C^* -algebras, these are equivalent to corresponding statements for projections, as in [2, Section 7].

Let $FP(R)$ be the class of finitely generated projective modules over a ring R . We say that $FP(R)$ satisfies the *refinement property* if whenever $A_1, A_2, B_1, B_2 \in FP(R)$ satisfy $A_1 \oplus A_2 \cong B_1 \oplus B_2$, there exist decompositions $A_i = A_{i1} \oplus A_{i2}$ for $i = 1, 2$ such that $A_{1j} \oplus A_{2j} \cong B_j$ for $j = 1, 2$.

It was proved in [2, Proposition 1.2] that every exchange ring satisfies the refinement property. Among C^* -algebras, it is worth to mention that every C^* -algebra with real rank zero [11] satisfies the refinement property. This is a theorem of Zhang [28, Theorem 3.2]. It can also be seen as a consequence of the above-mentioned result on exchange rings, since every C^* -algebra of real rank zero is an exchange ring [2, Theorem 7.2].

An Abelian monoid M is a *refinement monoid* if whenever $a + b = c + d$ in M , there exist $x, y, z, t \in M$ such that $a = x + y$ and $b = z + t$ while $c = x + z$ and $d = y + t$. It is clear that $V(R)$ is a refinement monoid if and only if the class $FP(R)$ satisfies the refinement property. We will show that this is the case when $R = L_K(E)$ or $R = C^*(E)$.

Now we discuss the concept of separative cancellation. We say that a ring R is *separative* in case it satisfies the following property: If $A, B, C \in FP(R)$ satisfy $A \oplus C \cong B \oplus C$ and C is isomorphic to direct summands of both nA and nB for some $n \in \mathbb{N}$, then $A \cong B$.

Many rings are separative. Indeed it is an outstanding open question to determine whether all exchange rings are separative. In the context of C^* -algebras, it is not known whether all C^* -algebras of real rank zero are separative. We will show that all graph C^* -algebras $C^*(E)$ and all Leavitt graph algebras $L_K(E)$ are separative.

This concept is closely related to the concept of weak cancellation, introduced by Brown in [10]. See also [12], where many extremally rich C^* -algebras are shown to have weak cancellation. Following [10] and [12], we say that a C^* -algebra A has *weak cancellation* if any pair of projections p, q in A that generate the same closed ideal I in A and have the same image in $K_0(I)$ must be Murray–von Neumann equivalent in A (hence in I). If $M_n(A)$ has weak cancellation for every n , then we say that A has *stable weak cancellation*. It is an open problem whether every extremally rich C^* -algebra satisfies weak cancellation. By [12, Theorem 2.11], every extremally rich C^* -algebra of real rank zero has stable weak cancellation.

If P and Q are projections in $M_\infty(A)$, we will use the symbol $P \sim Q$ to indicate that they are (Murray–von Neumann) equivalent, that is, there is a partial isometry W in $M_\infty(A)$ such that $W^*W = P$ and $WW^* = Q$. Similarly, we will write $P \lesssim Q$ in case P is equivalent to a projection Q' such that $Q' = Q'Q$. We will write $P \oplus Q$ for the block-diagonal matrix $\text{diag}(P, Q)$, and we will denote by $n \cdot P$ the direct sum of n copies of P .

Proposition 2.1 Let A be a C^* -algebra. Then A has stable weak cancellation if and only if A is separative.

Proof The proof is straightforward, taking into account the following fact: Two projections $P, Q \in M_\infty(A)$ whose respective entries generate the same closed ideal I of A have the same image in $K_0(I)$ if and only if there is a projection $E \in M_\infty(I)$ such that $P \oplus E \sim Q \oplus E$. (Note that a projection E belongs to $M_\infty(I)$ if and only if $E \lesssim n \cdot P$ and $E \lesssim n \cdot Q$ for some $n \geq 1$.) \square

There is a canonical pre-order on any Abelian monoid M , which is sometimes called the algebraic pre-order of M . This pre-order is defined by setting $x \leq y$ if and only if there is $z \in M$ such that $y = x + z$. This is the only pre-order that we will consider in this paper for a monoid.

An Abelian monoid M is said to be *separative* [2] in case M satisfies the following condition: If $a, b, c \in M$ satisfy $a + c = b + c$ and $c \leq na$ and $c \leq nb$ for some $n \in \mathbb{N}$, then $a = b$. It is clear that a ring R is separative if and only if the monoid $V(R)$ is separative.

3 Graph Algebras and Graph Monoids

A directed graph E consists of a vertex set E^0 , an edge set E^1 , and maps $r, s: E^1 \rightarrow E^0$ describing the range and source of edges. We say that E is a *row-finite graph* if each row in its adjacency matrix $A_E = (A(v, w))_{v, w \in E^0}$ has only a finite number of nonzero entries, where $A(v, w)$ is the number of edges going from v to w . This amounts to saying that each vertex in E emits only a finite number of edges.

Let $E = (E^0, E^1)$ be a row-finite graph, and let K be a field. We define the *graph K -algebra* $L_K(E)$ associated with E as the K -algebra generated by a set $\{p_v \mid v \in E^0\}$ together with a set $\{x_e, y_e \mid e \in E^1\}$, which satisfy the following relations:

- (1) $p_v p_{v'} = \delta_{v, v'} p_v$ for all $v, v' \in E^0$.
- (2) $p_{s(e)} x_e = x_e p_{r(e)} = x_e$ for all $e \in E^1$.
- (3) $p_{r(e)} y_e = y_e p_{s(e)} = y_e$ for all $e \in E^1$.
- (4) $y_e x_{e'} = \delta_{e, e'} p_{r(e)}$ for all $e, e' \in E^1$.
- (5) $p_v = \sum_{\{e \in E^1 \mid s(e) = v\}} x_e y_e$ for every $v \in E^0$ that emits edges.

Observe that relation (1) says that $\{p_v \mid v \in E^0\}$ is a set of pairwise orthogonal idempotents. Note also that the above relations imply that $\{x_e y_e \mid e \in E^1\}$ is a set of pairwise orthogonal idempotents in $L_K(E)$. If E is a finite graph then we have $\sum_{v \in E^0} p_v = 1$. In general the algebra $L_K(E)$ is not unital, but it can be written as a direct limit of unital graph algebras (with non-unital transition maps), so that it is an algebra with local units. To show this, we first observe the *functoriality property* of the construction, as follows. Recall that a graph homomorphism $f: E = (E^0, E^1) \rightarrow F = (F^0, F^1)$ is given by two maps $f^0: E^0 \rightarrow F^0$ and $f^1: E^1 \rightarrow F^1$ such that $r_F(f^1(e)) = f^0(r_E(e))$ and $s_F(f^1(e)) = f^0(s_E(e))$ for every $e \in E^1$. We say that a graph homomorphism f is *complete* in case f^0 is injective and f^1 restricts to a bijection from $s_E^{-1}(v)$ onto $s_F^{-1}(f^0(v))$ for every $v \in E^0$ such that v emits edges. Note that under the above assumptions, the map f^1 must also be injective. Let us consider the category \mathcal{G} whose objects are all the row-finite graphs and whose morphisms are

the complete graph homomorphisms. It is easy to check that the category \mathcal{G} admits direct limits. If $\{X_i\}_{i \in I}$ is a directed system in the category \mathcal{G} and $X = \varinjlim_{i \in I} X_i$, let us denote by $\psi_i: X_i \rightarrow X$ the canonical direct limit homomorphisms. Then the graphs $\psi_i(X_i)$ are complete subgraphs of X , $\psi_i(X_i)$ is a complete subgraph of $\psi_j(X_j)$ whenever $i \leq j$, and X is the union of the family of subgraphs $(\psi_i(X_i))_{i \in I}$ (that is, $X^0 = \cup_{i \in I} \psi_i^0(X_i^0)$ and $X^1 = \cup_{i \in I} \psi_i^1(X_i^1)$).

In order to simplify notation, the K -algebra $L_K(E)$ will be sometimes denoted by $L(E)$.

Lemma 3.1 Every row-finite graph E is a direct limit in the category \mathcal{G} of a directed system of finite graphs.

Proof Clearly, E is the union of its finite subgraphs. Let X be a finite subgraph of E . Define a finite subgraph Y of E as follows:

$$Y^0 = X^0 \cup \{r_E(e) \mid e \in E^1 \text{ and } s_E(e) \in X^0\}$$

and

$$Y^1 = \{e \in E^1 \mid s_E(e) \in X^0\}.$$

Then the vertices of Y that emit edges are exactly the vertices of X that emit edges in E , and if v is one of these vertices, then $s_E^{-1}(v) = s_Y^{-1}(v)$. This shows that the map $Y \rightarrow E$ is a complete graph homomorphism, and clearly $X \subseteq Y$. If Y_1 and Y_2 are two complete subgraphs of E and Y_1 is a subgraph of Y_2 , then the inclusion map $Y_1 \rightarrow Y_2$ is clearly a complete graph homomorphism.

Since the union of a finite number of finite complete subgraphs of E is again a finite complete subgraph of E , it follows that E is the direct limit in the category \mathcal{G} of the directed family of its finite complete subgraphs. \square

Lemma 3.2 The assignment $E \mapsto L_K(E)$ can be extended to a functor L_K from the category \mathcal{G} of row-finite graphs and complete graph homomorphisms to the category of K -algebras and (not necessarily unital) algebra homomorphisms. The functor L_K is continuous, that is, it commutes with direct limits. It follows that every graph algebra $L_K(E)$ is the direct limit of graph algebras corresponding to finite graphs.

Proof If $f: E \rightarrow F$ is a complete graph homomorphism, then f induces an algebra homomorphism $L(f): L_K(E) \rightarrow L_K(F)$, as follows. Set $L(f)(p_v) = p_{f^0(v)}$ and $L(f)(x_e) = x_{f^1(e)}$ and $L(f)(y_e) = y_{f^1(e)}$ for $v \in E^0$ and $e \in E^1$. Since f^0 is injective, relation (1) is preserved under $L(f)$. Relations (2), (3) are clearly preserved, relation (4) is preserved because f^1 is injective, and relation (5) is preserved because f^1 restricts to a bijection from $s_E^{-1}(v)$ onto $s_F^{-1}(f^0(v))$ for every $v \in E^0$ such that v emits edges.

The algebra $L_K(E)$ is the algebra generated by a universal family of elements $\{p_v, x_e, y_e \mid v \in E^0, e \in E^1\}$ satisfying relations (1)–(5). If $X = \varinjlim_{i \in I} X_i$ in the category \mathcal{G} , then, as observed above, we can think that $\{X_i\}_{i \in I}$ is a directed family of complete subgraphs of X , and the union of the graphs X_i is X . For a K -algebra A , a compatible set of K -algebra homomorphisms $L_K(X_i) \rightarrow A$, $i \in I$, determines, and is determined by, a set of elements $\{p'_v, x'_e, y'_e \mid v \in E^0, e \in E^1\}$ in A satisfying

conditions (1)–(5). It follows that $L_K(E) = \lim_{\rightarrow i \in I} L_K(X_i)$, as desired. The last statement now follows from Lemma 3.1. \square

The graph C^* -algebra $C^*(E)$ is the C^* -algebra generated by a universal Cuntz–Krieger E -family $\{P_v, S_e \mid v \in E^0, e \in E^1\}$, see [18, Theorem 1.2]. By definition, a Cuntz–Krieger E -family in a C^* -algebra A consists of a set $\{P_v \mid v \in E^0\}$ of pairwise orthogonal projections in A and a set $\{S_e \mid e \in E^1\}$ of partial isometries in A such that

$$S_e^* S_e = P_{r(e)} \quad \text{for } e \in E^1, \quad \text{and} \quad P_v = \sum_{\{e \in E^1 \mid s(e)=v\}} S_e S_e^* \quad \text{for } v \in E^0.$$

Therefore the same proof as in Lemma 3.2 can be applied to the case of C^* -algebras:

Lemma 3.3 The assignment $E \mapsto C^*(E)$ can be extended to a continuous functor from the category \mathcal{G} of row-finite graphs and complete graph homomorphisms to the category of C^* -algebras and $*$ -homomorphisms. Every graph C^* -algebra $C^*(E)$ is the direct limit of graph C^* -algebras associated with finite graphs.

Now we want to compute the monoid $V(L_K(E))$ associated with the finitely generated projective modules over the graph algebra $L_K(E)$. Though $L_K(E)$ is not in general a unital algebra, there is a well-defined monoid $V(L_K(E))$ associated with the finitely generated projective left modules over $L_K(E)$. We recall the general definition here.

Let I be a non-unital K -algebra, and consider any unital K -algebra R containing I as a two-sided ideal. We consider the class $FP(I, R)$ of finitely generated projective left R -modules P such that $P = IP$. Then $V(I)$ is defined as the monoid of isomorphism classes of objects in $FP(I, R)$, and does not depend on the particular unital ring R in which I sits as a two-sided ideal, as can be seen from the following alternative description: $V(I)$ is the set of equivalence classes of idempotents in $M_\infty(I)$, where $e \sim f$ in $M_\infty(I)$ if and only if there are $x, y \in M_\infty(I)$ such that $e = xy$ and $f = yx$. See [20, page 296].

The assignment $I \mapsto V(I)$ gives a functor from the category of non-unital rings to the category of Abelian monoids, that commutes with direct limits. Moreover, $L_K(E)$ being a ring with local units, it is well known that $K_0(L_K(E))$, the K_0 -group of the non-unital ring $L_K(E)$, is just the enveloping group of $V(L_K(E))$; see [20, Proposition 0.1].

Let M_E be the Abelian monoid given by the generators $\{a_v \mid v \in E^0\}$, with the relations:

$$a_v = \sum_{\{e \in E^1 \mid s(e)=v\}} a_{r(e)} \quad \text{for every } v \in E^0 \text{ that emits edges.} \tag{M}$$

Lemma 3.4 The assignment $E \mapsto M_E$ can be extended to a continuous functor from the category \mathcal{G} of row-finite graphs and complete graph homomorphisms to the category of Abelian monoids. It follows that every graph monoid M_E is the direct limit of graph monoids corresponding to finite graphs.

Proof Every complete graph homomorphism $f: E \rightarrow F$ induces a natural monoid homomorphism

$$M(f): M_E \rightarrow M_F,$$

and so we get a functor M from the category \mathcal{G} to the category of Abelian monoids. The fact that M commutes with direct limits is proven in the same way as in Lemma 3.2. \square

Theorem 3.5 *Let E be a row-finite graph. Then there is a natural monoid isomorphism $V(L_K(E)) \cong M_E$. Moreover, if E is finite, then the global dimension of $L_K(E)$ is ≤ 1 .*

Proof For each row-finite graph E , there is a unique monoid homomorphism $\gamma_E: M_E \rightarrow V(L(E))$ such that $\gamma_E(a_v) = [p_v]$. Clearly this defines a natural transformation from the functor M to the functor $V \circ L$; that is, if $f: E \rightarrow F$ is a complete graph homomorphism, then the following diagram commutes

$$\begin{array}{ccc} M_E & \xrightarrow{\gamma_E} & V(L(E)) \\ M(f) \downarrow & & \downarrow V(L(f)) \\ M_F & \xrightarrow{\gamma_F} & V(L(F)) \end{array}$$

We need to show that γ_E is a monoid isomorphism for every row-finite graph E . By using Lemmas 3.4 and 3.2, we see that it is enough to show that γ_E is an isomorphism for a finite graph E .

Let E be a finite graph and assume that $\{v_1, \dots, v_m\} \subseteq E^0$ is the set of vertices which emit edges. We start with an algebra

$$A_0 = \prod_{v \in E^0} K.$$

In A_0 we have a family $\{p_v : v \in E^0\}$ of orthogonal idempotents such that $\sum_{v \in E^0} p_v = 1$. Let us consider the two finitely generated projective left A_0 -modules $P = A_0 p_{v_1}$ and $Q = \oplus_{\{e \in E^1 | s(e)=v_1\}} A_0 p_{r(e)}$. There exists an algebra $A_1 := A_0 \langle i, i^{-1} : \overline{P} \cong \overline{Q} \rangle$ with a universal isomorphism $i: \overline{P} := A_1 \otimes_{A_0} P \rightarrow \overline{Q} := A_1 \otimes_{A_0} Q$, see [6, page 38]. Note that this algebra is precisely the algebra $L(X_1)$, where X_1 is the graph having $X_1^0 = E^0$, and where v_1 emits the same edges as it does in E , but all other vertices do not emit any edge. Namely the row $(x_e : s(e) = v_1)$ implements an isomorphism $\overline{P} = A_1 p_{v_1} \rightarrow \overline{Q} = \oplus_{\{e \in E^1 | s(e)=v_1\}} A_1 p_{r(e)}$ with inverse given by the column $(y_e : s(e) = v_1)^T$, which is clearly universal. By [6, Theorem 5.2], the monoid $V(A_1)$ is obtained from $V(A_0)$ by adjoining the relation $[P] = [Q]$. In our case we have that $V(A_0)$ is the free Abelian group on generators $\{a_v \mid v \in E^0\}$, where $a_v = [p_v]$, and so $V(A_1)$ is given by generators $\{a_v \mid v \in E^0\}$ and a single relation

$$a_{v_1} = \sum_{\{e \in E^1 | s(e)=v_1\}} a_{r(e)}.$$

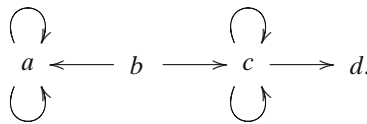
Now we proceed inductively. For $k \geq 1$, let A_k be the graph algebra $A_k = L(X_k)$, where X_k is the graph with the same vertices as E , but where only the first k vertices

v_1, \dots, v_k emit edges, and these vertices emit the same edges as they do in E . Then we assume by induction that $V(A_k)$ is the Abelian group given by generators $\{a_v \mid v \in E^0\}$ and relations

$$a_{v_i} = \sum_{\{e \in E^1 \mid s(e)=v_i\}} a_{r(e)},$$

for $i = 1, \dots, k$. Let A_{k+1} be the similar graph, corresponding to vertices v_1, \dots, v_k, v_{k+1} . Then we have $A_{k+1} = A_k \langle i, i^{-1} : \overline{P} \cong \overline{Q} \rangle$ for $P = A_k p_{v_{k+1}}$ and $Q = \bigoplus_{\{e \in E^1 \mid s(e)=v_{k+1}\}} A_k p_{r(e)}$, and so we can apply again Bergman’s Theorem [6, Theorem 5.2] to deduce that $V(A_{k+1})$ is the monoid with the same generators as before and the relations corresponding to v_1, \dots, v_k, v_{k+1} . It also follows from [6, Theorem 5.2] that the global dimension of $L(E)$ is ≤ 1 . This concludes the proof. \square

Example 3.6 Consider the following graph E :



Then M_E is the monoid generated by a, b, c, d with defining relations $a=2a, b=a+c, c=2c+d$. The Grothendieck group of M_E is infinite cyclic generated by the class of c . It follows that $K_0(L_K(E))$ is infinite cyclic generated by $[p_c]$, and $K_0(L_K(E)) = K_0(L_K(E))^+$.

4 Refinement

In this section we begin our formal study of the monoid M_E associated with a row-finite graph E , and we show that M_E is a refinement monoid. The main tool is a careful description of the congruence on the free Abelian monoid given by the defining relations of M_E .

Let F be the free Abelian monoid on the set E^0 . The nonzero elements of F can be written in a unique form up to permutation as $\sum_{i=1}^n x_i$, where $x_i \in E^0$. Now we will give a description of the congruence on F generated by the relations (M) on F . It will be convenient to introduce the following notation. For $x \in E^0$, write

$$\mathbf{r}(x) := \sum_{\{e \in E^1 \mid s(e)=x\}} r(e) \in F.$$

With this new notation relations (M) become $x = \mathbf{r}(x)$ for every $x \in E^0$ that emits edges.

Definition 4.1 Define a binary relation \rightarrow_1 on $F \setminus \{0\}$ as follows. Let $\sum_{i=1}^n x_i$ be an element in F as above and let $j \in \{1, \dots, n\}$ be an index such that x_j emits edges. Then $\sum_{i=1}^n x_i \rightarrow_1 \sum_{i \neq j} x_i + \mathbf{r}(x_j)$. Let \rightarrow be the transitive and reflexive closure of \rightarrow_1 on $F \setminus \{0\}$, that is, $\alpha \rightarrow \beta$ if and only if there is a finite string $\alpha = \alpha_0 \rightarrow_1 \alpha_1 \rightarrow_1 \dots \rightarrow_1 \alpha_i = \beta$. Let \sim be the congruence on F generated by the relation \rightarrow_1 (or, equivalently,

by the relation \rightarrow). Namely $\alpha \sim \alpha$ for all $\alpha \in F$ and, for $\alpha, \beta \neq 0$, we have $\alpha \sim \beta$ if and only if there is a finite string $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$, such that, for each $i = 0, \dots, n - 1$, either $\alpha_i \rightarrow_1 \alpha_{i+1}$ or $\alpha_{i+1} \rightarrow_1 \alpha_i$. The number n above will be called the *length* of the string.

It is clear that \sim is the congruence on F generated by relations (M), and so $M_E = F/\sim$.

The *support* of an element γ in F , denoted $\text{supp}(\gamma) \subseteq E^0$, is the set of basis elements appearing in the canonical expression of γ .

Lemma 4.2 Let \rightarrow be the binary relation on F defined above. Assume that $\alpha = \alpha_1 + \alpha_2$ and $\alpha \rightarrow \beta$. Then β can be written as $\beta = \beta_1 + \beta_2$ with $\alpha_1 \rightarrow \beta_1$ and $\alpha_2 \rightarrow \beta_2$.

Proof By induction, it is enough to show the result in the case where $\alpha \rightarrow_1 \beta$. If $\alpha \rightarrow_1 \beta$, then there is an element x in the support of α such that $\beta = (\alpha - x) + \mathbf{r}(x)$. The element x belongs either to the support of α_1 or to the support of α_2 . Assume, for instance, that the element x belongs to the support of α_1 . Then we set $\beta_1 = (\alpha_1 - x) + \mathbf{r}(x)$ and $\beta_2 = \alpha_2$. □

Note that the elements β_1 and β_2 in Lemma 4.2 are not uniquely determined by α_1 and α_2 in general, because the element $x \in E^0$ considered in the proof could belong to *both* the support of α_1 and the support of α_2 .

The following lemma gives the important “confluence” property of the congruence \sim on the free Abelian monoid F .

Lemma 4.3 Let α and β be nonzero elements in F . Then $\alpha \sim \beta$ if and only if there is $\gamma \in F$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$.

Proof Assume that $\alpha \sim \beta$. Then there exists a finite string $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$, such that, for each $i = 0, \dots, n - 1$, either $\alpha_i \rightarrow_1 \alpha_{i+1}$ or $\alpha_{i+1} \rightarrow_1 \alpha_i$. We proceed by induction on n . If $n = 0$, then $\alpha = \beta$ and there is nothing to prove. Assume the result is true for strings of length $n - 1$, and let $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$ be a string of length n . By induction hypothesis, there is $\lambda \in F$ such that $\alpha \rightarrow \lambda$ and $\alpha_{n-1} \rightarrow \lambda$. Now there are two cases to consider. If $\beta \rightarrow_1 \alpha_{n-1}$, then $\beta \rightarrow \lambda$ and we are done. Assume that $\alpha_{n-1} \rightarrow_1 \beta$. By definition of \rightarrow_1 , there is a basis element $x \in E^0$ in the support of α_{n-1} such that $\alpha_{n-1} = x + \alpha'_{n-1}$ and $\beta = \mathbf{r}(x) + \alpha'_{n-1}$. By Lemma 4.2, we have $\lambda = \lambda(x) + \lambda'$, where $x \rightarrow \lambda(x)$ and $\alpha'_{n-1} \rightarrow \lambda'$. If the length of the string from x to $\lambda(x)$ is positive, then we have $\mathbf{r}(x) \rightarrow \lambda(x)$ and so $\beta = \mathbf{r}(x) + \alpha'_{n-1} \rightarrow \lambda(x) + \lambda' = \lambda$. In case that $x = \lambda(x)$, then set $\gamma = \mathbf{r}(x) + \lambda'$. Then we have $\lambda \rightarrow_1 \gamma$ and so $\alpha \rightarrow \gamma$, and also $\beta = \mathbf{r}(x) + \alpha'_{n-1} \rightarrow \mathbf{r}(x) + \lambda' = \gamma$. This concludes the proof. □

We are now ready to show the refinement property of M_E .

Proposition 4.4 The monoid M_E associated with any row-finite graph E is a refinement monoid.

Proof Let $\alpha = \alpha_1 + \alpha_2 \sim \beta = \beta_1 + \beta_2$, with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F$. By Lemma 4.3, there is $\gamma \in F$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$. By Lemma 4.2, we can write $\gamma = \alpha'_1 + \alpha'_2 =$

$\beta'_1 + \beta'_2$, with $\alpha_i \rightarrow \alpha'_i$ and $\beta_i \rightarrow \beta'_i$ for $i = 1, 2$. Since F is a free Abelian monoid, F has the refinement property and so there are decompositions $\alpha'_i = \gamma_{i1} + \gamma_{i2}$ for $i = 1, 2$ such that $\beta'_j = \gamma_{1j} + \gamma_{2j}$ for $j = 1, 2$. The result follows. \square

5 Ideal Lattice

Let $E = (E^0, E^1)$ be a row-finite directed graph. In this section, we will establish the connection between saturated hereditary subsets of E^0 , order-ideals of the associated monoid M_E , and graded ideals of the graph algebra $L_K(E)$.

We start by recalling some basic concepts in graph theory, that will be needed later.

Let $E = (E^0, E^1)$ be a directed graph. For $n \geq 2$, we define

$$E^n := \{ \alpha = (\alpha_1, \dots, \alpha_n) \mid \alpha_i \in E^1 \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \text{ for } 1 \leq i \leq n - 1 \},$$

and $E^* = \bigcup_{n \geq 0} E^n$.

We define a relation \geq on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set H is *saturated* if every vertex which feeds into H and only into H is again in H ; that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$, then $v \in H$.

Definition 5.1 Let $v \in E^0$. We define the tree of v , to be the subset of E^0

$$T(v) = \{ w \in E^0 \mid \exists \alpha \in E^* \text{ with } s(\alpha) = v \text{ and } r(\alpha) = w \} = \{ w \in E^0 \mid v \geq w \}.$$

Clearly, the tree of v is the smallest hereditary subset of E^0 containing v .

We denote by \mathcal{H} the set of saturated hereditary subsets of the graph E .

Since the intersection of saturated sets is saturated, there is a smallest saturated subset \bar{S} containing any given subset S of E^0 . We will call \bar{S} the saturation of S . The saturation \bar{H} of a hereditary set H is again hereditary. Indeed, $\bar{H} = \bigcup_{n=0}^\infty \Lambda_n(H)$ is an increasing union of hereditary subsets $\Lambda_n(H)$, for $n \geq 0$, which are defined inductively as follows:

- (1) $\Lambda_0(H) = H$.
- (2) $\Lambda_n(H) = \{ y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(H) \} \cup \Lambda_{n-1}(H)$, for $n \geq 1$.

In particular this applies to the hereditary subsets of the form $T(v)$, where $v \in E^0$: The saturated hereditary subset of E generated by v is $\bar{T}(v) = \bigcup_{n=0}^\infty \Lambda_n(T(v))$.

An *order-ideal* of a monoid M is a submonoid I of M such that $x + y = z$ in M and $z \in I$ imply that both x, y belong to I . An order-ideal can also be described as a submonoid I of M , which is hereditary with respect to the canonical pre-order \leq on M : $x \leq y$ and $y \in I$ imply $x \in I$. Recall that the pre-order \leq on M is defined by setting $x \leq y$ if and only if there exists $z \in M$ such that $y = x + z$.

The set $\mathcal{L}(M)$ of order-ideals of M forms a (complete) lattice $(\mathcal{L}(M), \subseteq, \bar{\sum}, \cap)$.

Here, for a family of order-ideals $\{I_i\}$, we denote by $\bar{\sum} I_i$ the set of elements $x \in M$ such that $x \leq y$, for some y belonging to the algebraic sum $\sum I_i$ of the order-ideals I_i . Note that $\sum I_i = \bar{\sum} I_i$ whenever M is a refinement monoid.

Let F_E be the free Abelian monoid on E^0 , and recall that $M_E = F_E/\sim$. For $\gamma \in F_E$ we will denote by $[\gamma]$ its class in M_E . Note that any order-ideal I of M_E is generated as a monoid by the set $\{[v] \mid v \in E^0\} \cap I$.

The set \mathcal{H} of saturated hereditary subsets of E^0 is also a complete lattice $(\mathcal{H}, \subseteq, \bar{\cup}, \cap)$.

Proposition 5.2 Let E be a row-finite graph. Then, there are order-preserving mutually inverse maps

$$\varphi: \mathcal{H} \longrightarrow \mathcal{L}(M_E); \quad \psi: \mathcal{L}(M_E) \longrightarrow \mathcal{H},$$

where $\varphi(H)$ is the order-ideal of M_E generated by $\{[v] \mid v \in H\}$, for $H \in \mathcal{H}$, and $\psi(I)$ is the set of elements v in E^0 such that $[v] \in I$, for $I \in \mathcal{L}(M_E)$.

Proof The maps φ and ψ are obviously order-preserving. It will be enough to show the following facts:

- (1) For $I \in \mathcal{L}(M_E)$, the set $\psi(I)$ is a hereditary and saturated subset of E^0 .
- (2) If $H \in \mathcal{H}$ then $[v] \in \varphi(H)$ if and only if $v \in H$.

For, if (1) and (2) hold true, then ψ is well-defined by (1), and $\psi(\varphi(H)) = H$ for $H \in \mathcal{H}$, by (2). On the other hand, if I is an order-ideal of M_E , then obviously $\varphi(\psi(I)) \subseteq I$, and since I is generated as a monoid by $\{[v] \mid v \in E^0\} \cap I = [\psi(I)]$, it follows that $I \subseteq \varphi(\psi(I))$.

Proof of (1): Let I be an order-ideal of M_E , and set $H := \psi(I) = \{v \in E^0 \mid [v] \in I\}$. To see that H is hereditary, we have to prove that, whenever we have a path (e_1, e_2, \dots, e_n) in E with $s(e_1) = v$ and $r(e_n) = w$ and $v \in H$, then $w \in H$. If we consider the corresponding path $v \rightarrow_1 \gamma_1 \rightarrow_1 \gamma_2 \rightarrow_1 \dots \rightarrow_1 \gamma_n$ in F_E , we see that w belongs to the support of γ_n , so that $w \leq \gamma_n$ in F_E . This implies that $[w] \leq [\gamma_n] = [v]$, and so $[w] \in I$ because I is hereditary.

To show saturation, take v in E^0 such that $r(e) \in H$ for every $e \in E^1$ such that $s(e) = v$. We then have $\text{supp}(\mathbf{r}(v)) \subseteq H$, so that $[\mathbf{r}(v)] \in I$ because I is a submonoid of M_E . But $[v] = [\mathbf{r}(v)]$, so that $[v] \in I$ and $v \in H$.

Proof of (2): Let H be a saturated hereditary subset of E^0 , and let $I := \varphi(H)$ be the order-ideal of M_E generated by $\{[v] \mid v \in H\}$. Clearly $[v] \in I$ if $v \in H$. Conversely, suppose that $[v] \in I$. Then $[v] \leq [\gamma]$, where $\gamma \in F_E$ satisfies $\text{supp}(\gamma) \subseteq H$. Thus we can write $[\gamma] = [v] + [\delta]$ for some $\delta \in F_E$. By Lemma 4.3, there is $\beta \in F_E$ such that $\gamma \rightarrow \beta$ and $v + \delta \rightarrow \beta$. Since H is hereditary and $\text{supp}(\gamma) \subseteq H$, we get $\text{supp}(\beta) \subseteq H$. By Lemma 4.2, we have $\beta = \beta_1 + \beta_2$, where $v \rightarrow \beta_1$ and $\delta \rightarrow \beta_2$. Observe that $\text{supp}(\beta_1) \subseteq \text{supp}(\beta) \subseteq H$. Using that H is saturated, it is a simple matter to check that, if $\alpha \rightarrow_1 \alpha'$ and $\text{supp}(\alpha') \subseteq H$, then $\text{supp}(\alpha) \subseteq H$. Using this and induction, we obtain that $v \in H$, as desired. □

We next consider ideals in the algebra $L_K(E)$ associated with the graph E . For a general unital ring R , the lattice of order-ideals of $V(R)$ is isomorphic with the lattice of trace ideals of R ; see [4] and [15]. It is straightforward to see that this lattice isomorphism also holds when R is a ring with local units. In particular, the lattice of order-ideals of $V(L_K(E))$ is isomorphic with the lattice of trace ideals of $L_K(E)$. Since $V(L_K(E)) \cong M_E$ is a refinement monoid (Proposition 4.4), we see that the trace ideals of $L_K(E)$ are exactly the ideals generated by idempotents of $L_K(E)$. In

general not all the ideals in $L_K(E)$ will be generated by idempotents. For instance, if E is a single loop, then $L_K(E) = K[x, x^{-1}]$ and the ideal generated by $1 - x$ only contains the idempotent 0. However, it is possible to describe the ideals generated by idempotents by using the canonical grading of $L_K(E)$. Let us recall that $L_K(E)$ is generated by sets $\{p_v \mid v \in E^0\}$ and $\{x_e, y_e \mid e \in E^1\}$, which satisfy the following relations:

- (1) $p_v p_{v'} = \delta_{v,v'} p_v$ for all $v, v' \in E^0$.
- (2) $p_{s(e)} x_e = x_e p_{r(e)} = x_e$ for all $e \in E^1$.
- (3) $p_{r(e)} y_e = y_e p_{s(e)} = y_e$ for all $e \in E^1$.
- (4) $y_e x_{e'} = \delta_{e,e'} p_{r(e)}$ for all $e, e' \in E^1$.
- (5) $p_v = \sum_{\{e \in E^1 \mid s(e)=v\}} x_e y_e$ for every $v \in E^0$ that emits edges.

If we declare that the degree of x_e is 1 and the degree of y_e is -1 for all $e \in E^1$, and that the degree of each p_v is 0 for $v \in E^0$, then we obtain a well-defined degree on the algebra $L(E) = L_K(E)$, because all relations (1)–(5) are homogeneous. Thus $L(E)$ is a \mathbb{Z} -graded algebra:

$$L(E) = \bigoplus_{n \in \mathbb{Z}} L(E)_n; \quad L(E)_n L(E)_m \subseteq L(E)_{n+m}, \quad \text{for all } n, m \in \mathbb{Z}.$$

For a subset X of a \mathbb{Z} -graded ring $R = \bigoplus_{n \in \mathbb{Z}} R_n$, set $X_n = X \cap R_n$. An ideal I of R is said to be a *graded ideal* in case $I = \bigoplus_{n \in \mathbb{Z}} I_n$. Let us denote the lattice of graded ideals of a \mathbb{Z} -graded ring R by $\mathcal{L}_{gr}(R)$.

Recall that $v \in E^0$ is called a *sink* in case v does not emit any edge.

Theorem 5.3 *Let E be a row-finite graph. Then there are order-isomorphisms*

$$\mathcal{H} \cong \mathcal{L}(M_E) \cong \mathcal{L}_{gr}(L_K(E)),$$

where \mathcal{H} is the lattice of hereditary and saturated subsets of E^0 , $\mathcal{L}(M_E)$ is the lattice of order-ideals of the monoid M_E , and $\mathcal{L}_{gr}(L_K(E))$ is the lattice of graded ideals of the graph algebra $L_K(E)$.

Proof We have obtained an order-isomorphism $\mathcal{H} \cong \mathcal{L}(M_E)$ in Proposition 5.2. As we observed earlier there is an order-isomorphism $\mathcal{L}(M_E) = \mathcal{L}(V(L(E))) \cong \mathcal{L}_{idem}(L(E))$, where $\mathcal{L}_{idem}(L(E))$ is the lattice of ideals in $L(E)$ generated by idempotents. The isomorphism is given by the rule $I \mapsto \tilde{I}$, for every order-ideal I of M_E , where \tilde{I} is the ideal generated by all the idempotents $e \in L(E)$ such that $V(e) \in I$. (Here $V(e)$ denotes the class of e in $V(L(E)) = M_E$.) Given any order-ideal I of M_E , it is generated as a monoid by the elements $V(p_v) (= [v] = a_v)$ such that $V(p_v) \in I$, so that \tilde{I} is generated as an ideal by the idempotents p_v such that $p_v \in \tilde{I}$. In particular we see that every ideal of $L(E)$ generated by idempotents is a graded ideal.

It only remains to check that every graded ideal of $L(E)$ is generated by idempotents. For this, it will be convenient to recall the definition of the path algebra $P(E)$ associated with E . The algebra $P(E)$ is the algebra generated by a set $\{p_v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{x_e \mid e \in E^1\}$, which satisfy relation (2). A K -basis for $P(E)$ is given by the set of “paths” $\gamma = x_{e_1} x_{e_2} \cdots x_{e_r}$, such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, r - 1$. We put $s(\gamma) = s(e_1)$ and $r(\gamma) = r(e_r)$, and the *length* $|\gamma|$ of γ is defined to be r . It is easy to see that $P(E)$ is indeed a

subalgebra of $L(E)$. The algebra $P(E)^*$ is, by definition, the subalgebra of $L(E)$ generated by $\{p_v \mid v \in E^0\}$ and $\{y_e \mid e \in E^1\}$. Of course we can define an involution on $L(E)$ sending x_e to y_e , so that all p_v are projections: $p_v = p_v^2 = p_v^*$, and acting on K by any prescribed involution on K . Note that for $\gamma = x_{e_1}x_{e_2} \cdots x_{e_r}$, we have $\gamma^* = y_{e_r} \cdots y_{e_2}y_{e_1}$. Now elements in $L(E)$ can be described as linear combinations of elements of the form γv^* , where γ and v are paths on E with $r(\gamma) = r(v)$. It is clear that, for $n > 0$, we have $L(E)_n = \bigoplus_{|\gamma|=n} \gamma L(E)_0$, and similarly, $L(E)_{-n} = \bigoplus_{|\gamma|=n} L(E)_0 \gamma^*$.

Given a graded ideal J of $L(E)$, take any element $a \in J_n$, where $n > 0$. Then $a = \sum_{|\gamma|=n} \gamma a_\gamma$, for some $a_\gamma \in L(E)_0$. For a fixed path v of length n , we have $v^*a = a_v$, so that $a_v \in J_0$. We conclude that $J_n = L(E)_n J_0$, and similarly $J_{-n} = J_0 L(E)_{-n}$. Since J is a graded ideal, we infer that J is generated as ideal by J_0 , which is an ideal of $L(E)_0$.

To conclude the proof, we only have to check that every ideal of $L(E)_0$ is generated by idempotents. Indeed we will prove that $L(E)_0$ is a von Neumann regular ring, more precisely $L(E)_0$ is an ultramatricial K -algebra, i.e., a direct limit of matricial algebras over K [16], though not all the connecting homomorphisms are unital. (A matricial K -algebra is a finite direct product of full matrix algebras over K .)

By Lemma 3.2 we have $L(E) = \lim_{\substack{\longrightarrow \\ i \in I}} L(X_i)$ for a directed family $\{X_i \mid i \in I\}$ of finite graphs. Then $L(E)_0 = \lim_{\substack{\longrightarrow \\ i \in I}} L(X_i)_0$, and so we can assume that E is a finite graph.

Now for a finite graph E , all the transition maps are unital. They can be built in the following fashion. For each v in E^0 , and each $n \in \mathbb{Z}^+$, let us denote by $P(n, v)$ the set of paths $\gamma = x_{e_1} \cdots x_{e_n} \in P(E)$ such that $|\gamma| = n$ and $r(\gamma) = v$. The set of sinks will be denoted by $S(E)$. Now the algebra $L(E)_0$ admits a natural filtration by algebras $L_{0,n}$, for $n \in \mathbb{Z}^+$. Namely $L_{0,n}$ is the set of linear combinations of elements of the form γv^* , where γ and v are paths with $r(\gamma) = r(v)$ and $|\gamma| = |v| \leq n$. The algebra $L_{0,0}$ is isomorphic to $\prod_{v \in E^0} K$. In general the algebra $L_{0,n}$ is isomorphic to

$$\left[\prod_{i=0}^{n-1} \left(\prod_{v \in S(E)} M_{|P(i,v)|}(K) \right) \right] \times \left[\prod_{v \in E^0} M_{|P(n,v)|}(K) \right].$$

The transition homomorphism $L_{0,n} \rightarrow L_{0,n+1}$ is the identity on the factors $\prod_{v \in S(E)} M_{|P(i,v)|}(K)$, for $0 \leq i \leq n - 1$, and also on the factor $\prod_{v \in S(E)} M_{|P(n,v)|}(K)$ of the last term of the displayed formula. The transition homomorphism

$$\prod_{v \in E^0 \setminus S(E)} M_{|P(n,v)|}(K) \rightarrow \prod_{v \in E^0} M_{|P(n+1,v)|}(K)$$

is a block diagonal map induced by the following identification in $L(E)_0$: A matrix unit in a factor $M_{|P(n,v)|}(K)$, where $v \in E^0 \setminus S(E)$, is a monomial of the form γv^* , where γ and v are paths of length n with $r(\gamma) = r(v) = v$. Since v is not a sink, we can enlarge the paths γ and v using the edges that v emits, obtaining paths of length $n + 1$, and relation (5) in the definition of $L(E)$ gives $\gamma v^* = \sum_{\{e \in E^1 \mid s(e)=v\}} (\gamma x_e)(y_e v^*)$.

It follows that $L(E)_0$ is an ultramatricial K -algebra, and the proof is complete. \square

the Grothendieck group $G(M)$ of M is *unperforated*: for all $g \in G(M)$ and all positive integers n , we have $ng \geq 0 \implies g \geq 0$.

Proposition 6.4 Let E be a row-finite graph. Then the monoid M_E is unperforated.

Proof As in the proof of Theorem 6.3, we can reduce to the case of a finite graph E . In this case, the result follows from [9, Corollary 5.11(5)]. □

Corollary 6.5 Let E be a row-finite graph. Then $FP(L_K(E))$ satisfies the refinement property and $L_K(E)$ is a separative ring. Moreover, the monoid $V(L_K(E))$ is an unperforated monoid and $K_0(L_K(E))$ is an unperforated group.

Proof By Theorem 3.5, we have $V(L_K(E)) \cong M_E$. So the result follows from Proposition 4.4, Theorem 6.3 and Proposition 6.4. □

Another useful technique to deal with graph monoids of finite graphs consists in considering composition series of order-ideals in the monoid. These composition series correspond via Theorem 5.3 to composition series of graded ideals in $L_K(E)$, and, using [5, Theorem 4.1(b)], they also correspond to composition series of closed gauge-invariant ideals of the graph C^* -algebra $C^*(E)$. This approach will be used in the proof of Theorem 7.1 in our next section. It also leads to a different proof of the separativity of M_E (Theorem 6.3), that will be sketched in Remark 6.7.

Given an order-ideal S of a monoid M we define a congruence \sim_S on M by setting $a \sim_S b$ if and only if there exist $e, f \in S$ such that $a + e = b + f$. Let M/S be the factor monoid obtained from the congruence \sim_S ; see [2]. For large classes of rings R , one has $V(R/I) \cong V(R)/V(I)$ for any ideal I of R ; see [2, Proposition 1.4].

We need a monoid version of [5, Theorem 4.1(b)].

Lemma 6.6 Let E be a row-finite graph. For a saturated hereditary subset H of E^0 , consider the order-ideal $S = \varphi(H)$ associated with H , as in Proposition 5.2. Let $G = (G^0, G^1)$ be the graph defined as follows. Put $G^0 = E^0 \setminus H$ and $G^1 = \{e \in E^1 \mid r(e) \in G^0\}$. Then there is a natural monoid isomorphism $M_E/S \cong M_G$.

Proof Note that S is generated as a monoid by the elements a_v , with $v \in H$. There is a unique monoid homomorphism $\pi: F_E \rightarrow F_G$ sending v to 0 for $v \in H$ and v to v for $v \in E \setminus H$, where F_E (respectively F_G) is the free Abelian monoid on E , (respectively, G). The map π induces a surjective monoid homomorphism $\bar{\pi}: M_E \rightarrow M_G$, and it is clear that $\bar{\pi}$ factors through M_E/S , i.e., we have $M_E \rightarrow M_E/S \rightarrow M_G$. If $\pi(\alpha) \sim \pi(\beta)$ in F_G for $\alpha, \beta \in F_E$, then by Lemma 4.3 there is $\gamma \in F_G$ such that $\pi(\alpha) \rightarrow \gamma$ and $\pi(\beta) \rightarrow \gamma$. This means that there is a string $\pi(\alpha) = \gamma_0 \rightarrow_1 \gamma_1 \rightarrow_1 \dots \rightarrow_1 \gamma_r = \gamma$ in F_G , and similarly for $\pi(\beta) \rightarrow \gamma$. Let us consider the same strings, but now in F_E . We then get that $\alpha \rightarrow \gamma + \delta_1$, where δ_1 is supported on H , and similarly $\beta \rightarrow \gamma + \delta_2$, where δ_2 is supported on H . It follows that the following identity holds in M_E :

$$[\alpha] + [\delta_2] = [\gamma] + [\delta_1] + [\delta_2] = [\beta] + [\delta_1],$$

with $[\delta_1], [\delta_2] \in S$. We conclude that the map $M_E/S \rightarrow M_G$ is injective, and so it is a monoid isomorphism, as desired. \square

Let us call M_E *simple* if M_E has only the trivial order-ideals. This corresponds by Proposition 5.2 to the situation where the hereditary and saturated subset generated by any vertex of E is E^0 . It is well-known that this happens if and only if E is cofinal. Let $E^{\leq\infty}$ be the set of infinite paths in E together with the finite paths in E whose end point is a sink. Then E is said to be *cofinal* in case given a vertex v in E and a path γ in $E^{\leq\infty}$, there is a vertex w in the path γ such that $v \geq w$.

A finite path α of positive length is called a *loop* if $s(\alpha) = r(\alpha) = v$. A loop $\alpha = (e_1, e_2, \dots, e_n)$ is *simple* if all the vertices $s(e_i)$, $1 \leq i \leq n$, are distinct. For a subgraph G of E , an *exit* of G is an edge e in E with $s(e) \in G^0$ and $e \notin G^1$.

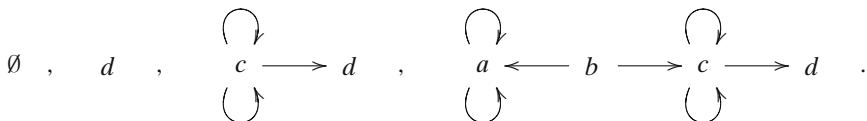
Remark 6.7 We are now ready to sketch a different proof of the separativity of M_E (Theorem 6.3), using the theory of order-ideals.

As in the proof of Theorem 6.3, we can assume that E is a finite graph. In this case it is obvious that E^0 has a finite number of saturated hereditary subsets, so M_E has a finite number of order-ideals. Take a finite chain $0 = S_0 \leq S_1 \leq \dots \leq S_n = M_E$ such that each S_i is an order-ideal of M_E , and all the quotients S_i/S_{i-1} are simple. By Proposition 5.2, we have $S_i \cong M_{H_i}$, for some finite graph H_i , and by Lemma 6.6, we have $S_i/S_{i-1} \cong M_{G_i}$ for some cofinal finite graph G_i . By Proposition 4.4, S_i is a refinement monoid for all i , so the Extension Theorem for refinement monoids [2, Theorem 4.5] tells us that S_i is separative if and only if so are S_{i-1} and S_i/S_{i-1} . It follows by induction that it is enough to show the case where E is a cofinal finite graph.

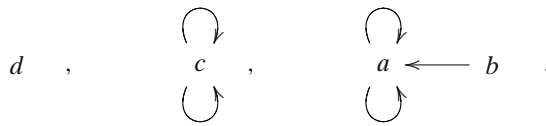
Let E be a cofinal finite graph. We distinguish three cases. First, suppose that E does not have loops. Then there is a sink v , and by cofinality for every vertex w of E there is a path from w to v . It follows that M_E is a free Abelian monoid of rank one (i.e., isomorphic to \mathbb{Z}^+), generated by a_v . In particular M_E is a separative monoid. Secondly, assume that E has a simple loop without exit, and let v be any vertex in this simple loop. By using the cofinality condition, it is easy to see that there are no other simple loops in E , and that every vertex in E connects to v . It follows again that M_E is a free Abelian monoid of rank one, generated by a_v .

Finally we consider the case where every simple loop has an exit. By cofinality, every vertex connects to every loop. Using this and the property that every loop has an exit, it is quite easy to show that for every nonzero element x in M_E there is a nonzero element y in M_E such that $x = x + y$. It follows that $M_E \setminus \{0\}$ is a group; see for example [2, Proposition 2.4]. In particular M_E is a separative monoid.

Example 6.8 We consider again the graph E described in Example 3.6. A composition series of order-ideals for M_E is obtained from the graph monoids corresponding to the following chain of saturated hereditary subsets of E :



By Lemma 6.6, the corresponding simple quotient monoids are the graph monoids corresponding to the following graphs:



7 The Monoid Associated with a Graph C^* -algebra

In this section, we will assume that $L(E) = L_{\mathbb{C}}(E)$ is the graph algebra of the graph E over the field \mathbb{C} of complex numbers, endowed with its natural structure of complex $*$ -algebra, so that $x_e^* = y_e$ for all $e \in E^1$, $p_v^* = p_v$ for all $v \in E^0$, and $(\xi a)^* = \bar{\xi} a^*$ for $\xi \in \mathbb{C}$ and $a \in L(E)$. There is a natural inclusion of complex $*$ -algebras $\psi: L(E) \rightarrow C^*(E)$, where $C^*(E)$ denotes the graph C^* -algebra associated with E .

Theorem 7.1 *Let E be a row-finite graph, and let $L(E) = L_{\mathbb{C}}(E)$ be the graph algebra over the complex numbers. Then the natural inclusion $\psi: L(E) \rightarrow C^*(E)$ induces a monoid isomorphism $V(\psi): V(L(E)) \rightarrow V(C^*(E))$. In particular the monoid $V(C^*(E))$ is naturally isomorphic with the monoid M_E .*

Proof The algebra homomorphism $\psi: L(E) \rightarrow C^*(E)$ induces the following commutative square:

$$\begin{CD} V(L(E)) @>V(\psi)>> V(C^*(E)) \\ @V\varphi_1VV @VV\varphi_2V \\ K_0(L(E)) @>K_0(\psi)>> K_0(C^*(E)). \end{CD}$$

The map $K_0(\psi)$ is an isomorphism by Theorem 3.5 and [22, Theorem 3.2]. Using Lemmas 3.2 and 3.3, we see that it is enough to show that $V(\psi)$ is an isomorphism for a finite graph E .

Assume that E is a finite graph. We first show that the map $V(\psi): V(L(E)) \rightarrow V(C^*(E))$ is injective. Suppose that P and Q are idempotents in $M_{\infty}(L(E))$ such that $P \sim Q$ in $C^*(E)$. By Theorem 3.5, we can assume that each of P and Q is equivalent in $M_{\infty}(L(E))$ to direct sums of “basic” projections, that is, projections of the form p_v , with $v \in E^0$. Let J be the closed ideal of $C^*(E)$ generated by the entries of P . Since $P \sim Q$, the closed ideal generated by the entries of P agrees with the closed ideal generated by the entries of Q and indeed it agrees with the closed ideal generated by the projections of the form p_w , where w ranges over the saturated hereditary subset H of E^0 generated by $\{v \in E^0 \mid P = \oplus p_v\}$ (see [5, Theorem 4.1]). It follows from Theorem 5.3 that P and Q generate the same ideal I_0 in $L(E)$. There is a projection $e \in L(E)$, which is the sum of the basic projections p_w , where w ranges in H , such that $I_0 = L(E)eL(E)$ and $eL(E)e = L(H)$ is also a graph algebra. Note that P and Q are full projections in $L(H)$, and so $[1_H] \leq m[P]$ and $[1_H] \leq m[Q]$ for some $m \geq 1$. Now consider the map $\psi_H: L(H) \rightarrow C^*(H)$. Since $V(\psi_H)([P]) = V(\psi_H)([Q])$ in $V(C^*(H))$ we get $K_0(\psi_H)(\varphi_1([P])) = K_0(\psi_H)(\varphi_1([Q]))$, and

since $K_0(\psi_H)$ is an isomorphism we get $\varphi_1([P]) = \varphi_1([Q])$. This means that there is $k \geq 0$ such that $[P] + k[1_H] = [Q] + k[1_H]$. But since $V(L(E))$ is separative and $[1_H] \leq m[P]$ and $[1_H] \leq m[Q]$, we get $[P] = [Q]$ in $V(L(E))$.

Now we want to see that the map $V(\psi): V(L(E)) \rightarrow V(C^*(E))$ is surjective. By [5, Theorem 4.1], there is a natural isomorphism between the lattice of saturated hereditary subsets of E^0 and the lattice of closed gauge-invariant ideals of $C^*(E)$. Thus, since E is finite, the number of closed gauge-invariant ideals of $C^*(E)$ is finite, and there is a finite chain $I_0 = \{0\} \leq I_1 \leq \dots \leq I_n = C^*(E)$ of closed gauge-invariant ideals such that each quotient I_{i+1}/I_i is gauge-simple. We proceed by induction on n . If $n = 1$ we have the case in which $C^*(E)$ is gauge-simple, and thus it is either purely infinite simple, or AF or Morita-equivalent to $C(\mathbb{T})$; see [5]. In either case the result follows. Note that in the purely infinite case, we use that $V(C^*(E)) = K_0(C^*(E)) \setminus \{0\} = K_0(L) \setminus \{0\} = V(L)$. Now assume that the result is true for graph C^* -algebras of (gauge) length $n - 1$ and let $A = C^*(E)$ be a graph C^* -algebra of length n . Let H be the saturated hereditary subset of E^0 corresponding to the ideal I_1 . Note that H is a minimal saturated hereditary subset of E^0 , and thus H is cofinal. Set $B = A/I_1$. By [5, Theorem 4.1(b)], we have $B \cong C^*(F)$, where $F^0 = E^0 \setminus H$ and $F^1 = \{e \in E^1 \mid r(e) \notin H\}$. Observe that by the induction hypothesis we know that every projection in B is equivalent to a finite orthogonal sum of basic projections of the form p_v , where v ranges in $F^0 = E^0 \setminus H$. Let $\pi: A \rightarrow B$ denote the canonical projection. Since I_1 is the closed ideal generated by its projections, there is an embedding $V(A)/V(I_1) \rightarrow V(B)$. This follows from [4, Proposition 5.3(c)], taking into account that every closed ideal generated by projections is an almost trace ideal. By induction hypothesis, $V(B) = V(C^*(F))$ is generated as a monoid by $[p_v]$, for $v \in E^0 \setminus H$, and so the map $V(A)/V(I_1) \rightarrow V(B)$ is also surjective, so that $V(B) \cong V(A)/V(I_1)$. In particular, $\pi(P) \sim \pi(Q)$ for two projections $P, Q \in M_\infty(A)$, if and only if there are projections $P', Q' \in M_\infty(I_1)$ such that $P \oplus P' \sim Q \oplus Q'$ in $M_\infty(A)$.

We first deal with the case where I_1 has stable rank one, which corresponds to the cases where I_1 is either AF or Morita-equivalent to $C(\mathbb{T})$. Note that in this case either H contains a sink v , or we have a simple loop without exit, in which case we select v as a vertex in this loop. Note that, by the cofinality of H , any projection in I_1 is equivalent to a projection of the form $k \cdot p_v$ for some $k \geq 0$. Now take any projection P in $M_\infty(A)$. Since $\pi(P) \sim \pi(p_{v_1} \oplus \dots \oplus p_{v_r})$ for some vertices v_1, \dots, v_r in $E \setminus H$, there are $a, b \geq 0$ such that

$$P \oplus a \cdot p_v \sim p_{v_1} \oplus \dots \oplus p_{v_r} \oplus b \cdot p_v.$$

Since the stable rank of $p_v A p_v$ is one, the projection p_v cancels in direct sums [23], and so, if $b \geq a$, we get

$$P \sim p_{v_1} \oplus \dots \oplus p_{v_r} \oplus (b - a)p_v,$$

so that P is equivalent to a finite orthogonal sum of basic projections. If $b < a$, then we have $P \oplus (a - b)p_v \sim p_{v_1} \oplus \dots \oplus p_{v_r}$. We claim that there is some $1 \leq i \leq r$ such that v is in the tree of v_i . For, assume to the contrary that $v \notin \bigcup_{i=1}^r T(v_i)$. We will see that v is not in the saturated hereditary subset of E generated by v_1, \dots, v_r . Note that the set $D = \bigcup_{i=1}^r T(v_i)$ is hereditary, and that the saturated hereditary subset of E generated by v_1, \dots, v_r is $\overline{D} = \bigcup_{j=0}^\infty \Lambda_j(D)$, see Section 5. Observe also that, since v is either a sink or belongs to a simple loop and H is cofinal, v belongs to the tree of any vertex in H , whence $H \cap D = \emptyset$. Let v' be a vertex in H . If $v' \in$

$\Lambda_1(D)$ then $s^{-1}(v') \neq \emptyset$ and $r(s^{-1}(v')) \subseteq D \cap H$. Since $H \cap D = \emptyset$, this is impossible. So $\Lambda_1(D) \cap H = \emptyset$. Indeed, an easy induction shows that $\Lambda_i(D) \cap H = \emptyset$ for all i , and so $\overline{D} \cap H = \emptyset$. But p_{ij} being equivalent to a subprojection of $p_{v_1} \oplus \cdots \oplus p_{v_r}$, the projection p_v belongs to the closed ideal of A generated by p_{v_1}, \dots, p_{v_r} , and so v belongs to \overline{D} . This contradiction shows that v belongs to the tree of some v_i , as claimed.

Now, the fact that v belongs to the tree of v_i implies that there is a projection Q which is a finite orthogonal sum of basic projections such that $p_{v_i} \sim p_v \oplus Q$. Therefore we get

$$P \oplus (a - b)p_v \sim (p_{v_1} \oplus \cdots \oplus p_{v_{i-1}} \oplus p_{v_{i+1}} \oplus \cdots \oplus p_{v_r} \oplus Q) \oplus p_v.$$

Since p_v can be cancelled in direct sums, we get

$$P \oplus (a - b - 1)p_v \sim (p_{v_1} \oplus \cdots \oplus p_{v_{i-1}} \oplus p_{v_{i+1}} \oplus \cdots \oplus p_{v_r} \oplus Q),$$

and so, using induction, we obtain that P is equivalent to a finite orthogonal sum of basic projections.

Finally we consider the case where I_1 is a purely infinite simple C^* -algebra. Recall that in this case I_1 has real rank zero [11], and that $V(I_1) \setminus \{0\}$ is a group. So there is a nonzero projection e in I_1 such that for every nonzero projection p in $M_\infty(I_1)$ there exists a nonzero projection $q \in I_1$ such that $p \oplus q \sim e$. Let P be a nonzero projection in $M_k(A)$, for some $k \geq 1$, and denote by I the closed ideal of A generated by (the entries of) P . If $I \cdot I_1 = 0$, then $I \cong (I + I_1)/I_1$, so that I is a closed ideal in the quotient C^* -algebra $B = A/I_1$. It follows then by our assumption on B that P is equivalent to a finite orthogonal sum of basic projections. Assume now that $I \cdot I_1 \neq 0$. Then there is a nonzero column $C = (a_1, a_2, \dots, a_k)^t \in A^k$ such that $C = PCe$. Consider the positive element $c = C^*C$, which belongs to eAe . Since $e \in I_1$ and I_1 has real rank zero, the C^* -algebra eAe has also real rank zero, so that we can find a nonzero projection $p \in cAc$. Take $x \in A$ such that $p = cxc$. By using standard tricks (see e.g., [24]), we can now produce a projection $P' \leq P$ such that $p \sim P'$. Namely, consider the idempotent $F = CpC^*CxC^*$ in $PM_k(A)P$. Then p and F are equivalent as idempotents, and F is equivalent to some projection P' in $PM_k(A)P$; see [24, Exercise 3.11(i)]. Since p and P' are equivalent as idempotents, they are also Murray–von Neumann equivalent, and see [24, Exercise 3.11(ii)], as desired. We have proved that there is a nonzero projection p in I_1 such that p is equivalent to a subprojection of P . Since I_1 is purely infinite simple, every projection in I_1 is equivalent to a subprojection of p , and so every projection in I_1 is equivalent to a subprojection of P .

Now we are ready to conclude the proof. There is a projection q in I_1 such that $P \oplus q$ is equivalent to a finite orthogonal sum of basic projections. Let q' be a nonzero projection in I_1 such that $q \oplus q' \sim e$, and observe that

$$P \oplus e \sim (P \oplus q) \oplus q',$$

so that $P \oplus e$ is also a finite orthogonal sum of basic projections. By the above argument, there is a projection e' such that $e' \leq P$ and $e \sim e'$. Write $P = e' + P'$. Then we have

$$P \oplus e \sim P' \oplus e' \oplus e \sim P' \oplus e \oplus e \sim P' \oplus e \sim P.$$

It follows that $P \sim P \oplus e$ and so P is equivalent to a finite orthogonal sum of basic projections. \square

Corollary 7.2 Let E be a row-finite graph. Then the monoid $V(C^*(E))$ is a refinement monoid and $C^*(E)$ has stable weak cancellation. Moreover, $V(C^*(E))$ is an unperforated monoid and $K_0(C^*(E))$ is an unperforated group.

Proof By Theorem 7.1, $V(C^*(E)) \cong M_E$, and so $V(C^*(E))$ is a refinement monoid by Proposition 4.4.

It follows from Theorem 6.3 that $V(C^*(E))$ is a separative monoid. By Proposition 2.1, this is equivalent to saying that $C^*(E)$ has stable weak cancellation. The statements about unperforation follow from Proposition 6.4. \square

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