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Trees having an even or quasi even degree sequence are graceful[☆]

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Abstract

A rooted tree with diameter D is said to have an even degree sequence if every vertex has even degree except for one root and the leaves, which are in the last level $\lfloor D/2 \rfloor$. The degree sequence is said to be quasi even if every vertex has even degree except for one root, every vertex in level $\lfloor D/2 \rfloor - 1$ and the leaves, which are in the last level $\lfloor D/2 \rfloor$. Hrnčiar and Haviar [P. Hrnčiar, A. Haviar, All trees of diameter five are graceful, Discrete Math. 233 (2001) 133–150] give a method to construct a graceful labeling for every tree with diameter five. Based upon their method we prove that every tree having an even or quasi even degree sequence is graceful. To do that we find for a tree of even diameter and rooted in its central vertex t of degree $\delta(t)$ up to $\delta(t)!$ graceful labelings if the tree has an even or quasi even degree sequence.

Keywords: Kotzig conjecture; Graceful labeling; Graceful trees

1. Introduction

Let G be a graph with m edges. A one-to-one function $f : V(G) \to \{0, 1, \ldots, m\}$ from the vertex set $V(G)$ of G to the set of the first $m + 1$ nonnegative numbers starting in zero is said to be a *vertex labeling of* G. If the absolute value $|f(u) - f(v)|$ is assigned to the edge uv as its label and the resulting edge labels are mutually distinct then the vertex labeling is called *a graceful labeling*. A graph possessing a graceful labeling is called *a graceful graph*.

In [10] Rosa published the conjecture of Ringel–Kotzig that every nontrivial tree is graceful. Among the trees known to be graceful are: caterpillars [10] (a caterpillar is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [7,8,11]; trees with diameter at most 5 [3,4,6,11]; trees with at most 27 vertices [1]; symmetrical trees (a rooted tree in which every level contains vertices of the same degree) [2,9]; for more information about graceful graphs see the dynamic survey by Gallian [5].

Let T be any tree and let us denote the degree of any vertex $u \in V(T)$ by $\delta_T(u)$ and its neighboring by $N_T(u)$. We also use $d_T(u, v)$ to denote the distance in T between any two vertices u and v. If T has even diameter D and it is

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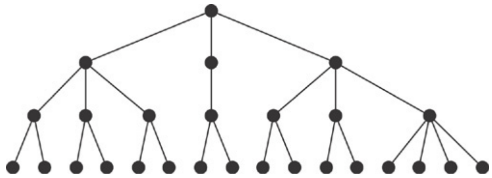

Fig. 1. A tree having a quasi even degree sequence.

rooted in its central vertex t, then $V(T) = \cup_{1\le i\le D/2} L_i \cup \{t\}$ where $L_i = \{v \in V(T) : d_T(v,t) = i\}$. If T has odd diameter D and it is rooted in its two (adjacent) central vertices a, b, then $V(T) = \cup_{1\le i\le (D-1)/2} L_i \cup \{a, b\}$ where $L_i = \{v \in V(T) : d_T(v,a) = i, d_T(v,b) = i+1\} \cup \{v \in V(T) : d_T(v,b) = i, d_T(v,a) = i+1\}$. In both cases the sets L_i are called *the levels* of the rooted tree.

Let us say that a rooted tree T with diameter $D(T)$ has an *even degree sequence* if every vertex has even degree except for one root and the leaves, which are in level $L_{\lfloor D(T)/2\rfloor}$. Analogously, a rooted tree T with diameter $D(T)$ is said to have a *quasi even degree sequence* if every vertex has even degree except for one root, the vertices in level $L_{\lfloor D(T)/2\rfloor-1}$ and the leaves, which are in level $L_{\lfloor D(T)/2\rfloor}$. See Fig. 1 as an example.

In this note we prove that every tree having an even or quasi even degree sequence is graceful. To do that we find for a tree of even diameter and rooted in its central vertex t of degree $\delta(t)$ up to $\delta(t)!$ graceful labelings if the tree has an even or quasi even degree sequence.

2. Results

We begin by obtaining a relationship between the parities of the number of edges m and the diameter of a tree T having either an even degree sequence or a quasi even degree sequence.

Lemma 2.1. *Let T be a tree of m edges and even diameter D. If T has an even degree sequence then $m + D/2$ is even. If T has a quasi even degree sequence then $m + D/2$ is odd.*

Proof. By definition of even or quasi even degree sequence the root t has odd degree $\delta_T(t)$, hence $|L_1| = \delta_T(t)$ is odd. As for all $i = 2, \ldots, D/2 - 1$, we have

$$|L_i| = \sum_{w\in L_{i-1}} (\delta_T(w) - 1) = \sum_{w\in L_{i-1}} \delta_T(w) - |L_{i-1}|$$

and $\delta_T(w)$ is even, then both numbers $|L_{i-1}|$ and $|L_i|$ must have the same parity. Therefore $|L_1|, \ldots, |L_{D/2-1}|$ are odd numbers. Thus, if T has an even degree sequence, then $|L_{D/2}|$ is also odd and therefore both $m = \sum_{i=1}^{D/2} |L_i|$ and $D/2$ have the same parity. Finally if T has a quasi even degree sequence, then $|L_{D/2}| = \sum_{w\in L_{D/2-1}} (\delta_T(w) - 1)$ is even because $\delta_T(w)$ is now odd, and hence m and $D/2$ have different parity. ■

We follow the terminology introduced by Hrnčiar and Haviari [6]. More precisely, let T be a tree and let $uu_1 \in E(T)$. We denote by T_{u,u_1} the subtree of T induced by the set of vertices

$$V(T_{u,u_1}) = \{w \in V(T) : w = u \text{ or } u_1 \text{ is on the } u - w \text{ path}\}.$$

Moreover we use the method introduced in Lemma 3 of [6] to transform a graceful tree T into a new graceful tree H of the same size. This new graceful tree H is obtained from T by shifting pairs of subtrees T_{u,u_1}, T_{u,u_2} from vertex u to some vertex $v \in V(T) \setminus (V(T_{u,u_1}) \cup V(T_{u,u_2}))$ such that $f(u) + f(v) = f(u_1) + f(u_2)$, a transfer that we will call a *pair transfer*. Or by transferring one subtree T_{u,u_1} from vertex u to vertex $v \in V(T) \setminus V(T_{u,u_1})$ if $2f(u_1) = f(u) + f(v)$, a transfer that we will call a *single transfer*. Clearly if the unique $u - v$ path in the tree does not contain a neighbor u_1 of u then $v \notin V(T_{u,u_1})$. This is guaranteed if uu_1 is an end-edge in the tree or if uv is an end-edge in the tree. In order to illustrate this method see Fig. 2 in which we perform first a single transfer $5 \to 1$ attaching to 1 the subtree $T_{5,3}$ because $5 + 1 = 2 \cdot 3$. And a pair transfer $5 \to 0$ attaching to 0 the subtrees $T_{5,1}$ and $T_{5,4}$ because $5 + 0 = 1 + 4$.

If there is no confusion we will not distinguish between a vertex and its label (for a given graceful labeling). When we do a transfer of a subtree T_{u,u_1} (with u_1 adjacent to u) from u to v we briefly speak about an $u \to v$ transfer.

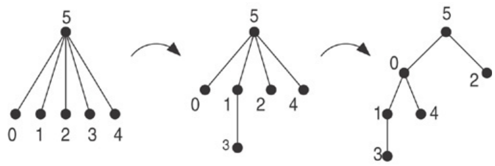

Fig. 2. Two kinds of transfers.

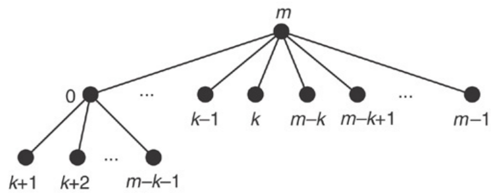

Fig. 3. Starting tree.

By using the above procedure every tree of diameter four or five was shown to have a graceful labeling [6]. Following the same lines of reasoning we prove that it is possible to obtain up to $\delta_T(t)!$ graceful labelings for a tree of even diameter D, rooted in its central vertex t, and having an even or quasi even degree sequence.

Theorem 2.1. *Let T be a tree of even diameter D and rooted in its central vertex t. Let us consider the set of $\delta_T(t)$ labels*

$$X = \left\{0, 1, \ldots, \frac{\delta_T(t)-1}{2}, m - \frac{\delta_T(t)-1}{2}, \ldots, m-1\right\}.$$

Suppose that T has either an even degree sequence or a quasi even degree sequence. Then any numbering of the vertex set $N_T(t)$ with labels in X can be extended to a graceful labeling f on $V(T)$ such that $f(t) = m$.

Proof. Notice that the root t has odd degree $\delta_T(t) \geq 3$ because t is a central vertex and because of the definition of even or quasi even degree sequence. Let $k = (\delta_T(t) - 1)/2$. We start by assigning $f(t) = m$ and distributing the set of labels $X = \{0, 1, 2, \ldots, k, m-k, m-k+1, \ldots, m-1\}$ among the neighbors of t in any order. If $D = 2$ then we are done. So assume $D \geq 4$. We obtain a graceful labeling of T by constructing a sequence of graceful trees of the same size as T

$$H^0, H^{m-1}, H^1, H^{m-2}, H^2, H^{m-3}, \ldots, T \tag{1}$$

using transfers of end-edges. As 0 has been assigned to one neighbor of the root t we know the value of $\delta_T(0)$. Let us study two cases according to the degree of 0 in T.

Case $\delta_T(0)$ even. The starting graceful tree H^0 is shown in Fig. 3 in which the children of 0 form the set $\{k+1, k+2, \ldots, m-k-1\}$. Notice that except for $m-k-1$ all the elements of this list can be associated in pairs whose sum is $m-1$, i.e., $(k+1)+(m-k-2) = (k+2)+(m-k-3) = \cdots = m-1$. Thus, denoting by $k_0 = \delta_T(0)/2$, the graceful tree H^{m-1} is obtained from H^0 by successive transfers $0 \to m-1$ shifting the subtrees $H^0_{0,k+k_0}$ and $H^0_{0,m-1-k-k_0}$, the subtrees $H^0_{0,k+k_0+1}$ and $H^0_{0,m-2-k-k_0}$, etc, such that the children of $m-1$ in H^{m-1} form the set of consecutive labels $\{k+k_0, \ldots, m-(k+k_0)-1\}$ and hence $\delta_{H^{m-1}}(0) = \delta_T(0)$, since 0 remains adjacent to $\{k+1, \ldots, k+k_0-1\} \cup \{m-(k+k_0), \ldots, m-k-1\}$. Now, we label the neighbors of 0 in level 2 of T as they have been labeled in H^{m-1}.

Similarly, for the second step in sequence (1), except for $k+k_0$ all the elements of the current list of children in H^{m-1} of $m-1$ can be associated in pairs whose sum is m, i.e., $(k+k_0+1)+(m-1-k-k_0) = (k+k_0+2)+(m-2-k-k_0) = \cdots = m$. So denoting by $k_{m-1} = \delta_T(m-1)/2$, the graceful tree H^1 is obtained from H^{m-1} by successive transfers $m-1 \to 1$ such that the children of 1 in H^1 form the set of consecutive labels $\{k+k_0+k_{m-1}, \ldots, m-(k+k_0+k_{m-1})\}$ which implies that the remaining children of $m-1$ in H^1 are $\{k+k_0, \ldots, k+k_0+k_{m-1}-1\} \cup \{m-(k+k_0+k_{m-1}-1), \ldots, m-(k+k_0)-1\}$. Then $\delta_{H^1}(m-1) = \delta_T(m-1)$. Next the children of $m-1$ are labeled in T as they have been labeled in H^1.

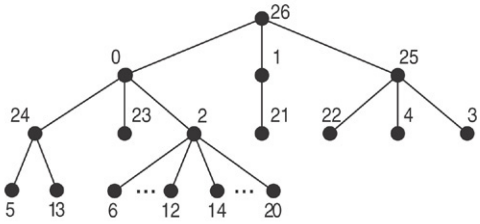

Fig. 4. First transfer between two vertices of odd degree.

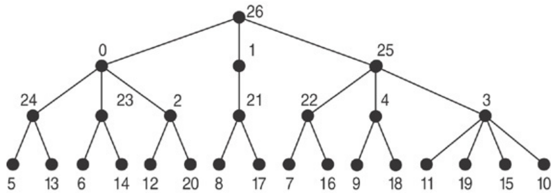

Fig. 5. Resulting graceful tree.

This procedure continues by constructing H^{β} from the preceding H^{α} in (1) as long as $\delta_T(\alpha)$ is even (stopping otherwise), by successive pair transfers (and perhaps one single transfer) $\alpha \to \beta$ in such a way that:

- The children of β in H^{β} form a set of consecutive labels.
- $\delta_{H^{\beta}}(\alpha) = \delta_T(\alpha)$.
- The children of α are labeled in T as they have been labeled in H^{β}.

If T has an even degree sequence then this procedure stops when all the vertices of T have been labeled and so we are done. So assume that T has a quasi even degree sequence, then the above procedure finishes in a graceful tree H^{α_0} such that $\delta_T(\alpha_0)$ is odd with α_0 belonging to the level $L_{D/2-1}$ of T. The children of α_0 in H^{α_0} form a consecutive list of labels. Let H^{β_0} be the successor tree in (1). If m is even then $D/2-1$ is even by Lemma 2.1, which implies that $\alpha_0 + \beta_0 = m$, because in this case all the levels L_j with j even start with a transfer $w \to w'$ such that $w + w' = m$. Similarly if m is odd then $D/2-1$ is odd by Lemma 2.1, which implies that $\alpha_0 + \beta_0 = m-1$, because in this case all the levels L_j with j odd start with a transfer $w \to w'$ such that $w + w' = m-1$. Now we construct the successor tree H^{β_0} in (1) by successive pair transfers $\alpha_0 \to \beta_0$ in such a way that α_0 remains adjacent to its first child if m is even, or its last child if m is odd, remains adjacent to vertex $\lfloor m/2 \rfloor = (\alpha_0 + \beta_0)/2$ and also remains adjacent to all the necessary pairs until completing its odd degree in T. After this operation the children of β_0 in H^{β_0} form two disjoint sets of consecutive labels, $\delta_{H^{\beta_0}}(\alpha_0) = \delta_T(\alpha_0)$ and now the children of α_0 are labeled in T as they have been labeled in H^{β_0}. We continue in this way by constructing H^{β} from the preceding H^{α} in (1) following this second procedure:

- The children of β in H^{β} form two disjoint sets of consecutive labels.
- $\delta_{H^{\beta}}(\alpha) = \delta_T(\alpha)$.
- The children of α are labeled in T as they have been labeled in H^{β}.

To illustrate these procedures let us consider the tree in Fig. 1. First of all we assign to the root the label $m = 26$ and distribute among its neighbors the labels 0, 25, 1 so we have $\delta_T(0) = 4$. The starting tree H^0 is in Fig. 3 in which the children of 0 are all the non-distributed labels. By applying the first procedure we obtain the tree of diameter 5 in Fig. 4, in which $\alpha_0 = 24$ and $\beta_0 = 2$. Then vertex 24 can give to vertex 2 two sets of consecutive labels, from 6 to 12 and from 14 to 20, in such a way that 13 and 5 remains adjacent to vertex 24. Now procedure 2 continues by transferring from 2 to 23 all the labels except labels 12 and 20, that is, the children of 23 are 6, 7, 8, 9, 10, 11 and 14, 15, 16, 17, 18, 19; 23 transfers to 3 all the labels except 6 and 14; 3 transfers to 22 all the labels except 11, 19, 10 and 15; 22 gives to 4 all the labels except 7 and 16, and finally 4 gives to 21 labels 8 and 17. The resulting tree is in Fig. 5. Another graceful labeling for this tree is in Fig. 6 in which we have started by distributing the labels 0, 1, 25 among the neighbors of 26 in a different way.

Case $\delta_T(0)$ odd. Then T has a quasi even degree sequence and $1 = D/2 - 1$, i.e. $D = 4$. We proceed following the second procedure defined in the above case until finishing. ∎

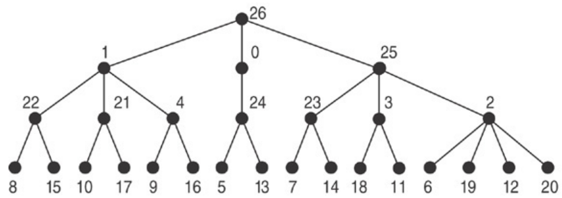

Fig. 6. Another graceful labeling.

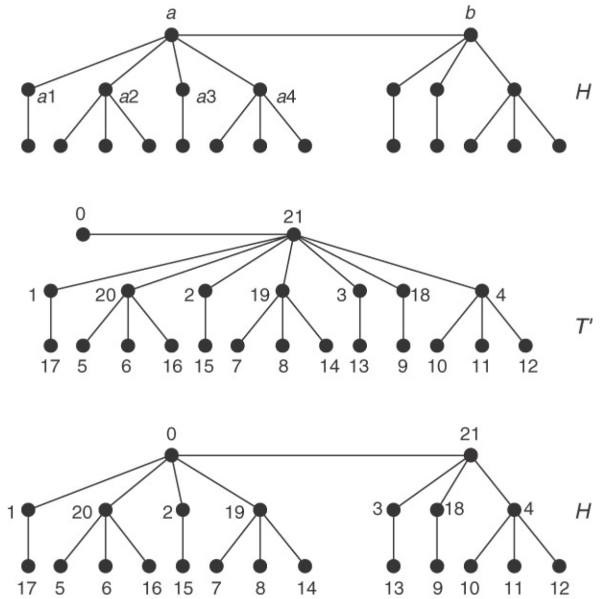

Fig. 7. Illustration.

Theorem 2.2. *Let H be a tree of odd diameter D, m edges and having an even degree sequence or a quasi even degree sequence. Then H is graceful.*

Proof. Let a and b be the two adjacent roots of H. By definition of even or quasi even degree sequence one root must have odd degree and the another one even degree. Without loss of generality we can suppose that $\delta_H(a)$ is odd and let us denote $N_H(a) - b = \{a_1, a_2, \ldots, a_{\delta_H(a)-1}\}$. Let us consider the tree T obtained from H by identifying a and b, say $a = b = t$, hence the degree of t is $\delta_T(t) = \delta_H(a) + \delta_H(b) - 2$ which is an odd number, the diameter of T is $D - 1$ even, and T has an even degree sequence or a quasi even degree sequence. Thus by applying Theorem 2.1, we can define a graceful labeling f of T satisfying that $f(t) = m - 1$ and

$$f(N_T(t)) = \left\{0, 1, \ldots, \frac{\delta_T(t) - 1}{2}, m - 1 - \frac{\delta_T(t) - 1}{2}, \ldots, m - 2\right\},$$

in such a way that the labels of the set $N_H(a) - b$ are

$$f(N_H(a) - b) = \{f(a_1), f(a_2), \ldots, f(a_{\delta_H(a)-1})\} \subset \{0, m - 2, 1, m - 3, \ldots\}.$$

That is, the labels of $N_H(a) - b$ can be ordered by pairs $(f(a_i), f(a_j))$ satisfying $f(a_i) + f(a_j) = m - 2$. Now let T' be a tree obtained from T by joining one new vertex t' and attaching to vertex t the pendant edge tt', hence $\delta_{T'}(t) = \delta_T(t) + 1$ is even and $|V(T')| = |V(H)|$. Let us define a labeling λ on $V(T')$ such that $\lambda(t') = 0$ and $\lambda(h) = f(h) + 1$ if $h \in V(T)$. So T' is a graceful tree of m edges and the neighbors of central vertex t, labeled $\lambda(t) = m$, satisfy that

$$u \in N_{T'}(t) \quad \text{iff} \quad \lambda(u) \in \left\{0, 1, \ldots, \frac{\delta_{T'}(t)}{2}, m - \frac{\delta_{T'}(t)}{2}, \ldots, m - 1\right\}.$$

Then the labels of $N_H(a) - b$ can be ordered by pairs satisfying $\lambda(a_i) + \lambda(a_j) = m$. Thus we obtain a graceful labeling of H from the labeling λ of T' by shifting $(\delta_H(a) - 1)/2$ pairs of subtrees T'_{t,a_i} and T'_{t,a_j} for $a_i, a_j \in N_H(a) - b$ such that $\lambda(a_i) + \lambda(a_j) = m$ from vertex t to the new vertex t' labeled with 0. By the way of illustration see Fig. 7. ■

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