IMA Journal of Applied Mathematics (2007) **72**, 191−205
doi:10.1093/imamat/hxm001
Advance Access publication on February 14, 2007

$\mathscr{C}^\infty$-symmetries and nonlocal symmetries of exponential type

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[Received on 6 July 2006; accepted on 27 November 2006]

Nonlocal symmetries generated by type I hidden symmetries are identified as specific $\mathscr{C}^\infty$-symmetries of an nth-order ordinary differential equation. The general method of reduction associated to these $\mathscr{C}^\infty$-symmetries allows us to give explicit transformations to reduce the order if $n > 1$. As a consequence, we give a complete classification of the equations of arbitrary order that admit this kind of nonlocal symmetries. We illustrate these results with several equations that have no Lie point symmetries. For $n = 1$, the method provides the linearization of first-order equations. This is applied to some examples of Riccati equations and Abel equations of the second kind.

Keywords: ordinary differential equations; hidden symmetries; $\mathscr{C}^\infty$-symmetries.

1. Introduction

It is well-known that the Lie groups of symmetry are a powerful tool to integrate ordinary differential equations. Canonical variables for a Lie point symmetry of a first-order differential equation provide the integration of the equation by quadrature. For higher-order equations, the technique of order reduction leads to the integration by quadrature in the case of solvable symmetry algebras. This can be done by using 'suitable' sequences of generators of the algebra (Olver, 1993; Ibragimov, 1995; Hydon, 2000). When different sequences are used to reduce the order, or the algebra of symmetry is nonsolvable, type I hidden symmetries could appear.

A type I hidden symmetry has been defined as a Lie symmetry V_2 that is lost (not inherited) when a Lie symmetry V_1 is used to reduce the order by one (Abraham-Shrauner & Guo, 1992). It occurs when the commutator $[V_1, V_2]$ does not belong to the subalgebra generated by V_1. In this case, V_2 will not be inherited by the reduced equation, but becomes a nonlocal symmetry of the exponential type. Although these nonlocal symmetries are not well-defined vector fields in the variables of the equation, they can be used to reduce the order (Olver, 1993).

There are many recent papers dealing with the application of nonlocal symmetries to integrate or reduce ordinary differential equations (Abraham-Shrauner *et al.*, 1995; Abraham-Shrauner & Guo, 1996; Abraham-Shrauner, 2002; Adam & Mahomed, 1988, 2002; Edelstein *et al.*, 2001; Geronimi *et al.*, 2001; Nucci & Leach, 2000). Usually, the determination of these nonlocal symmetries for

$$\Delta(x, u^{(n)}) = 0 \tag{1.1}$$

is done by increasing the order

$$\Delta(y, w^{(n+1)}) = 0 \tag{1.2}$$

and checking if there are lost symmetries in the reduction process. The main difficulty, as it is mentioned in Govinder & Leach (1995), is that there is no general method to construct such an equation (1.2).

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In Muriel & Romero (2001, Theorem 5.1), it was proved that exponential nonlocal symmetries are specific types of $\mathscr{C}^\infty$-symmetries. In this work, the following inverse problem is considered: we identify the subclass $\mathscr{H}$ of the $\mathscr{C}^\infty$-symmetries of an equation that corresponds to nonlocal symmetries generated by type I hidden symmetries. Since the $\mathscr{C}^\infty$-symmetries of an equation can be determined by a well-defined algorithm, we have a method to find these nonlocal symmetries. Let us remark a difference with the usual approaches: we first determine the $\mathscr{C}^\infty$-symmetries in $\mathscr{H}$, then the associated nonlocal symmetries and finally we could derive the corresponding equations of higher order (1.2) and the lost Lie symmetries (type I hidden symmetries). This last step is not necessary if one is only interested in determining nonlocal symmetries and using them to integrate the equation or reduce its order.

As a consequence, any process of integration or order reduction by means of nonlocal symmetries can be derived by the general method of reduction associated to a $\mathscr{C}^\infty$-symmetry in $\mathscr{H}$ (Muriel & Romero, 2001). In Section 3, we apply this method and give the explicit transformations that reduce the order of any equation that has a $\mathscr{C}^\infty$-symmetry in $\mathscr{H}$ (i.e. a nonlocal symmetry).

Several important results can be derived of these explicit transformations. In Section 4, we give a classification of the equations of arbitrary order that admit $\mathscr{C}^\infty$-symmetries of $\mathscr{H}$ and therefore type I hidden symmetries. The obtained families of equations include equations without Lie symmetries that can be integrated ((3.12) and (4.10)).

Another important consequence is derived when the method is applied to first-order equations. Any first-order equation that admits a $\mathscr{C}^\infty$-symmetry in $\mathscr{H}$ is linearizable. We also give the explicit changes of variables to transform the equation into the linear form. This is applied to general types of equations not given in an integrable form. In particular, several examples of Abel equations of the second kind can be integrated by our general method of linearization. Some of them are included in Polyanin & Zaitsev (2003); however, we also present new families of Abel equations of the second kind that can be integrated.

2. $\mathscr{C}^\infty$-symmetries and type I hidden symmetries

When an ordinary differential equation admits a Lie algebra of symmetry with two infinitesimal generators V_1 and V_2 such that $[V_1, V_2] = cV_2$, $c \in \mathbb{R}$, one ‘must’ reduce the order by using V_2; then, V_1 is inheritable as a Lie symmetry to the reduced equation. However, if $c \neq 0$ and if we use V_1, instead of V_2, to reduce the equation, then V_2 is lost as a Lie symmetry of the reduced equation, i.e. it is a hidden symmetry of type I. The first prolongation $V_2^{(1)}$ is not a well-defined vector field in the coordinates (x, u) of the reduced equation and it becomes an exponential vector field (see Olver, 1993, p. 181)

$$\mathrm{e}^{\int P(x,u)\mathrm{d}x}\left(\xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}\right). \tag{2.1}$$

These nonlocal symmetries are related with specific $\mathscr{C}^\infty$-symmetries of the reduced equation: the vector field $V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$ is a $\mathscr{C}^\infty$-symmetry of the reduced equation for the function $\lambda = P(x, u)$ (Muriel & Romero, 2001, Theorem 5.1).

In this Section, we consider the following inverse problem: to identify the $\mathscr{C}^\infty$-symmetries of an nth-order equation that can be derived from type I hidden symmetries of some $(n+1)$th-order equation when the Lie method of reduction is applied.

Let

$$\Delta(x, u^{(n)}) = 0 \tag{2.2}$$

be an nth-order ordinary differential equation. Suppose that Δ is a differentiable[1] function on $M^{(n)}$, where M is some open subset of the space $X \times U$ of the variables of the equation. Let us suppose that $V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$ is a $\mathscr{C}^\infty$-symmetry for some function λ that only depends on (x,u). It can be assumed that λ does depend on u, because otherwise fV would be a Lie symmetry for $f(x) = e^{\int \lambda(x)\mathrm{d}x}$ (Muriel & Romero, 2001, Lemma 5.1).

Let us consider the following transformation:

$$y_x = \lambda(x,u). \tag{2.3}$$

Since $\lambda_u \neq 0$, we can locally write $u = \varphi(x, y_x)$. By derivation, we get $u_x = D_x\varphi(x, y_x)$, where D_x denotes the total derivative operator with respect to x. By successive derivations, (2.2) can be written in terms of $\{x, y, y_1, \ldots, y_{n+1}\}$ as an $(n+1)$th-order differential equation in variables (x, y):

$$\widetilde{\Delta}(x, y^{(n+1)}) = 0. \tag{2.4}$$

It is clear that $V_1 = \frac{\partial}{\partial y}$ is a Lie point symmetry of (2.4). Since x and u are invariants of $V_1^{(1)}$, (2.2) corresponds to a reduced equation by the classical Lie method of reduction associated to V_1.

Our next goal is to search the conditions that V must verify to arise from another Lie symmetry V_2 of (2.4). First, such Lie point symmetry

$$V_2 = \widetilde{\xi}(x,y)\frac{\partial}{\partial x} + \widetilde{\eta}(x,y)\frac{\partial}{\partial y} \tag{2.5}$$

must verify $[V_1, V_2] = cV_2$, for some $c \in \mathbb{R}, c \neq 0$. It can be assumed that $c = 1$. This implies

$$\begin{aligned} \widetilde{\xi}(x,y) &= e^y a(x), \\ \widetilde{\eta}(x,y) &= e^y b(x), \end{aligned} \tag{2.6}$$

for some functions $a(x)$ and $b(x)$. By the prolongation formula, $V_2^{(1)} = V_2 + \widetilde{\eta}^{(1)}(x, y, y_x)\frac{\partial}{\partial y_x}$, where

$$\widetilde{\eta}^{(1)}(x, y, y_x) = e^y(-a(x){y_x}^2 + b(x)y_x - a'(x)y_x + b'(x)). \tag{2.7}$$

The vector field $e^{-y}V_2^{(1)}$ is projectable by means of $\pi(x, y, y_x) = (x, y_x)$ and

$$\pi_*(e^{-y}V_2^{(1)}) = a(x)\frac{\partial}{\partial x} + (-a(x){y_x}^2 + b(x)y_x - a'(x)y_x + b'(x))\frac{\partial}{\partial y_x}. \tag{2.8}$$

The vector field $\pi_*(e^{-y}V_2^{(1)})$, written in variables (x, u) by (2.3), must become V. Therefore, we deduce that ξ does not depend on u and $\xi = a$, and function b must verify the following relation:

$$b' + \lambda b = \xi\lambda^2 + \xi'\lambda + \xi\lambda_x + \eta\lambda_u. \tag{2.9}$$

In consequence, we have proved the following result.

[1]In this paper, the expression differentiable means $\mathscr{C}^\infty$ but the results can be checked in each case for lower order of differentiability.

THEOREM 2.1 Let V be a $\mathscr{C}^\infty$-symmetry for some function $\lambda = \lambda(x, u)$ of an equation $\Delta(x, u^{(n)}) = 0$. Consider the $(n+1)$th-order equation (2.4) obtained by the transformation $y_x = \lambda(x, u)$.

Then, V arises from a Lie symmetry V_2 of (2.4) if and only if V is of the form

$$V = \xi(x)\frac{\partial}{\partial x} + \left(\frac{b'(x) + \lambda b(x) - \xi(x)\lambda^2 - \xi'(x)\lambda - \xi(x)\lambda_x}{\lambda_u}\right)\frac{\partial}{\partial u}. \tag{2.10}$$

In this case, V_2 is given by $e^y(\xi(x)\frac{\partial}{\partial x} + b(x)\frac{\partial}{\partial y})$ and it is a hidden symmetry of type I.

In what follows, for a given equation, $\mathscr{H}$ will denote the subclass of the $\mathscr{C}^\infty$-symmetries of the form (2.10). The following property of the class $\mathscr{H}$ will be used later.

THEOREM 2.2 Let V be a $\mathscr{C}^\infty$-symmetry of (2.2) such that $V \in \mathscr{H}$. Then, $fV \in \mathscr{H}$ for any differentiable function f depending on x.

Proof. If $V \in \mathscr{H}$, then V is a $\mathscr{C}^\infty$-symmetry of (2.2), for $\lambda = \lambda(x, u)$, and $V = \xi(x)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}$, where

$$\eta = \frac{b' + \lambda b - \xi\lambda^2 - \xi'\lambda - \xi\lambda_x}{\lambda_u} \tag{2.11}$$

for some function b depending on x.

If $f = f(x)$, the vector field fV is a $\mathscr{C}^\infty$-symmetry of (2.2) for the function $\widetilde{\lambda} = \lambda - \frac{f'}{f}$ (Muriel & Romero, 2001, Lemma 5.1). Let us check that $\widetilde{\lambda}$ does only depend on (x, u). Let us prove that

$$f\eta = \frac{\widetilde{b}' + \widetilde{\lambda}\widetilde{b} - (f\xi)\widetilde{\lambda}^2 - (f\xi)'\widetilde{\lambda} - (f\xi)\widetilde{\lambda}_x}{\widetilde{\lambda}_u} \tag{2.12}$$

for some function $\widetilde{b} = \widetilde{b}(x)$. If $\widetilde{\lambda}$ is replaced by $\lambda - \frac{f'}{f}$, the right member of (2.12) becomes

$$\frac{\widetilde{b}' + \lambda(\widetilde{b} + \xi f') - \frac{f'}{f}\widetilde{b} + f(-\xi\lambda^2 - \xi'\lambda - \xi\lambda_x) + f'\xi' + \xi f'' - \xi\frac{f'^2}{f}}{\lambda_u}. \tag{2.13}$$

The comparison between (2.11) and (2.13) leads us to consider

$$\widetilde{b} = bf - \xi f'. \tag{2.14}$$

By (2.11), we have that (2.13) becomes $\frac{f}{\lambda_u}(b' + \lambda b - \xi\lambda^2 - \xi'\lambda - \xi\lambda_x) = f\eta$. Therefore, identity (2.12) is satisfied. □

3. Order reduction

In this section, we apply the method of reduction associated to a $\mathscr{C}^\infty$-symmetry $V \in \mathscr{H}$ (for the general case, see Muriel & Romero, 2003, Theorem 5.2). The following result about $\mathscr{C}^\infty$-symmetries will be used in what follows.

LEMMA 3.1 Let V be a $\mathscr{C}^\infty$-symmetry of $u_n = F(x, u^{(n-1)})$ for some function λ. If the equation is written in a new local system of coordinates $\{X, U^{(n)}\}$, $U_n = G(X, U^{(n-1)})$, then the vector field V, in variables $\{X, U\}$, is a $\mathscr{C}^\infty$-symmetry for the function $\lambda / D_x X$ in variables $\{X, U^{(n)}\}$.

Proof. Since V is a $\mathscr{C}^\infty$-symmetry of the equation, we have (Muriel & Romero, 2001)

$$[V^{[\lambda,(n-1)]}, A_{(x,u)}] = \lambda V^{[\lambda,(n-1)]} + \mu A_{(x,u)}, \tag{3.1}$$

where $A_{(x,u)} = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + F(x, u^{(n-1)}) \frac{\partial}{\partial u_{n-1}}$ is the vector field associated to the equation $u_n = F(x, u^{(n-1)})$. It can be checked that the vector field associated to the equation $U_n = G(X, U^{(n-1)})$ is the vector field $A_{(x,u)}/D_x(X)$ in variables $\{X, U^{(n)}\}$. By the properties of the Lie bracket, we deduce

$$\left[V^{[\lambda,(n-1)]}, \frac{A_{(x,u)}}{D_x(X)}\right] = \tilde{\lambda} V^{[\lambda,(n-1)]} + \tilde{\mu} \frac{A_{(x,u)}}{D_x(X)}, \tag{3.2}$$

where $\tilde{\lambda} = \lambda/D_x(X)$ and $\tilde{\mu} = \mu + V^{[\lambda,(n-1)]}(1/D_x(X))$. By writing (3.2) in terms of the variables $\{X, U^{(n)}\}$, we get the result. □

Let $V \in \mathscr{H}$ be a $\mathscr{C}^\infty$-symmetry of (2.2), of the form (2.10), for $\lambda = \lambda(x, u)$.

3.1 *Determination of two invariants $X = X(x, u)$ and $W = W(x, u, u_1)$ of $V^{[\lambda,(1)]}$*

Case 1: If $\xi(x) = 0$, V becomes $\left(\frac{b'+\lambda b}{\lambda_u}\right)\frac{\partial}{\partial u}$. Let us observe that $b(x) \neq 0$. We consider the following change of variables:

$$\begin{cases} X = x, \\ U = \frac{-b(x)}{\lambda(x,u)b(x)+b'(x)}. \end{cases} \tag{3.3}$$

It is clear that $V(X) = 0$ and $V(U) = -b(X)U$. This expression suggests to consider $\widehat{V} = -\frac{1}{b(x)}V$ instead of V. Clearly, $\widehat{V}$ in variables $\{X, U\}$ becomes $U\frac{\partial}{\partial U}$. By Lemma 3.1, the vector field $\widehat{V}$ is a $\mathscr{C}^\infty$-symmetry, for $\widehat{\lambda}(X, U) = -\frac{1}{U}$, of (2.2) transformed by (3.3):

$$\widehat{\Delta}(X, U^{(n)}) = 0. \tag{3.4}$$

A first-order invariant of $(U\frac{\partial}{\partial U})^{[\widehat{\lambda},(1)]}$ is given by

$$W = \frac{U_X - 1}{U} \tag{3.5}$$

and it is also an invariant of $V^{[\lambda,(1)]}$.

Case 2: For $\xi(x) \neq 0$, let us denote

$$A(x) = \exp\left(-\int \frac{b(x)}{\xi(x)} \mathrm{d}x\right) \tag{3.6}$$

and consider the following change of variables:

$$\begin{cases} X = \int \frac{A(x)}{\xi(x)}\mathrm{d}x + \frac{A(x)}{b(x)-\lambda(x,u)\xi(x)}, \\ U = \frac{A(x)}{b(x)-\lambda(x,u)\xi(x)}. \end{cases} \tag{3.7}$$

It can be checked that $V(X) = 0$ and $V(U) = -A(x)$. Therefore, we can consider $\widetilde{V} = -\frac{1}{A(x)} V$ instead of V. In variables $\{X, U\}$, $\widetilde{V}$ becomes $\frac{\partial}{\partial U}$. This vector field is a $\mathscr{C}^\infty$-symmetry, for $\widetilde{\lambda}(X, U) = \frac{U_X - 1}{U}$, of (2.2) transformed by (3.7) (Lemma 3.1):

$$\widetilde{\Delta}(X, U^{(n)}) = 0. \tag{3.8}$$

A first-order invariant of $(\frac{\partial}{\partial U})^{[\widetilde{\lambda},(1)]}$ and, therefore, of $V^{[\lambda,(1)]}$ is given by

$$W = \frac{U_X - 1}{U}. \tag{3.9}$$

3.2 *Reduction of order and recovery of solutions*

In both cases, we have determined two invariants X and W of $V^{[\lambda,(1)]}$. By derivation, $W_X = \frac{D_x W}{D_x X}$ is an invariant of $V^{[\lambda,(2)]}$ (Muriel & Romero, 2001, Theorem 3.1). If we continue the process, we get a complete system $\{X, W^{(n-1)}\}$ of invariants of $V^{[\lambda,(n)]}$. In terms of $\{X, W^{(n-1)}\}$, (3.4) or (3.8) becomes an $(n-1)$th-order equation of the form

$$W_{n-1} = F_{n-1}(X, W^{(n-2)}). \tag{3.10}$$

If $W(X) = G(X, C_1, \ldots, C_{n-1})$ is the general solution of (3.10), we solve the first-order and linear equation

$$\frac{U_X - 1}{U} = G(X, C_1, \ldots, C_{n-1}) \tag{3.11}$$

to obtain the general solution $U(X) = H(X, C_1, \ldots, C_n)$ of (3.4) or (3.8). Finally, we express this general solution in terms of the original variables, by means of (3.3) or (3.7), to get the general solution of the original equation.

3.3 *An example*

Let us consider the second-order differential equation

$$u_{xx} + \frac{x^2}{4u^3} + u + \frac{1}{2u} = 0. \tag{3.12}$$

This equation was proposed in Muriel & Romero (2001) as an example of an equation with no Lie symmetries, but integrable by using $\mathscr{C}^\infty$-symmetries. The vector field $V = u\frac{\partial}{\partial u}$ is a $\mathscr{C}^\infty$- symmetry for the function $\lambda = \frac{x}{u^2}$. It is clear that $\xi = 0$ does not depend on u and the corresponding condition (2.9)

$$b' + \frac{x}{u^2} b = u\frac{-2x}{u^3} \tag{3.13}$$

is satisfied, e.g. by the constant function $b = -2$. Theorem 2.1 implies that V arises from a type I hidden symmetry of the third-order equation

$$y_{xxx} = \frac{3x^2 y_{xx}^2 - 2x y_x y_{xx} + x^2 y_x^4 + 2x y_x^3 - y_x^2}{2x^2 y_x}, \tag{3.14}$$

where $y_x = \frac{x}{u^2}$. The corresponding type I hidden symmetry is given by $V_2 = -2\,\mathrm{e}^y \frac{\partial}{\partial y}$.

Since $\xi = 0$, we consider the change of variables (3.3) to reduce the order of the equation

$$\begin{cases} X = x, \\ U = \frac{-u^2}{x}. \end{cases} \tag{3.15}$$

In this case,

$$W = \frac{2u_x}{u} + \frac{x}{u^2} - \frac{1}{x} \tag{3.16}$$

and in terms of $\{X, W, W_1\}$, (3.12) becomes

$$W_1 = \frac{1}{2}\left(-W^2 - \frac{2W}{X} + \frac{1}{X^2} - 4\right). \tag{3.17}$$

This is a Bernoulli equation, whose general solution is $W(X) = -2\tan(X - c_1) - \frac{1}{X}$. We can recover the general solution of (3.12) by two ways:

1. From (3.16), we have

$$-2\tan(x - c_1) - \frac{1}{x} = \frac{2u_x}{u} + \frac{x}{u^2} - \frac{1}{x}. \tag{3.18}$$

This is also a Bernoulli equation; its general solution is given by

$$u(x) = \pm\cos(x - c_1)\sqrt{c_2 - \ln(\cos(x - c_1)) - x\tan(x - c_1)}. \tag{3.19}$$

2. Alternatively, we can calculate U from (3.5):

$$-2\tan(X - c_1) - \frac{1}{X} = \frac{U_X - 1}{U}. \tag{3.20}$$

This is a linear equation whose general solution is given by

$$U(X) = \frac{\cos(X - c_1)^2(C_2 + \ln(\cos(X - c_1)) + X\tan(X - c_1))}{X}. \tag{3.21}$$

By (3.15), we replace X by x and $U(X)$ by $\frac{-u^2}{x}$ and the general solution (3.19) is again obtained.

4. Classification of equations

As a consequence of the previous discussion, we can give the general form of an nth-order equation that admits a $\mathscr{C}^\infty$-symmetry of the type described in Theorem 2.1.

By successive derivations of W with respect to X, we can write $W_1, \dots, W_{n-1}$ in terms of $X, U, \dots, U_n$. For example,

$$W = \frac{U_1 - 1}{U}, \tag{4.1}$$

$$W_1 = \frac{U_2}{U} - \left(\frac{U_1 - 1}{U}\right)\frac{U_1}{U}, \tag{4.2}$$

$$W_2 = \frac{U_3}{U} - \left(\frac{3U_1 - 1}{U}\right)\frac{U_2}{U} + 2\left(\frac{U_1 - 1}{U}\right)\left(\frac{U_1}{U}\right)^2. \tag{4.3}$$

When these values are substituted into (3.10), we get an equation of order n in variables X and U. For $n = 1, 2, 3$, we get

$$U_1 = UF_0(X) + 1, \tag{4.4}$$

$$U_2 = UF_1\left(X, \frac{U_1-1}{U}\right) + \left(\frac{U_1-1}{U}\right)U_1, \tag{4.5}$$

$$U_3 = UF_2\left(X, \frac{U_1-1}{U}, \frac{U_2}{U} - \left(\frac{U_1-1}{U}\right)\frac{U_1}{U}\right) + \left(\frac{3U_1-1}{U}\right)U_2 - 2\left(\frac{U_1-1}{U}\right)\frac{U_1^2}{U}. \tag{4.6}$$

Equations of higher orders can be derived in a similar way. By the change of variables (3.3) or (3.7), we can obtain the general form of an equation of order n that admits a $\mathscr{C}^\infty$-symmetry V satisfying (2.9). The associated reduced equations of (4.4)–(4.6) are

$$W = F_0(X), \tag{4.7}$$

$$W_1 = F_1(X, W), \tag{4.8}$$

$$W_2 = F_2(X, W, W_1), \tag{4.9}$$

respectively, where $W = \frac{U_X-1}{U}$.

4.1 *Integration of equations not possessing Lie symmetries*

The families of equations (4.4)–(4.6) include equations that have no Lie symmetries and that are not obviously integrable by standard methods. As an example, let us consider the following second-order equation:

$$U_{XX} = \frac{((U_X)^3 - 2(U_X)^2 + U_X)X + U^3}{(UU_X - U)X}. \tag{4.10}$$

It can be checked that this equation has no Lie symmetries. We observe that (4.10) can be written in the form

$$U_{XX} = F\left(X, \frac{U_X - 1}{U}\right)U + U_X\frac{U_X - 1}{U} \tag{4.11}$$

for $F(a, b) = 1/(ab)$; therefore, (4.10) belongs to the family of equations (4.5).

If $W = \frac{U_X-1}{U}$, the corresponding reduced equation (4.8) is given by

$$W_X = \frac{1}{XW}, \tag{4.12}$$

which is separable. The general solution of (4.12) is given by $W^2 = \ln(X^2) + C_1, C_1 \in \mathbb{R}$. From the linear first order equation:

$$\frac{U_X - 1}{U} = \pm\sqrt{\ln(X^2) + C_1}, \tag{4.13}$$

we recover the general solution of (4.10):

$$U = e^{\pm f(X)}\left(\int e^{\mp f(X)}\,\mathrm{d}X + C_2\right), \tag{4.14}$$

where $f(X) = \int \sqrt{\ln(X^2) + C_1}\mathrm{d}X$.

5. Linearization of first-order differential equations

In general, canonical coordinates for a $\mathscr{C}^{\infty}$-symmetry of a first-order ordinary differential equation do not lead to the integration by quadratures. However, the $\mathscr{C}^{\infty}$-symmetries of the subclass $\mathscr{H}$ allow us to give the explicit changes of coordinates to transform the equation into a linear first-order equation.

Let

$$\Delta(x, u, u_x) = 0 \tag{5.1}$$

be a first-order ordinary differential equation. Let us suppose that (5.1) admits a $\mathscr{C}^{\infty}$-symmetry $V = \xi(x)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}$ for some function λ. In this case, it can always be assumed that λ does only depend on (x, u) because if λ depends on derivatives of u, then they can be expressed in terms of (x, u) by using (5.1) and its derivatives. Let us suppose that there exists some function b satisfying (2.9).

As a consequence of the discussion presented in Section 2, we have proved the following result.

COROLLARY 5.1 Any first-order equation that admits a $\mathscr{C}^{\infty}$-symmetry $V = \xi(x)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} \in \mathscr{H}$, for some function λ, is linearizable by (3.3) if $\xi = 0$ or by (3.7) if $\xi \neq 0$. The transformed linear equation is of the form $U_X = UF_0(X) + 1$, for some function $F_0(X)$.

This result let us obtain exact solutions of first-order equations which are not obviously integrable but have $\mathscr{C}^{\infty}$-symmetries of the subclass $\mathscr{H}$. In Section 6, we illustrate the method with several examples of some Riccati and Abel equations of the second kind.

5.1 *Exponential nonlocal symmetry approaches*

The integration of first-order ordinary differential equation by nonlocal symmetries has been studied by other authors (Adam & Mahomed, 2002; Edelstein *et al.*, 2001; Geronimi *et al.*, 2001, etc.). In what follows, we compare these approaches with the method presented in previous sections.

Theorem 2 in Adam & Mahomed (2002) gives a criterion that must satisfy a nonlocal symmetry

$$\widetilde{V} = \exp\left(\int N(x, u)\mathrm{d}x\right)\left(\xi(x)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}\right) \tag{5.2}$$

to transform a first-order equation into an integrable form: the equation

$$a'\xi\alpha^2 - a''\xi\alpha + a'\xi\alpha_x + a'\eta\alpha_u = 0 \tag{5.3}$$

must be satisfied for some function $a(x)$, where $\alpha = N + a''/a + \xi_x/\xi$.

This is derived by considering second-order equations that admit a non-Abelian algebra of dimension two. These equations admit two symmetries V_1 and V_2 such that $[V_1, V_2] = V_1$. When the order of the second-order equation is reduced by using V_2, then V_1 generates the nonlocal symmetry (5.2). This is valid when $V_2 \neq \rho(x, u)V_1$ for any function ρ (type I algebras; Ibragimov, 1995), but the authors in Adam & Mahomed (2002) mention that a similar theorem is not obvious for the case of algebras of the second type.

The following theorem, that can be proved by a straightforward procedure, let us obtain former results from our previous discussions.

THEOREM 5.2 If $\widetilde{V}$ in (5.2) satisfies (5.3), then $V = \xi(x)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}$ belongs to $\mathscr{H}$, for $\lambda = N(x, u)$ and $b = (\lambda - \alpha)\xi$.

Conversely, if $V = \xi(x)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} \in \mathscr{H}$ for $\lambda = \lambda(x, u)$ and some $b = b(x)$, and $\xi \neq 0$, then $\widetilde{V} = \exp\left(\int \lambda\, \mathrm{d}x\right)V$ satisfies (5.3) for $a = \int \frac{A(x)}{\xi(x)}\mathrm{d}x$ and $\alpha = \lambda - \frac{b}{\xi}$, where A is given by (3.6).

In consequence, Corollary 5.1 let us recover, in a simpler version, the result of Theorem 2 in Adam & Mahomed (2002) for first-order equations. It must be stressed that our result is also valid for the second type of 2D non-Abelian algebras.

6. Some examples of linearization of first-order equations

6.1 *Riccati equations*

Let us consider an arbitrary Riccati equation

$$u_x = f_0(x)u^2 + f_1(x)u + f_2(x). \tag{6.1}$$

The determining equation for the $\mathscr{C}^\infty$-symmetries of the form $V = \xi(x)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$, for $\lambda = \lambda(x,u)$, is given by

$$\begin{aligned}\lambda((f_0u^2 + f_1u + f_2)\xi - \eta) = \eta_x - ((f_0\xi)_x u^2 + (f_1\xi)_x u + (f_2\xi)_x)\\ +\eta_u(u^2 f_0 + f_1u + f_2) + (2u f_0 + f_1)\eta.\end{aligned} \tag{6.2}$$

The $\mathscr{C}^\infty$-symmetry V is of the form (2.10) if and only if

$$\eta(x,u) = \frac{b'(x) + \lambda b(x) - \xi(x)\lambda^2 - \xi'(x)\lambda - \xi(x)\lambda_x}{\lambda_u}, \tag{6.3}$$

for some function $b(x)$. When this value is substituted into the determining equation (6.2), we obtain a second-order partial differential equation for λ. To obtain particular solutions of this partial differential equation, we search solutions of the form $\lambda(x,u) = \lambda_1(x)u + \lambda_2(x)$. It can be assumed that $\lambda_2(x) = 0$ because by Theorem 2.2, $V \in \mathscr{H}$ for $\lambda = \lambda_1 u + \lambda_2$ if and only if $\mathrm{e}^{\int \lambda_2\, \mathrm{d}x} V \in \mathscr{H}$ for $\widetilde{\lambda} = \lambda_1 u$.

From the study of the resulting determining equations, we deduce the following result.

PROPOSITION 6.1 The unique $\mathscr{C}^\infty$-symmetries of the form $V = \eta(x,u)\frac{\partial}{\partial u} \in \mathscr{H}$ for $\lambda(x,u)$ linear on u are

$$V = f(x)b(x)(u - s(x))\frac{\partial}{\partial u} \quad \text{for } \lambda = f_0(x)u - \frac{f'(x)}{f(x)}, \tag{6.4}$$

where $s(x)$ is a particular solution of the Riccati equation.

The corresponding transformation (3.3) is given by

$$\begin{cases} X = x, \\ U = -\frac{1}{f_0(x)(u - s(x))}, \end{cases} \tag{6.5}$$

that transforms the Riccati equation (6.1) into the linear equation $U_X = 1 + F_0(X)U$, where $F_0(X) = -\frac{f_0'(x)}{f_0(x)} - f_1(x) - 2f_0(x)s(x)$.

6.1.1 *Some particular examples.* The examples in Adam & Mahomed (1988, 2002) of Riccati equations that are integrable by means of nonlocal symmetries are particular cases of Proposition 6.1: the equations

$$u_x + u^2 + xu - 1 = 0, \tag{6.6}$$

$$u_x + u^2 + u - \frac{1}{x} = 0 \tag{6.7}$$

admit $\widetilde{V} = (1 - xu)\exp(-\int u\,\mathrm{d}x)\frac{\partial}{\partial u}$ as a nonlocal symmetry. In consequence, $V = (1 - xu)\frac{\partial}{\partial u}$ is a $\mathscr{C}^\infty$-symmetry for $\lambda = -u$. Since λ is linear in u ($\lambda(x, u) = f_0(x)u$), then V is a $\mathscr{C}^\infty$-symmetry of the type considered in Proposition 6.1. In this case, $f(x) = 1, b(x) = x$ and $s(x) = \frac{1}{x}$ is a particular solution of (6.6). By means of (6.5),

$$\begin{cases} X = x, \\ U = \frac{x}{ux-1}, \end{cases} \tag{6.8}$$

the Riccati equations (6.6) and (6.7) become, respectively, the linear equations

$$U_X = 1 + \left(X + \frac{2}{X}\right)U, \tag{6.9}$$

$$U_X = 1 + \left(1 + \frac{2}{X}\right)U. \tag{6.10}$$

6.2 *Abel equations of the second kind*

Let us consider an arbitrary family of Abel equations of the second kind

$$uu_x = g(x)u + f(x). \tag{6.11}$$

Next, we determine some equations in this family that can be written in an integrable form by the method of Section 2.

A vector field $V = \xi(x)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}$ is a $\mathscr{C}^\infty$-symmetry of (6.11) for $\lambda = \lambda(x, u)$ if and only if

$$\begin{aligned} ((\eta - g)\xi u^2 - f\xi u)\lambda = &-(\eta_x + g\eta_u - g\xi_x - g_x\xi)u^2 \\ &-(f\eta_u - f\xi_x - f_x\xi)u - f\eta \end{aligned} \tag{6.12}$$

and $V \in \mathscr{H}$ if and only if

$$\eta(x, u) = \frac{b'(x) + \lambda b(x) - \xi(x)\lambda^2 - \xi'(x)\lambda - \xi(x)\lambda_x}{\lambda_u}, \tag{6.13}$$

for some function $b(x)$. When this value is substituted into (6.12), a second-order partial differential equation for λ is obtained. We have obtained several particular solutions:
Case I: For $\lambda = \lambda_1 x^m/u$, where $\lambda_1, m \in \mathbb{R}$ and $m \neq -1$, we obtain

$$\begin{aligned} \xi(x) &= p_2 x^{-m} + p_1 x, \\ \eta(x, u) &= \lambda_1 p_1 x^{m+1} + ((m + 1)p_1 - b)u + \lambda_1 p_2, \end{aligned} \tag{6.14}$$

where $p_1, p_2, f_1, g_1 \in \mathbb{R}$. Depending on the values of p_1 and b, we get four different families of functions g and f (see Table 1). In Tables 1 and 2, $k_1 = b/(p_1(m + 1))$ and $k_2 = b/(p_2(m + 1))$.
Case II: For $\lambda = \lambda_1 x^m/u$, where $\lambda_1 \in \mathbb{R}$ and $m = -1$, we obtain

$$\begin{aligned} \xi(x) &= x(p_2 \ln x + p_1), \\ \eta(x, u) &= \lambda_1 p_2 \ln x + (p_2 - b)u + \lambda_1 p_1, \end{aligned} \tag{6.15}$$

where $p_1, p_2, f_1, g_1 \in \mathbb{R}$. Depending on the values of p_2 and b, we get four different families of functions g and f (see Table 3). In Tables 3 and 4, $k_3 = b/p_2$ and $k_4 = b/p_1$.

TABLE 1 *Case I: $m \neq -1$. Families of functions g and f*

I-1	$p_1 \neq 0$, $b \neq 0$	$g(x) = x^m \left(\frac{p_1^{k_1} g_1}{(p_1 x^{m+1}+p_2)^{k_1}} + \frac{2\lambda_1 p_1}{b}(m+1) - \lambda_1 \right)$, $f(x) = -\frac{x^m}{b^2 p_1}(b\lambda_1 p_1^{k_1+1} g_1 (p_1 x^{m+1} + p_2)^{1-k_1} - b^2 p_1^{2k_1} f_1 (p_1 x^m + p_2)^{1-2k_1} + p_1 \lambda_1^2((m+1)p_1 - b)(p_1 x^{m+1} + p_2))$
I-2	$p_1 = 0$, $b \neq 0$	$g(x) = x^m \left(g_1 \, \mathrm{e}^{-k_2 x^{m+1}} - \lambda_1 \right)$, $f(x) = x^m \left(f_1 \, \mathrm{e}^{-2k_2 x^{m+1}} - \frac{\lambda_1 p_2}{b} \left(g_1 \, \mathrm{e}^{-k_2 x^{m+1}} - \lambda_1 \right) \right)$
I-3	$p_1 \neq 0$, $b = 0$	$g(x) = x^m \left(2\lambda_1 \ln\left(\frac{p_1 x^{m+1}+p_2}{p_1} \right) + g_1 \right)$, $f(x) = \frac{(p_1 x^{m+1}+p_2)}{p_1} x^m \left(f_1 - \frac{\lambda_1}{m+1} \ln\left(\frac{p_1 x^{m+1}+p_2}{p_1} \right) \left(g_1 - \lambda_1 + \lambda_1 \ln\left(\frac{p_1 x^{m+1}+p_2}{p_1} \right) \right) \right)$
I-4	$p_1 = 0$, $b = 0$	$g(x) = g_1 x^m$, $f(x) = f_1 x^m - \frac{\lambda_1(g_1-\lambda_1)}{m+1} x^{2m+1}$

TABLE 2 *Case I: $m \neq -1$. Linearization and $F_0(X) = \frac{A}{B+CX+DX^2+EX^3}$*

I-1	$X = \frac{\lambda_1(p_1 x^{m+1}+p_2)^{1-k_1}}{b(bu-\lambda_1(p_1 x^{m+1}+p_2))}$,	$U = \frac{u(p_1 x^{m+1}+p_2)^{-k_1}}{bu-\lambda_1(p_1 x^{m+1}+p_2)}$	$A = b\lambda_1^2 p_1$, $C = \lambda_1^2 p_1(b-(m+1)p_1)$, $D = b^2 g_1 \lambda_1 p_1^{k_1+1}$, $E = b^4 f_1 p_1^{2k_1}$
I-2	$X = \frac{\mathrm{e}^{-k_2 x^{m+1}} \lambda_1 p_2}{b(bu-\lambda_1 p_2)}$,	$U = \frac{\mathrm{e}^{-k_2 x^{m+1}} u}{bu-\lambda_1 p_2}$	$A = \lambda_1^2 p_2$, $C = \lambda_1^2 p_2$, $D = g_1 \lambda_1 p_2 b$, $E = f_1 b^3$
I-3	$X = \frac{\ln(p_1 x^{m+1}+p_2)}{p_1(m+1)} - \frac{u}{\lambda_1(p_1 x^{m+1}+p_2)}$,	$U = -\frac{u}{\lambda_1(p_1 x^{m+1}+p_2)}$	$A = \lambda_1^2(m+1)p_1$, $B = -f_1(m+1) - \lambda_1 \ln(p_1)(g_1 - \lambda_1 - \lambda_1 \ln(p_1))$, $C = \lambda_1^2(m+1)^2 p_1^2$, $D = \lambda_1(m+1) p_1(g_1 - \lambda_1 - 2\lambda_1 \ln(p_1))$
I-4	$X = \frac{1}{p_2}\left(\frac{x^{m+1}}{m+1} - \frac{u}{\lambda_1} \right)$,	$U = -\frac{u}{\lambda_1 p_2}$	$A = -\lambda_1^2 p_2$, $B = f_1$, $C = \lambda_1(\lambda_1 - g_1)p_2$

TABLE 3 *Case II: $m=-1$. Families of functions g and f*

II-1	$p_2\neq 0$, $b\neq 0$	$g(x)=\frac{e^{-\frac{p_1}{p_2}}g_1}{x}\left(\frac{p_2\ln x+p_1}{p_2}\right)^{-k_3}+\frac{2\lambda_1 p_2}{bx}-\frac{\lambda_1}{x}$, $f(x)=-\frac{p_2\lambda_1 e^{-\frac{p_1}{p_2}}g_1}{bx}\left(\frac{p_2\ln x+p_1}{p_2}\right)^{-k_3+1}+\frac{e^{-\frac{p_1}{p_2}}f_1}{x}\left(\frac{p_2\ln x+p_1}{p_2}\right)^{-2k_3+1}+\frac{\lambda_1^2(p_2\ln x+p_1)}{bx}(1-k_3)$
II-2	$p_2=0$, $b\neq 0$	$g(x)=g_1x^{-k_4-1}-\frac{\lambda_1}{x}$, $f(x)=-\frac{\lambda_1 p_1 g_1}{b}x^{-k_4-1}+f_1x^{-2k_4-1}+\frac{\lambda_1^2 p_1}{bx}$
II-3	$p_2\neq 0$, $b=0$	$g(x)=\frac{1}{x}\left(2\lambda_1\ln\left(\frac{p_2\ln x+p_1}{p_2}\right)+g_1\right)$, $f(x)=\frac{(p_1+p_2\ln(x))}{p_2x}\left(e^{-\frac{p_1}{p_2}}f_1-\lambda_1\ln(p_1+p_2\ln(x))+\lambda_1\ln(p_1+p_2\ln(x))(-g_1+\lambda_1+2\lambda_1\ln(p_2))\right)$
II-4	$p_2=0$, $b=0$	$g(x)=g_1/x$, $f(x)=\frac{1}{x}(\lambda_1^2-\lambda_1g_1)\ln x+\frac{f_1}{x}$

TABLE 4 *Case II: $m=-1$. Linearization and $F_0(X)=\frac{A}{B+CX+DX^2+EX^3}$*

II-1	$X=-\frac{\lambda_1(p_1+p_2\ln(x))^{1-k_3}}{-ub^2+\lambda_1 b(p_1+p_2\ln(x))}$,	$U=-\frac{u(p_1+p_2\ln(x))^{-k_3}}{\lambda_1(p_1+p_2\ln(x))-bu}$	$A=b\,e^{p_1/p_2}\lambda_1^2p_2$, $C=e^{p_1/p_2}\lambda_1^2(b-p_2)p_2$, $D=b^2g_1\lambda_1p_2^{k_3+1}$, $E=b^4f_1p_2^{2k_3}$
II-2	$X=\frac{\lambda_1p_1x^{-k_4}}{b(bu-\lambda_1p_1)}$,	$U=\frac{ux^{-k_4}}{bu-\lambda_1p_1}$	$A=\lambda_1^2p_1$, $C=\lambda_1^2p_1$, $D=bg_1\lambda_1p_1$, $E=b^3f_1$
II-3	$X=\frac{\ln(p_1+p_2\ln(x))}{p_2}-\frac{u}{\lambda_1(p_1+p_2\ln(x))}$,	$U=-\frac{u}{\lambda_1(p_1+p_2\ln(x))}$	$A=e^{p_1/p_2}\lambda_1^2p_2$, $B=-f_1$, $C=e^{p_1/p_2}\lambda_1p_2(g_1-\lambda_1-2\lambda_1\ln(p_2))$, $D=e^{p_1/p_2}\lambda_1^2p_2^2$
II-4	$X=\frac{\lambda_1\ln(x)-u}{\lambda_1p_1}$,	$U=-\frac{u}{\lambda_1p_1}$	$A=-\lambda_1^2p_1$, $B=f_1$, $C=\lambda_1(\lambda_1-g_1)p_1$

In Table 2 ($m \neq -1$) and Table 4 ($m = -1$), we present the explicit changes of variables to linearize the corresponding equations. In all cases, the resulting first-order linear equations are of the form $U_X = 1 + \frac{A}{B+CX+DX^2+EX^3}U$. The values of constants A, B, C, D and E that are not null are also given.

6.2.1 *Some particular examples.* Several of the exact solutions of Abel equations of the second kind that appear in Polyanin & Zaitsev (2003) can be obtained from the previous general method of linearization. For example, it can be checked that the functions g and f of the following equations are particular cases of the families indicated in each case.

I-1	$uu_x = (ax+3b)u + cx^3 - abx^2 - 2b^2x,$
	$uu_x = (3ax+b)u - a^2x^3 - abx^2 - \frac{2b^2}{9}x,$
	$2uu_x = ((3-m)x-1)u + (m+1)\left(x^3 - x^2 + \frac{2(m+1)}{(m+3)^2}x\right)$
I-2	$uu_x = (a\,e^x + b)u + c\,e^{2x} - ab\,e^x - b^2$
I-3	$uu_x = (2\ln x + a + 1)u + x(-\ln^2 x - a\ln x + b)$
I-4	$uu_x = u + Ax + B$

The Abel equations of the second kind considered in Examples 4–8 in Adam & Mahomed (2002) (some of them are also contained in Polyanin & Zaitsev, 2003) correspond to particular cases of functions appearing in Table 3.

7. Conclusions

In this work, we identify the nonlocal symmetries generated by type I hidden symmetries as a specific subclass $\mathscr{H}$ of the $\mathscr{C}^\infty$-symmetries of the given nth-order equation. This provides an algorithm to find this type of nonlocal symmetries. As a consequence of the general method of reduction associated to a $\mathscr{C}^\infty$-symmetry in $\mathscr{H}$, we can give the explicit transformations that reduce the order in the general case.

We have also provided a classification of the equations of arbitrary order that admits a $\mathscr{C}^\infty$-symmetry in $\mathscr{H}$ and hence, nonlocal symmetries derived by type I hidden symmetries. Several examples of equations without Lie symmetries are included in these families of equations.

For first-order equations, the method provides a general method of linearization. This can be applied to equations that are not obviously given in an integrable form, including examples of Riccati and Abel equations of the second kind.

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