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There's something about the diameter

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Abstract

We study diameter preserving linear bijections from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, Z)$ where X, Y are compact Hausdorff spaces and V, Z are Banach spaces. For instance, we obtain that if X has at least four points, Z is linearly isometric to V and either Z is a $\mathcal{C}_0(L)$ space or Z^* is strictly convex or smooth, then there is a diameter preserving linear bijection from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, Z)$ if and only if X is homeomorphic to Y. We also consider the case when X and Y are not compact but locally compact spaces.

Keywords: Diameter; Isometries; Banach–Stone

1. Introduction and notation

Although there is a vast literature concerning the Banach–Stone theorem and the study of surjective linear isometries between function spaces, only a few articles [1,2,4–6,8] are devoted to a similar aspect: the study of linear bijections between function spaces which preserve the diameter of the range. Such mappings are called diameter preserving linear bijections (dplb(s) from now on) and we shall show that they behave as well as (sometimes even better) the surjective linear isometries, in the sense that the existence of a dplb between two function spaces has remarkable consequences, some of them typical of a Banach–Stone-like theorem.

When it makes sense, we shall use the following notation:

The letters X, Y for compact or locally compact Hausdorff spaces, V, Z for Banach spaces, and Q, R, S, T, U for surjective linear isometries and diameter preserving linear bijections.

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$\mathrm{Hom}(Y, X)$ is the set of homeomorphisms from Y onto X.

B_Z and S_Z are, respectively, the closed unit ball and the unit sphere of Z. Besides, $\mathrm{ex}(B_Z)$ is the set of extreme points of B_Z.

δ_t is defined by $\delta_t(f) = f(t)$ for every f.

ξ_v is defined by $\xi_v(t) = v$ for every t.

$\xi_{X,V}$ is the set $\{f \in \mathcal{C}(X, V)\colon$ there exists $v \in V$ so that $f = \xi_v\}$, i.e., the set of constant functions. We write ξ if no confusion can arise.

$\hat{L}$ is the one-point compactification of the locally compact, Hausdorff space L.

$\#S$ is the cardinal of the set S.

$V \overset{1}{\sim} Z$ means that V is linearly isometric to Z.

Δ_X is the diagonal of the product space $X \times X$, i.e., $\{(x, y) \in X \times X\colon x = y\}$, with the induced topology.

$\dim V$ is the algebraic dimension of V. $\mathcal{L}(A)$ is the linear span of $A \subseteq V$. $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

It can be easily seen that $\rho([f]) = \mathrm{diam}(f(X))$ is a well-defined, complete norm in $\mathcal{C}(X, V)/\xi$ for every compact Hausdorff space X and every Banach space V. We will write $\mathcal{C}^{\rho}_{\xi}(X, V)$ instead of $(\mathcal{C}(X, V)/\xi, \rho)$. Given $[f] \in \mathcal{C}^{\rho}_{\xi}(X, V)$ and $x, x' \in X$, by $[f](x - x')$ we mean $f(x) - f(x')$, which is also well defined. Moreover, note that $[f]$ is completely determined if the values of $[f](x - x_0)$, where $x_0 \in X$ is fixed and x varies in the whole X, are given.

2. Results

The following proposition was proved in the article [1], under a different notation.

Proposition 2.1. *Let X, Y be compact Hausdorff spaces and V, Z Banach spaces. If $T: \mathcal{C}(X, V) \to \mathcal{C}(Y, Z)$ is a diameter preserving linear bijection then*:

(1) *The mapping $D = D(T): V \to Z$ is a linear bijection, where D is defined by $Dv = z$ if and only if $T\xi_v = \xi_z$.*
(2) *The mapping $\hat{T}: \mathcal{C}^{\rho}_{\xi}(X, V) \to \mathcal{C}^{\rho}_{\xi}(Y, Z)$, defined by $\hat{T}([f]) = [Tf]$, is a surjective linear isometry.*

Conversely, given a linear bijection $E: V \to Z$ and a surjective linear isometry $S: \mathcal{C}^{\rho}_{\xi}(X, V) \to \mathcal{C}^{\rho}_{\xi}(Y, Z)$, there exists a diameter preserving linear bijection $T: \mathcal{C}(X, V) \to \mathcal{C}(Y, Z)$ so that $S = \hat{T}$ and $E = D(T)$.

If we consider $\rho(f) = \mathrm{diam}(f(X))$, it satisfies $\|f\| \leqslant \rho(f)$ for every f which vanishes at some point and $\rho(f) \leqslant 2\|f\|$ for every f. The mapping ρ is not a norm but a seminorm, since $\rho(\xi_v) = 0$ for every $v \in V$. It is a norm in every $\mathcal{C}_0(L, V)$ space, where L is a locally compact, noncompact Hausdorff space. To obtain a norm related to ρ in $\mathcal{C}(X, V)$, we could take $\|\ \|_m$ given by $\|f\|_m = \max\{\|f\|_\infty, \rho(f)\}$.

Proposition 2.2. *Let X be a compact Hausdorff space with $\#X \geqslant 2$ and V a Banach space. Given $x_0 \in X$,*

(1) *If x_0 is isolated then*

$$\mathcal{C}_\xi^\rho(X,V) \overset{1}{\sim} \big(\mathcal{C}\big(X\setminus\{x_0\},V\big), \|\ \|_m\big).$$

(2) *If x_0 is not isolated then*

$$\mathcal{C}_\xi^\rho(X,V) \overset{1}{\sim} \big(\mathcal{C}_0\big(X\setminus\{x_0\},V\big), \rho\big).$$

Proof. (1) Consider $T:\mathcal{C}_\xi^\rho(X,V)\to(\mathcal{C}(X\setminus\{x_0\},V),\|\ \|_m)$ defined by $T([f])(x)=f(x)-f(x_0)$. It is well defined and linear. If $g\in\mathcal{C}(X\setminus\{x_0\},V)$, we can extend it to $f\in\mathcal{C}(X,V)$ by letting $f(x_0)=0$, it is clear that $T([f])=g$. So T is surjective. Besides, $\|T([f])\|_m=\max\{\sup\{\|f(x)-f(x_0)\|\colon x\in X\setminus\{x_0\}\},\sup\{\|f(x)-f(x')\|\colon x,x'\in X\setminus\{x_0\}\}\}=\rho([f])$.

(2) The same mapping T as in point (1) is well defined, linear and surjective. Also, $\rho(T[f])=\sup\{\|T([f])(x)-T([f])(x')\|\colon x,x'\in X\setminus\{x_0\}\}=\sup\{\|f(x)-f(x')\|\colon x,x'\in X\setminus\{x_0\}\}\overset{(*)}{=}\sup\{\|f(x)-f(x')\|\colon x,x'\in X\}=\rho([f])$. The equality $(*)$ is true because x_0 is a nonisolated point of X. □

Note that in case (2) of the previous proposition $\rho=\|\ \|_m$, so we could summarize both cases just by saying that for every $x\in X$ we have

$$\mathcal{C}_\xi^\rho(X,V) \overset{1}{\sim} \big(\mathcal{C}_0\big(X\setminus\{x\},V\big), \|\ \|_m\big).$$

The previous two propositions provide easily the following corollaries:

Corollary 2.3. *Let X, Y be locally compact, noncompact Hausdorff spaces and V, Z Banach spaces.*

- *There exists a diameter preserving linear bijection from $\mathcal{C}_0(X,V)$ onto $\mathcal{C}_0(Y,Z)$ if and only if there exists a surjective linear isometry from $\mathcal{C}_\xi^\rho(\hat{X},V)$ onto $\mathcal{C}_\xi^\rho(\hat{Y},Z)$.*
- *There exists a diameter preserving linear bijection from $\mathcal{C}(\hat{X},V)$ onto $\mathcal{C}(\hat{Y},Z)$ if and only if there exists a diameter preserving linear bijection from $\mathcal{C}_0(X,V)$ onto $\mathcal{C}_0(Y,Z)$ and* $\dim V=\dim Z$.

Corollary 2.4. *Let X, Y be compact Hausdorff spaces and V, Z Banach spaces.*

- *If $\#X=1$ then there exists a diameter preserving linear bijection from $\mathcal{C}(X,V)$ onto $\mathcal{C}(Y,Z)$ if and only if $\#Y=1$ and* $\dim V=\dim Z$.
- *If $\#X=2$ then there exists a diameter preserving linear bijection from $\mathcal{C}(X,V)$ onto $\mathcal{C}(Y,Z)$ if and only if there exists a surjective linear isometry from V onto $\mathcal{C}_\xi^\rho(Y,Z)$ and* $\dim V=\dim Z$.

Henceforth we will consider only the cases when $\min\{\#X,\#Y\}\geqslant 3$. Now we shall study the extreme points of the unit ball of $\mathcal{C}_\xi^\rho(X,V)^*$. First we need the following result, which is well known and can be found, for instance, in [3].

Lemma 2.5. *If M is a linear subspace of $(\mathcal{C}(X,V),\|\ \|_\infty)$ then $\mathrm{ex}(B_{M^*})\subseteq\{(v^*\delta_x)|_M\colon x\in X, v^*\in\mathrm{ex}(B_{V^*})\}$. If $M=\mathcal{C}(X,V)$ then the equality holds.*

Our next result could be proved by using tensor products (see [7], for instance), but we prefer a direct approach.

Proposition 2.6. *Let X be a compact Hausdorff space and V a Banach space. The extreme points of the unit ball of $\mathcal{C}_\xi^\rho(X,V)^*$ are precisely the mappings $v^*(\delta_{x_1}-\delta_{x_2})$, where $v^*\in \mathrm{ex}(B_{V^*})$ and $\{x_1,x_2\}\subseteq X$ with $x_1\neq x_2$.*

Proof. Let $R:\mathcal{C}_\xi^\rho(X,V)\to(\mathcal{C}(X\times X,V),\|\ \|_\infty)$ be defined by $R[f](x_1,x_2)=f(x_1)-f(x_2)$, it is easy to see that R is a linear isometry. Therefore $\mathcal{C}_\xi^\rho(X,V)$ is linearly isometric to a subspace M of $(\mathcal{C}(X\times X,V),\|\ \|_\infty)$, applying Lemma 2.5 we deduce that if φ is an extreme point of $\mathcal{C}_\xi^\rho(X,V)^*$ then $\varphi=v^*(\delta_{x_1}-\delta_{x_2})$, where $v^*\in\mathrm{ex}(B_{V^*})$ and $\{x_1,x_2\}\subseteq X$ with $x_1\neq x_2$.

Conversely, take $v^*\in\mathrm{ex}(B_{V^*})$ and $\{x_1,x_2\}\subseteq X$ with $x_1\neq x_2$. Consider $\varphi=v^*(\delta_{x_1}-\delta_{x_2})$ and suppose that $\varphi=\frac{1}{2}(\varphi_1+\varphi_2)$, $\|\varphi_1\|=\|\varphi_2\|=1$. By virtue of the identity $\psi(v)=\mathrm{Re}\,\psi(v)-i\,\mathrm{Re}\,\psi(iv)$, we can assume that φ, φ_1 and φ_2 are real-valued. Let $f\in\mathcal{C}(X,V)$ be so that $f(x_1)=f(x_2)=0$ and $\rho(f)\leqslant 1/2$. Given $\varepsilon>0$ there exists $v\in S_V$ so that $v^*(v)\geqslant 1-\varepsilon/4$. There also exist disjoint neighbourhoods U_1 of x_1 and U_2 of x_2 with $\|f(x)\|\leqslant\varepsilon/4$ if $x\in U_1\cup U_2$. Consider $f_0:X\to[-v/2,v/2]$ continuous and so that $f_0(x_1)=v/2$, $f_0(x_2)=-v/2$ and $f_0(X\setminus(U_1\cup U_2))=\{0\}$. Then we obtain, for every $b\in S_{\mathbb{K}}$, that $\|f_0+bf\|\leqslant 1/2+\varepsilon/4$ and therefore $\rho(f_0+bf)\leqslant 1+\varepsilon/2$. Since $\rho(f_0)=1$, for every $b\in S_{\mathbb{K}}$ we have

$$1\geqslant\varphi_1\big([f_0-bf]\big)+\varphi_1\big([bf]\big)$$

and

$$1+\frac{\varepsilon}{2}\geqslant\varphi_2\big([f_0-bf]\big)$$

which implies

$$\varphi_1\big([bf]\big)\leqslant\varepsilon.$$

It is now clear that $\varphi_1([f])=\varphi_2([f])=0$ for every $f\in\mathcal{C}(X,V)$ such that $f(x_1)=f(x_2)$.

Now, for each $i\in\{1,2\}$ we define $v_i^*:V\to\mathbb{K}$ by $v_i^*(f(x_1)-f(x_2))=\varphi_i([f])$, where f is any function of $\mathcal{C}(X,V)$. If $g(x_1)-g(x_2)=f(x_1)-f(x_2)$ then $\varphi_i([f-g])=0$ and $\varphi_i([f])=\varphi_i([g])$, so v_i^* is well defined. Moreover, v_i^* is linear and $\|v_i^*\|=\|\varphi_i\|=1$. If $f\in\mathcal{C}(X,V)$ then $v^*(f(x_1)-f(x_2))=\frac{1}{2}(\varphi_1(f)+\varphi_2(f))=\frac{1}{2}(v_1^*(f(x_1)-f(x_2))+v_2^*(f(x_1)-f(x_2)))$; therefore $v^*=\frac{1}{2}(v_1^*+v_2^*)$, which immediately implies $v^*=v_1^*=v_2^*$ and thus $\varphi=\varphi_1=\varphi_2$. □

For our main theorem we need a few lemmas to simplify the managing of such extreme points.

Lemma 2.7. *Let X be a compact Hausdorff space and V a Banach space. Considering $\mathcal{C}_\xi^\rho(X,V)^*$, if $v^*(\delta_{x_1}-\delta_{x_2})=w^*(\delta_{x_3}-\delta_{x_4})$, $x_1\neq x_2$, $x_3\neq x_4$ and $\{v^*,w^*\}\subseteq S_{V^*}$ then $v^*\in\{-w^*,w^*\}$ and $\{x_1,x_2\}=\{x_3,x_4\}$.*

Proof. If $\ker v^*\neq\ker w^*$ then there exist $v\in\ker v^*\setminus\ker w^*$ and $f\in\mathcal{C}(X,V)$ so that $f(x_3)-f(x_4)=v$ and $f(X)\subseteq\mathcal{L}(v)$, this implies $w^*(\delta_{x_3}-\delta_{x_4})([f])\neq 0$ and $v^*(\delta_{x_1}-\delta_{x_2})([f])=0$, which is a contradiction.

Thus, there exists $\alpha\in S_{\mathbb{K}}$ so that $w^*=\alpha v^*$. Besides, if we had $\#\{x_1,x_2,x_3,x_4\}\geqslant 3$ we would easily get a contradiction by choosing the appropriate $f\in\mathcal{C}(X,V)$; therefore, $\{x_1,x_2\}=\{x_3,x_4\}$ and in consequence $\alpha\in\{-1,1\}$. □

Lemma 2.8. *Let X be a compact Hausdorff space and V a Banach space. In $\mathcal{C}_\xi^\rho(X,V)^*$, suppose $v^*(\delta_{x_1}-\delta_{x_2})+w^*(\delta_{x_3}-\delta_{x_4})=u^*(\delta_{x_5}-\delta_{x_6})$ where $\{v^*,w^*,u^*\}\subseteq \mathrm{ex}(B_{V^*})$ and $\{x_1,x_2,x_3,x_4,x_5,x_6\}\subseteq X$ with $x_1\neq x_2$, $x_3\neq x_4$, $x_5\neq x_6$. Assume also that $v^*(\delta_{x_1}-\delta_{x_2})$ and $w^*(\delta_{x_3}-\delta_{x_4})$ are linearly independent.*

Then either

(1) $\#\{\ker u^*,\ker v^*,\ker w^*\}=3$*, and there exists* $r^*\in\{-w^*,w^*\}$ *so that* $v^*+r^*\in\mathrm{ex}(B_{V^*})$ *and*

$$\begin{aligned}v^*(\delta_{x_1}-\delta_{x_2})&=v^*(\delta_{x_1}-\delta_{x_2}),\\ w^*(\delta_{x_3}-\delta_{x_4})&=r^*(\delta_{x_1}-\delta_{x_2}),\\ u^*(\delta_{x_5}-\delta_{x_6})&=(v^*+r^*)(\delta_{x_1}-\delta_{x_2})\end{aligned}$$

or

(2) $\#\{\ker u^*,\ker v^*,\ker w^*\}=1$*, and there exists* $r^*\in\{-v^*,v^*\}$ *and* $i\in\{1,2\}$ *so that*

$$\begin{aligned}v^*(\delta_{x_1}-\delta_{x_2})&=r^*(\delta_{x_i}-\delta_{x_{3-i}}),\\ w^*(\delta_{x_3}-\delta_{x_4})&=r^*(\delta_{x_3}-\delta_{x_i}),\\ u^*(\delta_{x_5}-\delta_{x_6})&=r^*(\delta_{x_3}-\delta_{x_{3-i}}).\end{aligned}$$

Proof. Given $w\in\ker w^*$, there exists $f\in\mathcal{C}(X,V)$ so that $f(x_1)-f(x_2)=w$, $f(x_5)-f(x_6)=\pm w$ and $f(X)\subseteq\ker w^*$. Therefore, for every $w\in\ker w^*$ we have $u^*(w)=\pm v^*(w)$ and this implies $u^*=\pm v^*$ in $\ker w^*$. Analogously $u^*=\pm w^*$ in $\ker v^*$ and thus $u^*=\pm v^*\pm w^*$ in $\ker v^*\cup\ker w^*$.

(1) If $\#\{\ker u^*,\ker v^*,\ker w^*\}=3$ then $u^*=\pm v^*\pm w^*$ in V, let us suppose for instance that $u^*=v^*+w^*$, the other cases are analogous. Then we have

$$v^*(\delta_{x_1}-\delta_{x_2}+\delta_{x_6}-\delta_{x_5})=w^*(\delta_{x_4}-\delta_{x_3}+\delta_{x_5}-\delta_{x_6})$$

and, since $\ker v^*\neq\ker w^*$, we deduce that $\delta_{x_1}-\delta_{x_2}+\delta_{x_6}-\delta_{x_5}=0$ and $\delta_{x_4}-\delta_{x_3}+\delta_{x_5}-\delta_{x_6}=0$. Therefore $x_1=x_3=x_5$ and $x_2=x_4=x_6$.

(2) If $\#\{\ker u^*,\ker v^*,\ker w^*\}\leqslant 2$, suppose for instance that $\ker u^*=\ker w^*$, this implies that $v^*=0$ in $\ker w^*$ and therefore $\ker v^*=\ker w^*$. In any other case we can proceed analogously, deducing that $\#\{\ker u^*,\ker v^*,\ker w^*\}=1$.

There exist $\alpha,\beta\in S_{\mathbb{K}}$ so that $w^*=\alpha v^*$ and $u^*=\beta v^*$, so

$$v^*(\delta_{x_1}-\delta_{x_2})+\alpha v^*(\delta_{x_3}-\delta_{x_4})=\beta v^*(\delta_{x_5}-\delta_{x_6}).$$

This implies that, in $\mathcal{C}(X)^*$,

$$(\delta_{x_1}-\delta_{x_2})+\alpha(\delta_{x_3}-\delta_{x_4})=\beta(\delta_{x_5}-\delta_{x_6})$$

and from this it is easy to deduce that $\#\{x_1,x_2,x_3,x_4,x_5,x_6\}\leqslant 3$. Moreover, since $v^*(\delta_{x_1}-\delta_{x_2})$ and $w^*(\delta_{x_3}-\delta_{x_4})$ are linearly independent we obtain $\#\{x_1,x_2,x_3,x_4,x_5,x_6\}=3$. We can assume, renaming if necessary, that $x_3\notin\{x_1,x_2\}$, $x_3=x_5$ and take $\alpha'\in\{-\alpha,\alpha\}$ and $\beta'\in\{-\beta,\beta\}$ so that

$$(\delta_{x_1}-\delta_{x_2})+\alpha'(\delta_{x_3}-\delta_{x_4})=\beta'(\delta_{x_3}-\delta_{x_6}).$$

If it were $x_4=x_6$ we would easily obtain a contradiction, so there exists $i\in\{1,2\}$ so that $x_4=x_i$ and $x_6=x_{3-i}$. Taking $f,g\in\mathcal{C}(X)$ so that $f(x_1)-f(x_2)=g(x_1)-g(x_2)=1$, $f(x_3)-$

$f(x_i)=1$ and $g(x_3)-g(x_i)=-1$ and applying the previous equality to f and g we obtain that there are only two possibilities:

$$\text{If } i=1, \quad \left\{\begin{array}{c} 1+\alpha'=2\beta' \\ 1-\alpha'=0 \end{array}\right\} \implies \alpha'=\beta'=1.$$

$$\text{If } i=2, \quad \left\{\begin{array}{c} 1+\alpha'=0 \\ 1-\alpha'=-2\beta' \end{array}\right\} \implies \alpha'=\beta'=-1.$$

Therefore $\alpha, \beta \in \{-1, 1\}$ and it is routine to check that in each of the four possible values of the pair (α, β) we get the desired conclusion. □

Definition 2.9. Let X be a compact Hausdorff space and V a Banach space. Given $\{v^*, w^*, u^*\} \subseteq \mathrm{ex}(B_{V^*})$ and $\{x_1, x_2, x_3, x_4, x_5, x_6\} \subseteq X$ as in Lemma 2.8, we will say that $(v^*(\delta_{x_1}-\delta_{x_2}), w^*(\delta_{x_3}-\delta_{x_4}), u^*(\delta_{x_5}-\delta_{x_6}))$ is an H-3-uple if it satisfies (1) in that lemma and a C-3-uple if it satisfies (2).

In the previous definition, the C of "C-3-uple" stands for "Classic" and the H of "H-3-uple" stands for "Hexagonal." This terms will be justified after Theorem 2.11.

Lemma 2.10. *Let V, Z be Banach spaces. If there exists $F:\mathrm{co}(\mathrm{ex}(B_{Z^*})) \to \mathrm{co}(\mathrm{ex}(B_{V^*}))$ which is a linear w^*–w^* homeomorphism so that $F(\mathrm{ex}(B_{Z^*})) = \mathrm{ex}(B_{V^*})$ then there exists a surjective linear isometry $G: V \to Z$ so that $G^*|_{\mathrm{co}(\mathrm{ex}(B_{Z^*}))} = F$.*

Proof. Given $z^* \in B_{Z^*}$, there exists $(z^*_\alpha)_{\alpha\in\Lambda} \xrightarrow{w^*} z^*$, a net in $\mathrm{co}(\mathrm{ex}(B_{Z^*}))$. Since $F(z^*_\alpha) \in B_{V^*}$ for every α, there exists a subnet $(z^*_\beta)_{\beta\in\Gamma}$ so that $F(z^*_\beta) \xrightarrow{w^*} v^* \in B_{V^*}$. We define $\overline{F}(z^*) = v^*$, it is straightforward to see that this extension of F to B_{Z^*} is well defined. So we have $\overline{F}: B_{Z^*} \to B_{V^*}$ which is w^*–w^* continuous and linear; analogously with $\overline{F^{-1}}: B_{V^*} \to B_{Z^*}$. Moreover, $\overline{F} \circ \overline{F^{-1}} = \mathrm{Id}_{B_{V^*}}$ in $\mathrm{co}(\mathrm{ex}(B_{V^*}))$, so $\overline{F} \circ \overline{F^{-1}} = \mathrm{Id}_{B_{V^*}}$ in B_{V^*}. Analogously $\overline{F^{-1}} \circ \overline{F} = \mathrm{Id}_{B_{Z^*}}$ in B_{Z^*}, therefore $\overline{F}: B_{Z^*} \to B_{V^*}$ is a w^*–w^* homeomorphism.

By the properties of F we also know that $\overline{F}(rB_{Z^*}) = rB_{V^*}$ for every $r \in [0,1]$, so $\|\overline{F}(z^*)\| \leqslant \|z^*\|$ for every $z^* \in B_{Z^*}$. Same for $\overline{F}^{-1}$, so $\overline{F}$ is an isometry. The natural extension $\tilde{F}: Z^* \to V^*$ given by $\tilde{F}(x) = \|x\|\overline{F}(\frac{x}{\|x\|})$ if $x \neq 0$ is thus a w^*–w^* homeomorphism and a linear isometry, so there exists a surjective linear isometry $G: V \to Z$ with $G^* = \tilde{F}$. □

In the next theorem we consider a surjective linear isometry from $\mathcal{C}^\rho_\xi(X, V)$ onto $\mathcal{C}^\rho_\xi(Y, Z)$. By means of Corollary 2.3, this is a way of obtaining results on dplbs in the compact as well as in the locally compact, noncompact case.

Theorem 2.11. *Let X, Y be compact Hausdorff spaces with at least three points and V, Z Banach spaces. If $S:\mathcal{C}^\rho_\xi(X, V) \to \mathcal{C}^\rho_\xi(Y, Z)$ is a surjective linear isometry then there exist partitions $\{H_Z, C_Z\}$ of $\mathrm{ex}(B_{Z^*})$ and $\{H_V, C_V\}$ of $\mathrm{ex}(B_{V^*})$ satisfying:*

(1) *For every $z^* \in \mathrm{ex}(B_{Z^*})$, $v^* \in \mathrm{ex}(B_{V^*})$, $y, y' \in Y$ and $x, x' \in X$ satisfying $S^*z^*(\delta_y - \delta_{y'}) = v^*(\delta_x - \delta_{x'})$ we have $z^* \in H_Z$ if and only if $v^* \in H_V$, and thus $z^* \in C_Z$ if and only if $v^* \in C_V$.*
(2) *For every $z_1^* \in H_Z$ and $z_2^* \in C_Z$ we have $\|z_1^* \pm z_2^*\| = 2$; besides, $\mathcal{L}(H_Z) \cap \mathrm{ex}(B_{Z^*}) = H_Z$, $\mathcal{L}(C_Z) \cap \mathrm{ex}(B_{Z^*}) = C_Z$ and analogously for V.*

(3) *If* $H_Z \neq \emptyset$ *then there exists a partition* $(Z_a)_{a\in A}$ *of* H_Z *so that every* (Z_a, w^*) *is homeomorphic to* $(X \times X) \setminus \Delta_X$; *moreover, the homeomorphism is provided by a* w^*–w^* *linear homeomorphism from* $\mathrm{co}(Z_a)$ *to* $\mathrm{co}(\mathrm{ex}(B_{\mathcal{C}^\rho_\xi(X,\mathbb{K})^*}))$. *Besides, if* $\#X \geqslant 4$ *then for every* $a, b \in A$ *with* $a \neq b$ *there exist* $z_1^* \in Z_a$ *and* $z_2^* \in Z_b$ *so that* $\|z_1^* \pm z_2^*\| = 2$. *Analogously for* H_V *and* Y.

(4) *If* $C_Z \neq \emptyset$ *then* C_Z *is* w^*–w^* *homeomorphic to* C_V *and* X *is homeomorphic to* Y.

(5) *If* $C_Z = \emptyset$ *and* $\#A = 1$ *then there exist surjective linear isometries* $Q : V \to \mathcal{C}^\rho_\xi(Y, \mathbb{K})$ *and* $R : Z \to \mathcal{C}^\rho_\xi(X, \mathbb{K})$ *so that for every* $f \in \mathcal{C}(X, V)$, $y, y' \in Y$ *and* $x, x' \in X$ *we have*

$$R\big(S[f](y - y')\big)(x - x') = Q\big([f](x - x')\big)(y - y').$$

(6) *If* $H_Z = \emptyset$ *then there exist*:
- *A mapping* $t : \mathrm{ex}(B_{Z^*}) \to \mathrm{Hom}(Y, X)$ *which is* w^**-pointwise continuous* (*in particular,* Y *and* X *are homeomorphic*);
- *A* w^*–w^* *homeomorphism* $F : \mathrm{ex}(B_{Z^*}) \to \mathrm{ex}(B_{V^*})$,

so that for every $f \in \mathcal{C}(X, V)$, $y, y' \in Y$ *and* $z^* \in \mathrm{ex}(B_{Z^*})$ *we have*

$$z^*\big(S[f](y - y')\big) = F(z^*)\big([f]\big(t_{z^*}(y) - t_{z^*}(y')\big)\big),$$

where $t_{z^*} = t(z^*)$.

Proof. Many of the results obtained for X and V can be obtained for Y and Z just by considering $(S^*)^{-1}$; we shall not mention these cases.

We take $H_Z = \{z^* \in \mathrm{ex}(B_{Z^*})$: There exists 3 distinct points $y_1, y_2, y_3 \in Y$ so that $(S^*z^*(\delta_{y_1} - \delta_{y_2}), S^*z^*(\delta_{y_3} - \delta_{y_1}), S^*z^*(\delta_{y_3} - \delta_{y_2}))$ is an H-3-uple$\}$ and $C_Z = \{z^* \in \mathrm{ex}(B_{Z^*})$: There exists 3 distinct points $y_1, y_2, y_3 \in Y$ so that $(S^*z^*(\delta_{y_1} - \delta_{y_2}), S^*z^*(\delta_{y_3} - \delta_{y_1}), S^*z^*(\delta_{y_3} - \delta_{y_2}))$ is a C-3-uple$\}$.

By Lemma 2.8, it is clear that $H_Z \cup C_Z = \mathrm{ex}(B_{Z^*})$.

If $z^* \in H_Z$ then, according to the definitions, there exist $v^*, r^* \in \mathrm{ex}(B_{V^*})$, $x_1, x_2 \in X$ and $y_1, y_2, y_3 \in Y$ with $\#\{y_1, y_2, y_3\} = 3$ so that $v^* + r^* \in \mathrm{ex}(B_{V^*})$, $S^*z^*(\delta_{y_1} - \delta_{y_2}) = v^*(\delta_{x_1} - \delta_{x_2})$ and $S^*z^*(\delta_{y_3} - \delta_{y_1}) = r^*(\delta_{x_1} - \delta_{x_2})$. For every $y_i \in Y \setminus \{y_1, y_2, y_3\}$, if we apply Lemma 2.8 both to $S^*z^*(\delta_{y_i} - \delta_{y_1}) + S^*z^*(\delta_{y_1} - \delta_{y_2})$ and $S^*z^*(\delta_{y_i} - \delta_{y_1}) + S^*z^*(\delta_{y_1} - \delta_{y_3})$ and take into account that $v^* \neq \pm r^*$, we deduce that $S^*z^*(\delta_{y_i} - \delta_{y_1}) = v_i^*(\delta_{x_1} - \delta_{x_2})$ for certain $v_i^* \in \mathrm{ex}(B_{V^*})$. Consequently, for every $y_j \in Y \setminus \{y_i\}$ we have $S^*z^*(\delta_{y_i} - \delta_{y_j}) = (v_i^* - v_j^*)(\delta_{x_1} - \delta_{x_2})$ and $v^*_{(i,j)} = v_i^* - v_j^*$ must be an extreme point of B_{V^*}. In addition, this implies that $z^* \notin C_Z$ and therefore $H_Z \cap C_Z = \emptyset$.

(1) Now if we take $x_k \in X \setminus \{x_1, x_2\}$ then we deduce in the same way that there exists $z_k^* \in \mathrm{ex}(B_{Z^*}) \setminus \{z^*, -z^*\}$ so that $S^*z_k^*(\delta_{y_i} - \delta_{y_j}) = v^*_{(i,j)}(\delta_{x_k} - \delta_{x_1})$ and so for $x_l \in X \setminus \{x_1, x_2, x_k\}$ there exists $z_l^* \in \mathrm{ex}(B_{Z^*}) \setminus \{z^*, -z^*\}$ so that $S^*(z_k^* - z_l^*)(\delta_{y_i} - \delta_{y_j}) = v^*_{(i,j)}(\delta_{x_k} - \delta_{x_l})$, let us call $z^*_{(k,l)} = z_k^* - z_l^*$, which must be an extreme point of B_{Z^*}. This also implies $z^*_{(k,l)} \in H_Z$ and $v^*_{(i,j)} \in H_V$.

(2) By the linearity of S^*, $\mathcal{L}(H_Z) \cap \mathrm{ex}(B_{Z^*}) = H_Z$ and so $\mathcal{L}(C_Z) \cap \mathrm{ex}(B_{Z^*}) = C_Z$. On the other hand, take $z_1^* \in H_Z$ and $z_2^* \in C_Z$. There exists $v_2^* \in \mathrm{ex}(B_{V^*})$, $\{y_1, y_2, y_3\} \subset Y$ and $\{x_1, x_2, x_3\} \subset X$ so that $S^*z_2^*(\delta_{y_i} - \delta_{y_j}) = v_2^*(\delta_{x_i} - \delta_{x_j})$ and $\{x_4, x_5\} \subset X$ and $v^*_{(i,j)} \in \mathrm{ex}(B_{V^*})$ so that $z_1^*(\delta_{y_i} - \delta_{y_j}) = v^*_{(i,j)}(\delta_{x_4} - \delta_{x_5})$ for every $\{i, j\} \subset \{1, 2, 3\}$. Note we can take the same $\{y_1, y_2, y_3\}$ for both z_1^* and z_2^* just because $C_Z \cap H_Z = \emptyset$. Therefore, there must exist $\{i, j\}$ so that $\|S^*(z_1^*(\delta_{y_i} - \delta_{y_j})) + S^*(z_2^*(\delta_{y_i} - \delta_{y_j}))\| = 2$, which implies $\|z_1^* + z_2^*\| = 2$.

(3) Assume $H_Z \neq \emptyset$. Then $H_V \neq \emptyset$; from point (1) we have the equality $S^* z^*_{(k,l)}(\delta_{y_i} - \delta_{y_j}) = v^*_{(i,j)}(\delta_{x_k} - \delta_{x_l})$ and we will say that the equivalence class of z^* is $\{z^*_{(k,l)}\colon (x_k, x_l) \in (X \times X) \setminus \Delta_X\}$. These classes define partitions $(Z_a)_{a\in A}$ of H_Z and analogously $(V_a)_{a\in A}$ of H_V, both indexed by the same set A, so that for every $a \in A$ we can define $\tilde{Q}:\mathrm{ex}(B_{\mathcal{C}^\rho_\xi(Y,\mathbb{K})^*}) \to V_a$ by $\tilde{Q}\lambda(\delta_{y_i} - \delta_{y_j}) = \lambda v^*_{(i,j)}$ and $\tilde{R}:\mathrm{ex}(B_{\mathcal{C}^\rho_\xi(X,\mathbb{K})^*}) \to Z_a$ by $\tilde{R}\lambda(\delta_{x_k} - \delta_{x_l}) = \lambda z^*_{(k,l)}$, being $\tilde{Q}$ and $\tilde{R}$ bijections.

If $\delta_{x_\alpha} - \delta_{x'_\alpha} \xrightarrow{w^*} \delta_x - \delta_{x'}$ then for every $g \in \mathcal{C}(Y, Z)$ and every $v^*_{(i,j)}$ we have $v^*_{(i,j)}(S^{-1}[g](x_\alpha - x'_\alpha)) \to v^*_{(i,j)}(S^{-1}[g](x - x'))$, i.e., $\tilde{R}(\delta_{x_\alpha} - \delta_{x'_\alpha})(\delta_{y_i} - \delta_{y_j}) \xrightarrow{w^*} \tilde{R}(\delta_x - \delta_{x'})(\delta_{y_i} - \delta_{y_j})$, which in turn implies $\tilde{R}(\delta_{x_\alpha} - \delta_{x'_\alpha}) \xrightarrow{w^*} \tilde{R}(\delta_x - \delta_{x'})$. Thus $\tilde{R}$ is w^*–w^* continuous; in addition, $\{0\} \cup \mathrm{ex}(B_{\mathcal{C}^\rho_\xi(X,\mathbb{K})^*})$ is w^*-closed in $B_{\mathcal{C}^\rho_\xi(X,\mathbb{K})^*}$, which implies that $\tilde{R}$ is a w^*–w^* homeomorphism. Note also that $(\mathrm{ex}(B_{\mathcal{C}^\rho_\xi(X,\mathbb{K})^*}), w^*)$ is naturally homeomorphic to $(X \times X) \setminus \Delta_X$. By similar techniques, it is not difficult to prove that $\tilde{R}$ can be linearly extended to a w^*–w^* homeomorphism $\tilde{R}:\mathcal{L}(\mathrm{ex}(B_{\mathcal{C}^\rho_\xi(X,\mathbb{K})^*})) \to \mathcal{L}(Z_a)$. Analogously for $\tilde{Q}$. Now if $4 = \#\{x_1, x_2, x_3, x_4\} \subseteq X$ and $a, b \in A$, $a \neq b$, we can consider $z^*_{a,(1,2)} = \tilde{R}_a(\delta_{x_1} - \delta_{x_2})$ and $z^*_{b,(3,4)} = \tilde{R}_b(\delta_{x_3} - \delta_{x_4})$ and for every $y_i, y_j \in Y$ we have $\|z^*_{a,(1,2)} \pm z^*_{b,(3,4)}\| = \|(z^*_{a,(1,2)} \pm z^*_{b,(3,4)})(\delta_{y_i} - \delta_{y_j})\| = \|\tilde{Q}_a(\delta_{y_i} - \delta_{y_j})(\delta_{x_1} - \delta_{x_2}) \pm \tilde{Q}_b(\delta_{y_i} - \delta_{y_j})(\delta_{x_3} - \delta_{x_4})\| = 2$.

(4) Given $z^* \in C_Z$ and since $H_Z \cap C_Z = \emptyset$, we have that for every $y_1, y_2, y_3 \in Y$ with $\#\{y_1, y_2, y_3\} = 3$ there exist $x_1, x_2, x_3 \in X$ with $\#\{x_1, x_2, x_3\} = 3$ and $v^* \in C_V$ so that

$$S^*z^*(\delta_{y_1} - \delta_{y_2}) = v^*(\delta_{x_1} - \delta_{x_2}),$$
$$S^*z^*(\delta_{y_3} - \delta_{y_1}) = v^*(\delta_{x_3} - \delta_{x_1}),$$
$$S^*z^*(\delta_{y_3} - \delta_{y_2}) = v^*(\delta_{x_3} - \delta_{x_2}).$$

Now if $y \in Y \setminus \{y_1, y_2, y_3\}$ then it can be seen, as it was done before, that $S^*z^*(\delta_y - \delta_{y_3}) = v^*(\delta_x - \delta_{x_3})$ for certain $x \in X$. We deduce that there exists a bijection $F: C_Z \to C_V$ and for every $z^* \in C_Z$ a bijection $t_{z^*}: Y \to X$ satisfying, for every $y_1, y_2 \in Y$,

$$S^*z^*(\delta_{y_1} - \delta_{y_2}) = F(z^*)(\delta_{t_{z^*}(y_1)} - \delta_{t_{z^*}(y_2)}).$$

Now, fix $z^* \in C_Z$ and let $(y_\alpha)_{\alpha\in\Lambda}$ be a net in Y convergent to $y_0 \in Y$. Then $z^*(\delta_{y_\alpha} - \delta_{y_0}) \xrightarrow{w^*} 0$ and so $F(z^*)(\delta_{t_{z^*}(y_\alpha)} - \delta_{t_{z^*}(y_0)}) \xrightarrow{w^*} 0$. Every subnet of $(t_{z^*}(y_\alpha))_{\alpha\in\Lambda}$ has a convergent subnet, let us say $(t_{z^*}(y_\beta))_{\beta\in\Gamma} \to t_0 \in X$. Hence, for every $f \in \mathcal{C}(X, V)$ we have that $F(z^*)(f(t_{z^*}(y_\beta)) - f(t_0)) \to 0$ and $F(z^*)(f(t_{z^*}(y_\alpha)) - f(t_{z^*}(y_0))) \to 0$. This immediately implies $t_{z^*}(y_0) = t_0$. Since every subnet of $(t_{z^*}(y_\alpha))_{\alpha\in\Lambda}$ has a subnet which is convergent to $t_{z^*}(y_0)$, the net $(t_{z^*}(y_\alpha))_{\alpha\in\Lambda}$ converges to $t_{z^*}(y_0)$. This proves that t_{z^*} is continuous and hence a homeomorphism.

Suppose that $(z^*_\alpha)_{\alpha\in\Lambda}$ is a net in C_Z which is w^*-convergent to $z^*_0 \in C_Z$. Then, $z^*_\alpha(\delta_{y_1} - \delta_{y_2}) \xrightarrow{w^*} z^*_0(\delta_{y_1} - \delta_{y_2})$ for every $y_1, y_2 \in Y$. Therefore, $F(z^*_\alpha)(\delta_{t_{z^*_\alpha}(y_1)} - \delta_{t_{z^*_\alpha}(y_2)}) \xrightarrow{w^*} F(z^*_0)(\delta_{t_{z^*_0}(y_1)} - \delta_{t_{z^*_0}(y_2)})$. Consider a subnet of $(F(z^*_\alpha))_{\alpha\in\Lambda}$, let us say $(F(z^*_\beta))_{\beta\in\Gamma}$; this subnet has a subnet, let us say $(F(z^*_\gamma))_{\gamma\in\Delta}$, so that $t_{z^*_\gamma}(y_1) \to x_1$, $t_{z^*_\gamma}(y_2) \to x_2$ and $F(z^*_\gamma) \xrightarrow{w^*} u^*$ for certain $x_1, x_2 \in X$ and $u^* \in B_{Z^*}$. Let us call $x_3 = t_{z^*_0}(y_1)$ and $x_4 = t_{z^*_0}(y_2)$, we have $F(z^*_\gamma)(\delta_{x_1} - \delta_{x_2}) \xrightarrow{w^*} F(z^*_0)(\delta_{x_3} - \delta_{x_4})$ and $F(z^*_\gamma)(\delta_{x_1} - \delta_{x_2}) \xrightarrow{w^*} u^*(\delta_{x_1} - \delta_{x_2})$. Since $F(z^*_0)(\delta_{x_3} - \delta_{x_4}) = u^*(\delta_{x_1} - \delta_{x_2})$, if $x_3 \neq x_4$ then $u^* \neq 0$ and $\{x_1, x_2\} = \{x_3, x_4\}$. This implies $F(z^*_0) = \pm u^*$; suppose that

$F(z_0^*) = -u^*$, then $x_1 = x_4$ and $x_2 = x_3$. If we take $y_3 \notin \{y_1, y_2\}$ and $x_5 = t^*_{z_0}(y_3)$ then $x_5 \notin \{x_3, x_4\}$ and there exists a subnet of $(t_{z^*_\gamma}(y_3))_{\gamma\in\Delta}$, let us say $(t_{z^*_\mu}(y_3))_{\mu\in\Sigma}$, so that $t_{z^*_\mu}(y_3) \to x_6$ for certain $x_6 \in X$. As above, we obtain that $F(z_0^*)(\delta_{x_3} - \delta_{x_5}) = u^*(\delta_{x_1} - \delta_{x_6})$ but the set $\{x_1, x_3, x_5\} = \{x_3, x_4, x_5\}$ has three elements and this produces an immediate contradiction. Therefore $F(z_0^*) = u^*$, $x_1 = x_3$ and $x_2 = x_4$.

We have proved that every subnet of $(F(z^*_\alpha))_{\alpha\in\Lambda}$ has a subnet convergent to $F(z_0^*)$, and that for every $y_1 \in Y$, every subnet of $(t_{z^*_\alpha}(y_1))_{\alpha\in\Lambda}$ has a subnet convergent to $t_{z^*_0}(y_1)$. Thus $F(z_\alpha) \xrightarrow{w^*} F(z_0^*)$ and for every $y \in Y$, $t_{z^*_\alpha}(y) \to t_{z^*_0}(y)$. We could do with the mapping F^{-1} the same as with F, so we deduce that $F : C_Z \to C_V$ is a w^*–w^* homeomorphism and that $t : C_Z \to \mathrm{Hom}(Y, X)$, defined by $t(z^*) = t_{z^*}$, is w^*-pointwise continuous.

(5) Call $A = \{a\}$, in this case $Z_a = H_Z = \mathrm{ex}(B_{Z^*})$, $V_a = H_V = \mathrm{ex}(B_{V^*})$. By Lemma 2.10 and point (3), there exist $Q : V \to \mathcal{C}^\rho_\xi(Y)$ and $R : Z \to \mathcal{C}^\rho_\xi(X)$ surjective linear isometries so that

$$S^*\big(R^*(\delta_x - \delta_{x'})\big)(\delta_y - \delta_{y'}) = Q^*(\delta_y - \delta_{y'})(\delta_x - \delta_{x'})$$

for every $x, x' \in X$ and $y, y' \in Y$. In other words,

$$R\big(S[f](y - y')\big)(x - x') = Q\big([f](x - x')\big)(y - y')$$

for every $[f] \in \mathcal{C}^\rho_\xi(X, V)$, $y, y' \in Y$ and $x, x' \in X$.

(6) In this case $C_Z = \mathrm{ex}(B_{Z^*})$, $C_V = \mathrm{ex}(B_{V^*})$ and the result follows easily from the proof of point (4). □

Note that the simplest non-trivial example of a $\mathcal{C}^\rho_\xi(X, V)$ space is $\mathcal{C}^\rho_\xi(\{1, 2, 3\}, \mathbb{R})$, whose unit ball is a regular hexagon. That is why the "H" in H_Z. On the other hand, when $C_Z = \mathrm{ex}(B_{Z^*})$ the surjective linear isometry S exhibits somehow a typical Banach–Stone behaviour, so we use the term "Classic."

Throughout this paper the set A in the previous theorem will be denoted by $A(S)$ or simply A if no confusion can arise. It is logical to wonder whether A can have any cardinality or not. The next example provides an affirmative answer.

Example 2.12. Let Γ be a set, by $\mathcal{B}(\Gamma, V)$ we mean the space of bounded mappings from Γ to V with the supremum norm. Now consider two compact spaces X and Y having at least three points each, and define $S : \mathcal{C}^\rho_\xi(X, \mathcal{B}(\Gamma, \mathcal{C}^\rho_\xi(Y))) \to \mathcal{C}^\rho_\xi(Y, \mathcal{B}(\Gamma, \mathcal{C}^\rho_\xi(X)))$ by $S[f](y - y')(\gamma)(x - x') = [f](x - x')(\gamma)(y - y')$, where $\gamma \in \Gamma$. It is easy to prove that S is a surjective linear isometry for which $H_Z = \mathrm{ex}(B_{Z^*})$ and $\#A = \#\Gamma$.

Taking $V = Z = \mathbb{K}$ in Theorem 2.11 we deduce that there is a dplb from $\mathcal{C}(X, \mathbb{K})$ onto $\mathcal{C}(Y, \mathbb{K})$ if and only if $\mathcal{C}^\rho_\xi(X, \mathbb{K})$ is linearly isometric to $\mathcal{C}^\rho_\xi(Y, \mathbb{K})$ if and only if X is homeomorphic to Y, a result that had also been obtained in [2]. This yields the following corollary:

Corollary 2.13. *Let X, Y be compact Hausdorff spaces with at least three points and V, Z be linearly isometric Banach spaces. Suppose that there exists a surjective linear isometry $S : \mathcal{C}^\rho_\xi(X, V) \to \mathcal{C}^\rho_\xi(Y, Z)$ so that either $C_Z(S) \neq \emptyset$ or $\#A(S) = 1$. Then X is homeomorphic to Y.*

Proof. Both cases are derived from Theorem 2.11. Indeed, we already knew that if $C_Z \neq \emptyset$ then X is homeomorphic to Y. If $C_Z = \emptyset$ and $\#A = 1$ then $\mathcal{C}^\rho_\xi(X, \mathbb{K}) \overset{1}{\sim} Z \overset{1}{\sim} V \overset{1}{\sim} \mathcal{C}^\rho_\xi(Y, \mathbb{K})$, which implies X is homeomorphic to Y. □

A natural question appears from the previous theorem.

Question 2.14. Let X, Y be compact Hausdorff spaces with at least three points and V, Z be linearly isometric Banach spaces. Suppose that there exists a diameter preserving linear bijection $T:\mathcal{C}(X,V)\to\mathcal{C}(Y,Z)$. Is X necessarily homeomorphic to Y?

Theorem 2.11 motivates the following definitions:

Definitions 2.15. Let X, Y be compact Hausdorff spaces with at least three points and V, Z Banach spaces. If $S:\mathcal{C}_\xi^\rho(X,V)\to\mathcal{C}_\xi^\rho(Y,Z)$ is a surjective linear isometry then we shall say that S is

(1) an H-isometry if $C_Z=\emptyset$;
(2) a C-isometry if $H_Z=\emptyset$;
(3) an HC-isometry in any other case, i.e., when C_Z and H_Z are nonempty.

Accordingly, a dplb T will be H, C or HC depending on what kind of isometry $\hat{T}$ is.

With this new notation and as an immediate corollary from Theorem 2.11 and Corollary 2.13 we obtain:

Corollary 2.16. *Let X, Y be compact Hausdorff spaces with at least three points and V, Z Banach spaces. Suppose $S:\mathcal{C}_\xi^\rho(X,V)\to\mathcal{C}_\xi^\rho(Y,Z)$ is a surjective linear isometry. Then*

(1) *If S is of type C or HC then X is homeomorphic to Y.*
(2) *If Z or V is a $\mathcal{C}_0(L)$ space then S is of type C.*
(3) *If Z^* has no isometric copy of $(\mathbb{R}^2,\|\ \|_\infty)$ (in particular, when Z^* is strictly convex or smooth) then S is of type C or H. If moreover S is of type H and $\#X\geqslant 4$ then $\#A(S)=1$. Analogously for V^* and Y.*

It is easy to construct examples of C-dplbs and we have already given an example of H-dplb, now we construct an example of HC-dplb by joining the other two types.

Example 2.17. Let $S:\mathcal{C}(X,V_1)\to\mathcal{C}(Y,Z_1)$ be an H-dplb and $U:\mathcal{C}(X,V_2)\to\mathcal{C}(Y,Z_2)$ be a C-dplb. Then an easy computation shows that $T:\mathcal{C}(X,V_1\oplus_\infty V_2)\to\mathcal{C}(Y,Z_1\oplus_\infty Z_2)$ defined by $Tf(y)=(Sf_1(y),Uf_2(y))$ is an HC-dplb. Indeed, $(H_{Z_1\oplus_\infty Z_2},w^*)$ is homeomorphic to $(\mathrm{ex}(B_{Z_1^*}),w^*)$ and $(C_{Z_1\oplus_\infty Z_2},w^*)$ is homeomorphic to $(\mathrm{ex}(B_{Z_2^*}),w^*)$.

Although Theorem 2.11 does not provide too much information on HC-dplbs, we have enough signs to suspect that the previous example is essentially unique. In other words,

Conjecture 2.18. *Let X, Y be compact Hausdorff spaces with at least three points and V, Z Banach spaces. If $T:\mathcal{C}(X,V)\to\mathcal{C}(Y,Z)$ is an HC-diameter preserving linear bijection then there exist non-trivial subspaces Z_1,Z_2 of Z and V_1,V_2 of V with $Z=Z_1\oplus_\infty Z_2$ and $V=V_1\oplus_\infty V_2$ and two diameter preserving linear bijections $S:\mathcal{C}(X,V_1)\to\mathcal{C}(Y,Z_1)$ and $U:\mathcal{C}(X,V_2)\to\mathcal{C}(Y,Z_2)$ so that S is an H-dplb, U is a C-dplb and for every $f\in\mathcal{C}(X,V)$ we have $\hat{T}[f]=\hat{S}[f_1]+\hat{U}[f_2]$, where the f_i are the component functions of f.*

Now we provide a characterization of the H-dplbs for which the set A is unitary.

Proposition 2.19. *Let X, Y be compact Hausdorff spaces and V, Z Banach spaces. There exists an H-diameter preserving linear bijection, with $\#A = 1$, from $\mathcal{C}(X,V)$ onto $\mathcal{C}(Y,Z)$ if and only if $Z \overset{1}{\sim} \mathcal{C}_\xi^\rho(X)$, $V \overset{1}{\sim} \mathcal{C}_\xi^\rho(Y)$ and there exists a linear bijection $D : \mathcal{C}_\xi^\rho(Y) \to \mathcal{C}_\xi^\rho(X)$. In such case, if $R : Z \to \mathcal{C}_\xi^\rho(X)$ and $Q : V \to \mathcal{C}_\xi^\rho(Y)$ are surjective linear isometries and $u : \mathcal{C}(X, \mathcal{C}_\xi^\rho(Y)) \to \mathcal{C}_\xi(X)$ is a linear mapping so that $u(\xi_w) = Dw$ for every $w \in \mathcal{C}_\xi^\rho(Y)$ and $x_0 \in X$, $y_0 \in Y$ are fixed then $T : \mathcal{C}(X,V) \to \mathcal{C}(Y,Z)$ defined by $R(Tf(y))(x - x_0) = Q(f(x) - f(x_0))(y - y_0) + u(Qf)(x - x_0)$ is a diameter preserving linear bijection. Conversely, every H-diameter preserving linear bijection from $\mathcal{C}(X,V)$ onto $\mathcal{C}(Y,Z)$ so that $\#A = 1$ has this form.*

Proof. The "only if" part is deduced from Proposition 2.1 plus Theorem 2.11. For the converse statement apply Proposition 2.1 and the fact that $S : \mathcal{C}_\xi^\rho(X, \mathcal{C}_\xi^\rho(Y)) \to \mathcal{C}_\xi^\rho(Y, \mathcal{C}_\xi^\rho(X))$ defined by $S[f](y - y')(x - x') = [f](x - x')(y - y')$ is a surjective linear isometry.

Now suppose we have D, R, Q, u and T as in the statement.

It is clear that T is linear. Let us prove it is diameter preserving. $\rho(Tf) = \sup\{\|Tf(y) - Tf(y')\|\colon y, y' \in Y\} = \sup\{\|R(Tf(y) - Tf(y'))\|\colon y, y' \in Y\} = \sup\{|R(Tf(y) - Tf(y'))(x - x')|\colon y, y' \in Y,\ x, x' \in X\} = \sup\{|Q(f(x) - f(x'))(y - y')|\colon x, x' \in X,\ y, y' \in Y\} = \sup\{\|Q(f(x) - f(x'))\|\colon x, x' \in X\} = \sup\{\|f(x) - f(x')\|\colon x, x' \in X\} = \rho(f)$.

Injectivity. Suppose $Tf = 0$, then $Q(f(x) - f(x_0))(y - y_0) + u(Qf)(x - x_0) = 0$ for every x, y. This implies $u(Qf)(x - x_0) = 0$ for every x, thus $u(Qf) = 0$. Now $Q(f(x) - f(x_0))(y - y_0) = 0$ for every x, y, which implies $f(x) - f(x_0) = 0$ for every x and so $f = \xi_{f(x_0)}$. Since $u(\xi_{Q(f(x_0))}) = D(Q(f(x_0))) = 0$ and D is a linear bijection, we deduce $Q(f(x_0)) = 0$ and so $f = 0$.

Surjectivity. We define $S : \mathcal{C}(Y,Z) \to \mathcal{C}(X,V)$ by $Q(Sg(x))(y - y_0) = R(g(y) - g(y_0))(x - x_0) + D^{-1}(R(g(y_0)) - u(f_g))(y - y_0)$, where $f_g(x)(y - y_0) = R(g(y) - g(y_0))(x - x_0)$. Let us see that $TS = \mathrm{Id}$. Note that $QSg = f_g + \xi_{D^{-1}(R(g(y_0)) - u(f_g))} R(TSg(y))(x - x_0) = Q(Sg(x) - Sg(x_0))(y - y_0) + u(QSg)(x - x_0) = R(g(y) - g(y_0))(x - x_0) + u(QSg)(x - x_0) = R(g(y) - g(y_0))(x - x_0) + u(f_g)(x - x_0) + u(\xi_{D^{-1}(R(g(y_0)) - u(f_g))})(x - x_0) = R(g(y) - g(y_0))(x - x_0) + u(f_g)(x - x_0) + (R(g(y_0)) - u(f_g))(x - x_0) = R(g(y))(x - x_0)$. □

In the next proposition we summarize all the information we have collected on C-dplbs. Note that in this case we do not know:

(1) what intrinsic conditions must be imposed on V, Z to assure the existence of a C-dplb, and
(2) what conditions must be imposed on the mappings F, t and L_{z^*} (which will be presented at once) to assure the bijectivity of T instead of requiring it.

Proposition 2.20. *Let X, Y be compact homeomorphic infinite Hausdorff spaces and V, Z be Banach spaces. Suppose that $L_{z^*} : \mathcal{C}(X,V) \to \mathbb{K}$ is a linear map for every $z^* \in \mathrm{ex}(B_{Z^*})$, $F : \mathrm{ex}(B_{Z^*}) \to \mathrm{ex}(B_{V^*})$ is a bijection, $t : \mathrm{ex}(B_{Z^*}) \to \mathrm{Hom}(Y,X)$ is a mapping and $T : \mathcal{C}(X,V) \to \mathcal{C}(Y,Z)$ is a linear bijection satisfying $z^*(Tf(y)) = F(z^*)(f(t_{z^*}(y))) + L_{z^*}(f)$ for every $z^* \in \mathrm{ex}(B_{Z^*})$, $f \in \mathcal{C}(X,V)$ and $x \in X$. Then T is diameter preserving, t is w^*-pointwise continuous and F is a w^*–w^* homeomorphism. Conversely, for every C-diameter* *preserving linear bijection* $T:\mathcal{C}(X,V)\to\mathcal{C}(Y,Z)$ *there exist* L_{z^*}, F *and* t *as above. Moreover, in this situation we have*:

(1) *F sends w^*-vanishing nets to w^*-vanishing nets.*
(2) *If* $z_1^*,z_2^*,\dots,z_n^*\in \mathrm{ex}(B_{Z^*})$ *are linearly independent,* $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{K}\setminus\{0\}$ *and* $z_0^*=\sum_{i=1}^n\alpha_i z_i^*\in\mathrm{ex}(B_{Z^*})$ *then* $t_{z_0^*}=t_{z_1^*}=\cdots=t_{z_n^*}$. *As a consequence, F can be extended to a linear bijection* $\overline{F}:\mathcal{L}(\mathrm{ex}(B_{Z^*}))\to\mathcal{L}(\mathrm{ex}(B_{V^*}))$.
(3) *$\overline{F}$ is an isometry. Consequently, if Z^* and V^* have the Bade property then* $Z^*\overset{1}{\sim}V^*$.
(4) *If V or Z is strictly convex then t is constant.*
(5) *If any of these hypotheses holds*:
  (a) $\#t(\mathrm{ex}(B_{Z^*}))<\infty$,
  (b) $V\overset{1}{\sim}\mathcal{C}_0(L)$ *and* $Z\overset{1}{\sim}\mathcal{C}_0(M)$ *for certain locally compact spaces L, M,*
  then there exists a surjective linear isometry $G:V\to Z$ *so that* $G^*|_{\mathrm{ex}(B_{Z^*})}=F$, *in particular* $V\overset{1}{\sim}Z$.

Proof. T is trivially diameter preserving, since for every $z\in Z$ we have $\|z\|=\sup\{z^*(z):\ z^*\in\mathrm{ex}(B_{Z^*})\}$. We have already shown in Theorem 2.11 that t is w^*-pointwise continuous and F is a w^*–w^* homeomorphism. The converse is easily deduced from Proposition 2.1 plus Theorem 2.11, just fix $y\in Y$ and define $L_{z^*}(f)=z^*(Tf(y))-F(z^*)(f(t_{z^*(y)}))$. In addition,

(1) For every $z^*\in\mathrm{ex}(B_{Z^*})$, $x_1,x_2\in X$ and $f\in\mathcal{C}(X,V)$ we have

$$z^*\bigl(Tf\bigl(t_{z^*}^{-1}(x_1)\bigr)-Tf\bigl(t_{z^*}^{-1}(x_2)\bigr)\bigr)=F(z^*)\bigl(f(x_1)-f(x_2)\bigr).$$

Take $z_\alpha^*\xrightarrow{w^*}0$, we can suppose $F(z_\alpha^*)\xrightarrow{w^*}v^*\in B_{V^*}$. For every $v\in V$ there exist f,x_1,x_2 with $f(x_1)=v$ and $f(x_2)=0$, so $F(z_\alpha^*)(v)=F(z^*\alpha)(f(x_1)-f(x_2))=z_\alpha^*(Tf(t_{z^*}^{-1}(x_1))-Tf(t_{z^*}^{-1}(x_2)))\to 0$. Therefore $v^*=0$.

(2) In this situation we have $F(z_0^*)(\delta_{t_{z_0^*}(y_1)}-\delta_{t_{z_0^*}(y_2)})=\sum_{i=1}^n\alpha_iF(z_i^*)(\delta_{t_{z_i^*}(y_1)}-\delta_{t_{z_i^*}(y_2)})$ for every $y_1,y_2\in Y$. Since Y is infinite, we can choose y_1,y_2 so that $t_{z_i^*}(y_1)\neq t_{z_j^*}(y_2)$ for every $i,j\in\{0,\dots,n\}$ and therefore we can construct, for every $v\in S_V$, $f\in\mathcal{C}(X,V)$ with $\rho(f)=1$, $f(t_{z_i^*}(y_1))=v/2$ and $f(t_{z_i^*}(y_2))=-v/2$ for every $i\in\{0,\dots,n\}$. This implies $F(z_0^*)=\sum_{i=1}^n\alpha_iF(z_i^*)$ and we can divide $\{1,\dots,n\}$ in a partition $\{J_1,\dots,J_k\}$ of sets which are minimal with the property $\sum_{i\in J_k}\alpha_iF(z_i^*)(\delta_{t_{z_0^*}(y_1)}-\delta_{t_{z_0^*}(y_2)}-(\delta_{t_{z_i^*}(y_1)}-\delta_{t_{z_i^*}(y_2)}))=0$ for every $y_1,y_2\in Y$. For every k, either

$$\sum_{i\in J_k}\alpha_iF\bigl(z_i^*\bigr)=0\quad\text{and}\quad t_{z_i^*}=t_{z_j^*}\quad\text{for every } i,j\in J_k,$$

or

$$t_{z_0^*}=t_{z_i^*}\quad\text{for every } i\in J_k.$$

But in the first situation we have $\sum_{i\in J_k}\alpha_iF(z_i^*)(\delta_{t_{z_i^*}(y_1)}-\delta_{t_{z_i^*}(y_2)})=0$ for every $y_1,y_2\in Y$ and thus $\sum_{i\in J_k}\alpha_iz_i^*(\delta_{y_1}-\delta_{y_2})=0$ for every $y_1,y_2\in Y$, which contradicts the hypothesis of linear independence. Therefore $t_{z_0^*}=t_{z_i^*}$ for every $i\in\{1,\dots,n\}$. In particular, we have proved that F is linear where it makes sense, so it can be linearly extended as desired.

(3) If $z_1^*,z_2^*,\dots,z_n^*$ are extreme points of B_{Z^*} and $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{K}$ then for every $y_1,y_2\in Y$ we have $\hat{T}^*(\sum_{i=1}^n\alpha_iz_i^*)(\delta_{y_1}-\delta_{y_2})=\sum_{i=1}^n\alpha_iF(z_i^*)(\delta_{t_{z_i^*}}(y_1)-\delta_{t_{z_i^*}}(y_2))$ and thus

$\|\sum_{i=1}^{n}\alpha_i z_i^*\| = \|\sum_{i=1}^{n}\alpha_i F(z_i^*)(\delta_{t_{z_i^*}}(y_1) - \delta_{t_{z_i^*}}(y_2))\|$ regardless of y_1, y_2. Moreover, we can choose y_1, y_2 so that $t_{z_i^*}(y_1) \neq t_{z_j^*}(y_2)$ for every $i, j \in \{1, \ldots, n\}$ and therefore we can construct, for every $v \in S_V$, $f \in \mathcal{C}(X, V)$ with $\rho(f) = 1$, $f(t_{z_i^*}(y_1)) = v/2$ and $f(t_{z_i^*}(y_2)) = -v/2$ for every $i \in \{1, \ldots, n\}$. This implies $\|\sum_{i=1}^{n}\alpha_i z_i^*\| \geqslant \sup_{v\in S_V}\|\sum_{i=1}^{n}\alpha_i F(z_i^*)(v)\| = \|\sum_{i=1}^{n}\alpha_i F(z_i^*)\|$. Thus $\overline{F}$ is continuous with $\|\overline{F}\| = 1$. The same reasoning works for $\overline{F}^{-1}$, so we deduce that $\overline{F}$ is a surjective linear isometry.

(4) Let Z be strictly convex. If t is not constant, there exist $z_1^*, z_2^* \in \mathrm{ex}(B_{Z^*})$, $v_1^*, v_2^* \in \mathrm{ex}(B_{V^*})$, $y_1, y_2 \in Y$ and $x_1, x_2, x_3, x_4 \in X$ with $z_1^* \neq z_2^*$, $v_1^* \neq v_2^*$, $T^* z_1^*(\delta_{y_1} - \delta_{y_2}) = v_1^*(\delta_{x_1} - \delta_{x_2})$, $T^* z_2^*(\delta_{y_1} - \delta_{y_2}) = v_2^*(\delta_{x_3} - \delta_{x_4})$ and $\#(\{x_1, x_2\} \cap \{x_3, x_4\}) \leqslant 1$. Moreover, by an argument of w^*-density we can assume that v_1^* and v_2^* attain their norm, i.e., there exist $v_1, v_2 \in S_V$ with $v_1^*(v_1) = v_2^*(v_2) = 1$.

Suppose that $\#(\{x_1, x_2\} \cap \{x_3, x_4\}) = 1$. We can assume $x_2 = x_4$; the other possibilities are similar. It is easy to construct $f_1, f_2 \in \mathcal{C}(X, V)$ satisfying $f_1(x_1) = v_1$, $f_1(x_2) = -v_1$, $f_2(x_3) = v_2$, $f_2(x_2) = 0$, $\rho(f_1) = 2$, $\rho(f_2) = 1$ and $\rho(f_1 \pm f_2) \leqslant 2$. Now take $a = \frac{1}{2}(T(f_1 + f_2)(y_1) - T(f_1 + f_2)(y_2))$ and $b = \frac{1}{2}(T(f_1 - f_2)(y_1) - T(f_1 - f_2)(y_2))$, we have $\|a\| \leqslant 1$, $\|b\| \leqslant 1$, $z_1^*(a + b) = v_1^*(2v_1) = 2$ and $z_2^*(a - b) = v_2^*(v_2) = 1$, which implies $\|a + b\| = 2$ and $\|a - b\| \geqslant 1$, therefore contradicting the strict convexity of Z. The case $\#(\{x_1, x_2\} \cap \{x_3, x_4\}) = 0$ is easier since we have less restrictions on the choice of f_1 and f_2.

(5)(a) Suppose $\sum_{i=1}^{n_\beta}\alpha_{i,\beta} z_{i,\beta}^* \xrightarrow{w^*} 0$, then $\sum_{i=1}^{n_\beta}\alpha_{i,\beta} F(z_{i,\beta}^*)(f(t_{z_{i,\beta}^*}(y_1)) - f(t_{z_{i,\beta}^*}(y_2))) \to 0$ for every $f \in \mathcal{C}(X, V)$ and $y_1, y_2 \in Y$. Let us call $\{t_1, \ldots, t_n\}$ the range of t. We can choose y_1, y_2 so that $t_i(y_1) \neq t_j(y_2)$ for every $i, j \in \{1, \ldots, n\}$. Given $v \in V$, there exists $f \in \mathcal{C}(X, V)$ so that $f(t_i(y_1)) = v$ and $f(t_i(y_2)) = 0$ for every $i \in \{1, \ldots, n\}$. This implies that $\sum_{i=1}^{n_\beta}\alpha_{i,\beta} F(z_{i,\beta}^*) \xrightarrow{w^*} 0$. We have just proved that $\overline{F}$, considered from $\mathrm{co}(\mathrm{ex}(B_{Z^*}))$ onto $\mathrm{co}(\mathrm{ex}(B_{V^*}))$, is w^*–w^* continuous. Analogously for $\overline{F}^{-1}$, now apply Lemma 2.10.

(5)(b) We can assume without loss of generality $V = \mathcal{C}_0(L)$ and $Z = \mathcal{C}_0(M)$. There is a w^*–w^* homeomorphism $F : \mathrm{ex}(B_{\mathcal{C}_0(M)^*}) \to \mathrm{ex}(B_{\mathcal{C}_0(L)^*})$, so the mapping $H : M \to L$ given by $Hm = l$ if and only if $F\delta_m = \delta_l$ is also a homeomorphism. Then $G : \mathcal{C}_0(L) \to \mathcal{C}_0(M)$ given by $G\varphi = \varphi \circ H$ is a surjective linear isometry, and it is trivial that G^* is an extension of F. □

Both Theorems 2.19 and 2.20 can be stated almost identically for the case when X and Y are locally compact, noncompact Hausdorff spaces. Concretely, all points (1)–(5) in Theorem 2.20 remain valid in this case. Note also that point (4) in that theorem works for a property (easily shown to be) weaker than being strictly convex: If $a, b \in S_Z$ and $\|a + b\| = 2$ then $\|a - b\| < 1$.

It is a natural question whether $V \overset{1}{\sim} Z$ whenever there is a C-dplb from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, Z)$. This and similar questions are posed in the next section.

3. Additional questions

Apart from the already stated Problem 2.14 and Conjecture 2.18, a number of questions arises naturally. They can be summarized as follows.

Let X, Y be homeomorphic compact Hausdorff spaces with at least three points, and V, Z Banach spaces. Which relations of implication, apart from the obvious ones, does there exist between the following propositions?

(1) There exists an HC-dplb from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, Z)$.
(2) $Z \overset{1}{\sim} V$.
(3) There exists a C-dplb from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, Z)$.
(4) $Z^* \overset{1}{\sim} V^*$.

Everybody knows that $2 \Rightarrow 4$ and $4 \not\Rightarrow 2$. Besides, it is quite clear that $2 \not\Rightarrow 1$, $4 \not\Rightarrow 1$, $2 \Rightarrow 3$ and $3 \not\Rightarrow 1$. To see $4 \not\Rightarrow 3$ just consider $V = c$, $Z = c_0$ and Proposition 2.20, point (5). So what remains is: $1 \Rightarrow 2$?, $1 \Rightarrow 3$?, $1 \Rightarrow 4$?, where the second one is a formally weaker version of Conjecture 2.18; and $3 \Rightarrow 2$?, $3 \Rightarrow 4$?, which have been shown to happen in several cases in Proposition 2.20.

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