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Exchange Leavitt path algebras and stable rank ☆

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Abstract

We characterize those Leavitt path algebras which are exchange rings in terms of intrinsic properties of the graph and show that the values of the stable rank for these algebras are 1, 2 or ∞. Concrete criteria in terms of properties of the underlying graph are given for each case.

Keywords: Graph; Leavitt path algebra; Exchange ring; Stable rank

0. Introduction

For a row-finite graph E, the Leavitt path algebra $L(E)$ is the algebraic analogue of the Cuntz–Krieger algebra $C^*(E)$ described in [17]. The pioneering papers in which $L(E)$

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is introduced and studied are [1,2,7]. In [7], Ara, Moreno and Pardo carry out a study of the monoid $V(L(E))$. Concretely they show that there is an explicitly described natural isomorphism between the lattice of graded ideals of $L(E)$ and the lattice of order ideals of $V(L(E))$. In [1,2] Abrams and Aranda Pino provide characterizations of the simplicity and purely infinite simplicity, respectively, of the Leavitt path algebra $L(E)$ in terms of properties involving the graph E only.

An associative unital ring R is said to be an exchange ring if R_R has the exchange property introduced by Crawley and Jónsson. The structure of exchange rings has been intensively investigated by several authors; in the not necessarily unital case, their study was initiated by Ara in [3]. On the other hand, the concept of stable rank, introduced by Bass for unital rings (see, e.g., [9]), is very useful in treating the stabilization problem in K-theory. In [20], Vaserstein presents the definition of stable rank for a not necessarily unital ring. For the more specific case of C^*-algebras, the exchange property is closely related with the real rank, because a C^*-algebra has real rank zero if and only if it is an exchange ring [6, Theorem 7.2].

Following the philosophy of [1,2,7], the aim of this paper is to study the exchange property for Leavitt path algebras and, within this class of Leavitt path algebras, their stable rank.

Some of the motivating ideas for our characterization of the exchange property are contained in the works of Jeong and Park [13] and Bates, Hong, Raeburn and Szymański [10], while ideas regarding the stable rank grew from the paper by Deicke, Hong and Szymański [11]. The proofs presented here significantly differ from those of the analytic setting of C^*-algebras and the arguments are necessarily different in the purely algebraic context since many of the tools used there are not available in our setting.

The paper is divided into seven sections. After some preliminaries, we begin by stating basic properties concerning special subsets of graphs. In particular, we study the ideals generated by hereditary and saturated subsets of vertices and cofinality of the graph.

Condition (K), studied in the third section, plays a central role in the paper. On the one hand, it is precisely the condition we need to impose on E so that $L(E)$ is exchange; on the other hand, the development of results concerning the stable rank of $L(E)$ occur under this hypothesis.

The main result characterizing exchange Leavitt path algebras appears in Section 4:

Theorem 4.5. *For a graph E, the following conditions are equivalent*:

(1) *$L(E)$ is an exchange ring.*
(2) *E/H satisfies Condition* (L) *for every hereditary saturated subset H of E^0.*
(3) *E satisfies Condition* (K).
(4) $\mathcal{L}_{\mathrm{gr}}(L(E)) = \mathcal{L}(L(E))$.
(5) *E_H and E/H satisfy Condition* (K) *for every hereditary saturated subset H of E^0.*
(6) *E_H and E/H satisfy Condition* (K) *for some hereditary saturated subset H of E^0.*

The rest of the sections are devoted to computing the stable rank in Leavitt path algebras satisfying Condition (K). The first step towards this aim is done in Section 5, first by investigating the absence of unital purely infinite simple quotients of $L(E)$ (Proposition 5.4) and secondly by relating prime graded ideals with maximal tails (Proposition 5.6). Then, in Section 6, we calculate the stable rank for Leavitt path algebras which do not have nonzero bounded graph traces and for which every vertex lying on a closed simple path is left infinite (Corollary 6.8). The paper finishes in Section 7 with a criterion to compute the stable rank for exchange Leavitt path algebras:

Theorem 7.6. *Let E be a graph satisfying Condition* (K). *Then, the values of the stable rank of $L(E)$ are*:

(1) $\mathrm{sr}(L(E)) = 1$ *if E is acyclic.*
(2) $\mathrm{sr}(L(E)) = \infty$ *if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks.*
(3) $\mathrm{sr}(L(E)) = 2$ *otherwise.*

1. Preliminaries

Throughout this paper, we describe Leavitt path algebras following the presentation of [7, Sections 2 and 4] but using the notation of [1] for the elements.

A (*directed*) *graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*.

A vertex which emits no edges is called a *sink*. A graph E is *finite* if E^0 is a finite set. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. A *path* μ in a graph E is a sequence of edges $\mu = (\mu_1, \dots, \mu_n)$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \dots, n-1$. In such a case, $s(\mu) := s(\mu_1)$ is the *source* of μ and $r(\mu) := r(\mu_n)$ is the *range* of μ. An edge e is an *exit* for a path μ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$. If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is a called a *cycle*. If $v = s(\mu) = r(\mu)$ and $s(\mu_i) \neq v$ for every $i > 1$, then μ is a called a *closed simple path based at v*. We denote by $CSP_E(v)$ the set of closed simple paths in E based at v. For a path μ we denote by μ^0 the set of its vertices, i.e., $\{s(\mu_1), r(\mu_i) \mid i = 1, \dots, n\}$. For $n \geqslant 2$ we define E^n to be the set of paths of length n, and $E^* = \bigcup_{n \geqslant 0} E^n$ the set of all paths. We define a relation $\geqslant$ on E^0 by setting $v \geqslant w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geqslant w$ and $v \in H$ imply $w \in H$. A hereditary set is *saturated* if every vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. The set $T(v) = \{w \in E^0 \mid v \geqslant w\}$ is the *tree* of v, and it is the smallest hereditary subset of E^0 containing v. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x \in X} T(x)$. Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0. The *hereditary saturated closure* of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X. It is shown in [7] that the hereditary saturated closure of a set X is $\bar{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

(1) $\Lambda_0(X) = T(X)$,
(2) $\Lambda_n(X) = \{y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, for $n \geqslant 1$.

Let $E = (E^0, E^1, r, s)$ be a graph, and let K be a field. We define the *Leavitt path K-algebra $L_K(E)$* associated with E ($L(E)$ when the base field is understood) as the K-algebra generated by a set $\{v \mid v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e\in E^1 \mid s(e)=v\}} ee^*$ for every $v \in E^0$ that emits edges.

Note that the relations above imply that $\{ee^* \mid e \in E^1\}$ is a set of pairwise orthogonal idempotents in $L(E)$. Note also that if E is a finite graph then we have $\sum_{v\in E^0} v = 1$. In general the algebra $L(E)$ is not unital, but it can be written as a direct limit of unital Leavitt path algebras (with not unital transition maps), so that it is an algebra with local units. Throughout this paper, we will be concerned only with row-finite graphs.

2. Basic properties of graphs

Let E be a graph. For any subset H of E^0, we will denote by $I(H)$ the ideal of $L(E)$ generated by H.

Lemma 2.1. *If H is a subset of E^0, then $I(H) = I(\bar{H})$, and $\bar{H} = I(H) \cap E^0$.*

Proof. Take $G = I(H) \cap E^0$. By [1, Lemma 3.9], $G \in \mathcal{H}$. Thus, by minimality, we get $H \subseteq \bar{H} \subseteq G$, whence $I(H) \subseteq I(\bar{H}) \subseteq I(G)$. Since $G \subseteq I(H)$, we have $I(G) \subseteq I(H)$, so we get the desired equality. The second statement holds by [7, Proposition 4.2 and Theorem 4.3], as desired. □

For a graph E and a hereditary subset H of E^0, we denote by E/H the *quotient graph*

$$\left(E^0 \setminus H, \left\{e \in E^1 \mid r(e) \notin H\right\}, r|_{(E/H)^1}, s|_{(E/H)^1}\right),$$

and by E_H the *restriction graph*

$$\left(H, \left\{e \in E^1 \mid s(e) \in H\right\}, r|_{(E_H)^1}, s|_{(E_H)^1}\right).$$

Thus both E/H and E_H are simply the full subgraphs of E^0 generated by $E^0 \setminus H$ and H, respectively. Observe that while $L(E_H)$ can be seen as a subalgebra of $L(E)$, the same cannot be said about $L(E/H)$.

Now, we recall that $L(E)$ has a $\mathbb{Z}$-grading. For every $e \in E^1$, set the degree of e as 1, the degree of e^* as -1, and the degree of every element in E^0 as 0. Then we obtain a well-defined degree on the Leavitt path K-algebra $L(E)$, thus, $L(E)$ is a $\mathbb{Z}$-graded algebra:

$$L(E) = \bigoplus_{n\in\mathbb{Z}} L(E)_n, \qquad L(E)_n L(E)_m \subseteq L(E)_{n+m}, \quad \text{for all } n, m \in \mathbb{Z}.$$

An ideal I of a $\mathbb{Z}$-graded algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a *graded ideal* in case $I = \bigoplus_{n\in\mathbb{Z}}(I \cap A_n)$.

Remark 2.2. An ideal J of $L(E)$ is graded if and only if it is generated by idempotents; in fact, $J = I(H)$, where $H = J \cap E^0 \in \mathcal{H}_E$. (See the proofs of [7, Proposition 4.2 and Theorem 4.3].)

Lemma 2.3. *Let E be a graph and consider a proper $H \in \mathcal{H}_E$. Define $\Psi : L(E) \to L(E/H)$ by setting $\Psi(v) = \chi_{(E/H)^0}(v)v$, $\Psi(e) = \chi_{(E/H)^1}(e)e$ and $\Psi(e^*) = \chi_{((E/H)^1)^*}(e^*)e^*$ for every vertex v and every edge e, where $\chi_{(E/H)^0} : E^0 \to K$ and $\chi_{(E/H)^1} : E^1 \to K$ denote the characteristic functions. Then*:

(1) *The map Ψ extends to a K-algebra epimorphism of $\mathbb{Z}$-graded algebras with* $\operatorname{Ker}(\Psi) = I(H)$ *and therefore $L(E)/I(H) \cong L(E/H)$.*
(2) *If X is hereditary in E, then $\Psi(X) \cap (E/H)^0$ is hereditary in E/H.*
(3) *For $X \supseteq H$, $X \in \mathcal{H}_E$ if and only if $\Psi(X) \cap (E/H)^0 \in \mathcal{H}_{(E/H)}$.*
(4) *For every $X \supseteq H$, $\overline{\Psi(X) \cap (E/H)^0} = \Psi(\bar{X}) \cap (E/H)^0$.*

Proof. (1) It was shown in [1, Proof of Theorem 3.11] that Ψ extends to a K-algebra morphism. By definition, Ψ is $\mathbb{Z}$-graded and onto. Moreover, $I(H) \subseteq \operatorname{Ker}(\Psi)$.

Since Ψ is a graded morphism, $\operatorname{Ker}(\Psi) \in \mathcal{L}_{\mathrm{gr}}(L(E))$. By [7, Theorem 4.3], there exists $X \in \mathcal{H}_E$ such that $\operatorname{Ker}(\Psi) = I(X)$. By Lemma 2.1, $H = I(H) \cap E^0 \subseteq I(X) \cap E^0 = X$. Hence, $I(H) \neq \operatorname{Ker}(\Psi)$ if and only if there exists $v \in X \setminus H$. But then $\Psi(v) = v \neq 0$ and $v \in \operatorname{Ker}(\Psi)$, which is impossible.

(2) It is clear by the definition of Ψ.

(3) Since Ψ is a graded epimorphism, there is a bijection between graded ideals of $L(E/H)$ and graded ideals of $L(E)$ containing $I(H)$. Thus, the result holds by [7, Theorem 4.3].

(4) It is immediate by part (3). □

Recall that a ring R is said to be an *idempotent ring* if $R = R^2$. For an idempotent ring R we denote by R-Mod the full subcategory of the category of all left R-modules whose objects are the "unital" nondegenerate modules. Here a left R-module M is said to be *unital* if $M = RM$, and M is said to be *nondegenerate* if, for $m \in M$, $Rm = 0$ implies $m = 0$. Note that if R has an identity then R-Mod is the usual category of left R-modules.

We will use the well-known definition of a Morita context in the case where the rings R and S do not necessarily have an identity. Let R and S be idempotent rings. We say that $(R, S, M, N, \varphi, \psi)$ is a (*surjective*) *Morita context* if ${}_RM_S$ and ${}_SN_R$ are unital bimodules and $\varphi : N \otimes_R M \to S$, $\psi : M \otimes_S N \to R$ are surjective S-bimodule and R-bimodule maps, respectively, satisfying the compatibility relations: $\varphi(n \otimes m)n' = n\psi(m \otimes n')$, $m'\varphi(n \otimes m) = \psi(m' \otimes n)m$ for every $m, m' \in M$, $n, n' \in N$.

In [12] (see Proposition 2.5 and Theorem 2.7) it is proved that if R and S are two idempotent rings, then R-Mod and S-Mod are equivalent categories if and only if there exists a (surjective) Morita context $(R, S, M, N, \varphi, \psi)$. In this case, we will say that the rings R and S are *Morita equivalent* and we will refer to as the (surjective) Morita context (R, S, M, N).

Lemma 2.4. *Let E be a graph and $H \subseteq E^0$ a proper hereditary subset. Then $L(E_H)$ is Morita equivalent to $I(H)$.*

Proof. Define Λ as $\mathbb{N}$ if H is an infinite set or as $\{1, \ldots, \mathrm{card}(H)\}$ otherwise. Let $H = \{v_i \mid i \in \Lambda\}$, and consider the ascending family of idempotents $e_n = \sum_{i=1}^{n} v_i$ $(n \in \Lambda)$. By [1, Lemma 1.6], $\{e_n \mid n \in \Lambda\}$ is a set of local units for $L(E_H)$, so that $L(E_H) = \bigcup_{i \in \Lambda} e_i L(E) e_i$. Since $I(H)$ is generated by the idempotents $v_i \in H$, it is an idempotent ring. Moreover, $I(H) = \bigcup_{i \in \Lambda} L(E) e_i L(E)$. It is not difficult to see that

$$\left(\sum_{i \in \Lambda} e_i L(E) e_i, \sum_{i \in \Lambda} L(E) e_i L(E), \sum_{i \in \Lambda} L(E) e_i, \sum_{i \in \Lambda} e_i L(E)\right)$$

is a (surjective) Morita context for the idempotent rings $L(E_H) = \sum_{i \in \Lambda} e_i L(E) e_i$ and $I(H) = \sum_{i \in \Lambda} L(E) e_i L(E)$, hence $I(H)$ is Morita equivalent to $L(E_H)$. □

Under certain conditions we will see in Section 5 that $I(H)$ is not only Morita equivalent to a Leavitt path algebra; in fact it is isomorphic to a Leavitt path algebra.

The proof of the following lemma is straightforward.

Lemma 2.5. *Let $H \in \mathcal{H}_E$, and let $X \subseteq H$ be any subset. Then, $X \in \mathcal{H}_E$ if and only if $X \in \mathcal{H}_{E_H}$.*

Lemma 2.6. *Let E be a graph and $H \in \mathcal{H}_E$. Then, the canonical map*

$$K_0\big(L(E)\big) \to K_0\big(L(E)/I(H)\big)$$

is an epimorphism.

Proof. If $H = E^0$ or $H = \emptyset$, the result follows trivially. Now, suppose H is a proper subset of E^0. By Lemma 2.3(1) we have $L(E)/I(H) \cong L(E/H)$. By [7, Lemma 5.6],

$$V\big(L(E)\big)/V\big(I(H)\big) \cong V\big(L(E/H)\big) \cong V\big(L(E)/I(H)\big).$$

Since $L(E)$ and $L(E/H)$ have a countable unit, we have that $K_0(L(E)) = \mathrm{Grot}(V(L(E)))$ and $K_0(L(E/H)) = \mathrm{Grot}(V(L(E/H)))$. Hence, the canonical map $K_0(L(E)) \to K_0(L(E)/I(H))$ is clearly an epimorphism, as desired. □

We denote by E^∞ the set of infinite paths $\gamma = (\gamma_n)_{n=1}^{\infty}$ of the graph E and by $E^{\leqslant\infty}$ the set E^∞ together with the set of finite paths in E whose end vertex is a sink. We say that a vertex v in a graph E is *cofinal* if for every $\gamma \in E^{\leqslant\infty}$ there is a vertex w in the path γ such that $v \geqslant w$. We say that a graph E is *cofinal* if so are all the vertices of E.

Observe that if a graph E has cycles, then E cofinal implies that every vertex connects to a cycle (in fact to any cycle).

Lemma 2.7. *If E is cofinal and $v \in E^0$ is a sink, then*:

(1) *The only sink of E is v.*
(2) *For every $w \in E^0$, $v \in T(w)$.*
(3) *E contains no infinite paths. In particular, E is acyclic.*

Proof. (1) It is obvious from the definition.

(2) Since $T(v) = \{v\}$, the result follows from the definition of $T(v)$ by considering the path $\gamma = v \in E^{\leqslant\infty}$.

(3) If $\alpha \in E^{\infty}$, then there exists $w \in \alpha^0$ such that $v \geqslant w$, which is impossible. Thus, in particular, E contains no closed simple paths, and therefore no cycles. $\square$

The next result is known in the case of graphs without sinks. Since we have no knowledge of the existence of a (published) version of the result in the general case, we give a proof for the sake of completeness.

Lemma 2.8. *A graph E is cofinal if and only if $\mathcal{H} = \{\emptyset, E^0\}$.*

Proof. Suppose E to be cofinal. Let $H \in \mathcal{H}$ with $\emptyset \neq H \neq E^0$. Fix $v \in E^0 \setminus H$ and build a path $\gamma \in E^{\leqslant\infty}$ such that $\gamma^0 \cap H = \emptyset$: If v is a sink, take $\gamma = v$. If not, then $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \nsubseteq H$; otherwise, H saturated implies $v \in H$, which is impossible. Hence, there exists $e_1 \in s^{-1}(v)$ such that $r(e_1) \notin H$. Let $\gamma_1 = e_1$ and repeat this process with $r(e_1) \notin H$. By recurrence either we reach a sink or we have an infinite path γ whose vertices are not in H, as desired. Now consider $w \in H$. By the hypothesis, there exists $z \in \gamma$ such that $w \geqslant z$, and by hereditariness of H we get $z \in H$, contradicting the definition of γ.

Conversely, suppose that $\mathcal{H} = \{\emptyset, E^0\}$. Take $v \in E^0$ and $\gamma \in E^{\leqslant\infty}$, with $v \notin \gamma^0$ (the case $v \in \gamma^0$ is obvious). By hypothesis the hereditary saturated subset generated by v is E^0, i.e., $E^0 = \bigcup_{n \geqslant 0} \Lambda_n(v)$. Consider m the minimum n such that $\Lambda_n(v) \cap \gamma^0 \neq \emptyset$, and let $w \in \Lambda_m(v) \cap \gamma^0$. If $m > 0$, then by minimality of m it must be $s^{-1}(w) \neq \emptyset$ and $r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$. The first condition implies that w is not a sink and since $\gamma = (\gamma_n) \in E^{\leqslant\infty}$, there exists $i \geqslant 1$ such that $s(\gamma_i) = w$ and $r(\gamma_i) = w' \in \gamma^0$, the latter meaning that $w' \in r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$, contradicting the minimality of m. Therefore $m = 0$ and then $w \in \Lambda_0(v) = T(v)$, as we needed. $\square$

3. Condition (K)

We begin this section by recalling the two following well-known notions which will play a central role in the sequel. The name of Condition (L) was given in [14], while Condition (K) was formulated in [15].

(1) A graph E satisfies Condition (L) if every closed simple path has an exit, equivalently [1, Lemma 2.5], if every cycle has an exit.
(2) A graph E satisfies Condition (K) if for each vertex v on a closed simple path there exists at least two distinct closed simple paths α, β based at v, or, following [2], $V_1 = \emptyset$.

Remark 3.1.

(1) Notice that if E satisfies Condition (K) then it satisfies Condition (L).
(2) According to [2, Lemma 7], if $L(E)$ is simple then it satisfies Condition (K).

It is not difficult to see that if E satisfies Condition (L) then so does E_H, whereas E/H need not. Condition (K) has a better behavior as is shown in the following result.

Lemma 3.2. *Let E be a graph and H a hereditary subset of E^0. If E satisfies Condition* (K)*, so do E_H and E/H.*

Proof. We will see $CSP_E(v) = CSP_{E_H}(v)$ and $CSP_E(w) = CSP_{E/H}(w)$ for every $v \in H$ and $w \in E^0 \setminus H$. Clearly, $CSP_{E_H}(v) \subseteq CSP_E(v)$; conversely, let $\alpha \in CSP_E(v)$, and suppose $\alpha = (\alpha_1, \dots, \alpha_n)$. Since H is hereditary and $s(\alpha_1) = v \in H$, we get $r(\alpha_1) = s(\alpha_2) \in H$. Thus, by recurrence, $\alpha \in CSP_{E_H}(v)$ and the result holds.

Now, let $v \in E^0 \setminus H$ and consider $\alpha = (\alpha_1, \dots, \alpha_n) \in CSP_E(v)$. Since $r(\alpha_n) = v \notin H$ we get $\alpha_n \in (E/H)^1$. If $\alpha_{n-1} \notin (E/H)^1$ then $r(\alpha_{n-1}) = s(\alpha_n) \in H$ and H hereditary implies $v = r(\alpha_n) \in H$, a contradiction. By recurrence, $\alpha \in CSP_{E/H}(v)$; since the converse is immediate, the result follows. □

For a graded algebra A, denote by $\mathcal{L}(A)$ and $\mathcal{L}_{\mathrm{gr}}(A)$ the lattices of ideals and graded ideals, respectively, of A. The following proposition provides a description of the ideals of $L(E)$ for E a graph satisfying Condition (K).

Proposition 3.3. *If a graph E satisfies Condition* (K) *then, for every ideal J of $L(E)$, $J = I(H)$, where $H = J \cap E^0$ is a hereditary saturated subset of E^0. In particular, $\mathcal{L}_{\mathrm{gr}}(L(E)) = \mathcal{L}(L(E))$.*

Proof. Let J be a nonzero ideal of $L(E)$. By [1, Lemma 3.9] (which can be applied because E satisfies Condition (L) by Remark 3.1(1)) and [2, Proposition 6], $H = J \cap E^0 \neq \emptyset$ is a hereditary saturated subset of E^0. Therefore, and taking into account Remark 2.2, $I(H)$ is a graded ideal of $L(E)$ contained in J.

Suppose $I(H) \neq J$. Then, by Lemma 2.3(1),

$$0 \neq J/I(H) \lhd L(E)/I(H) \cong L(E/H).$$

Thus, E/H satisfies Condition (L) by Lemma 3.2 and Remark 3.1(1). Now, consider the isomorphism (of K-algebras) $\bar{\Psi} : L(E)/I(H) \to L(E/H)$ given by $\bar{\Psi}(x + I(H)) = \Psi(x)$ (for Ψ as in Lemma 2.3). By [2, Proposition 6], $\emptyset \neq \bar{\Psi}(J/I(H)) \cap (E^0 \setminus H) = \Psi(J) \cap (E^0 \setminus H)$, so there exists $v \in J \cap (E^0 \setminus H)$ with $\Psi(v) \in \Psi(J)$. But $v \in E^0 \cap J = H$ and, on the other hand, $v = \Psi(v) \in E^0 \setminus H$, which is impossible.

To finish, take into account that J is an ideal generated by idempotents and apply Remark 2.2. □

In Theorem 4.5 the converse of the previous result is proved.

Corollary 3.4. *If E satisfies Condition* (K) *then for every ideal I of $L(E)$ the canonical map*

$$K_0\big(L(E)\big) \to K_0\big(L(E)/I\big)$$

is an epimorphism.

Proof. By Proposition 3.3, $I = I(H)$ for the hereditary saturated subset $H = I \cap E^0$ of E^0. Then, the result holds by Lemma 2.6. □

Recall that a *matricial algebra* is a finite direct product of full matrix algebras over K, while a *locally matricial algebra* is a direct limit of matricial algebras.

The following result can be obtained as a corollary of Proposition 3.3. However we do not include its proof because it can be reached by slightly modifying that of [14, Corollary 2.3].

Corollary 3.5. *If E is a finite acyclic graph, then $L(E)$ is a K-matricial algebra.*

Corollary 3.6. *If E is an acyclic graph, then $L(E)$ is a locally matricial K-algebra.*

Proof. By [7, Lemma 2.2], $L(E) \cong \varinjlim L(X_n)$, where X_n is a finite subgraph of E for all $n \geqslant 1$. Hence, X_n is a finite acyclic graph for every $n \geqslant 1$, whence the result holds by Corollary 3.5. □

Recall that a *graph homomorphism* $f: E = (E^0, E^1) \to F = (F^0, F^1)$ is given by two maps $f^0: E^0 \to F^0$ and $f^1: E^1 \to F^1$ such that $r_F(f^1(e)) = f^0(r_E(e))$ and $s_F(f^1(e)) = f^0(s_E(e))$ for every $e \in E^1$. A graph homomorphism f is said to be *complete* in case f^0 is injective and f^1 restricts to a bijection from $s_E^{-1}(v)$ onto $s_F^{-1}(f^0(v))$ for every $v \in E^0$ that emits edges. Note that under the assumptions above, the map f^1 must also be injective.

Lemma 3.7. *If E is a graph satisfying Condition* (K) *then there exists an ascending family $\{X_n\}_{n\geqslant 0}$ of finite subgraphs such that*:

(1) *For every $n \geqslant 0$, X_n satisfies Condition* (K).
(2) *For every $n \geqslant 0$, the inclusion map $X_n \subseteq E$ is a complete graph homomorphism.*
(3) $E = \bigcup_{n\geqslant 0} X_n$.

Proof. We will construct X_n by recurrence on n. First, we enumerate $E^0 = \{v_n \mid n \geqslant 0\}$. Then, we define $X_0 = \{v_0\}$. Clearly, X_0 satisfies Condition (K) and also $X_0 \subseteq E$ is a complete graph homomorphism.

Now, suppose we have constructed $X_0, X_1, \ldots, X_n$ satisfying (1) and (2). Consider the graph $\tilde{X}_{n+1}$ with: (a) $\tilde{X}^1_{n+1} = X^1_n \cup \{e \in E^1 \mid s(e) \in X^0_n\}$; (b) $\tilde{X}^0_{n+1} = X^0_n \cup \{v_{n+1}\} \cup \{r(e) \mid e \in \tilde{X}^1_{n+1}\}$. Clearly, $\tilde{X}_{n+1}$ is finite and satisfies (2). If it also satisfies (1), we define $X_{n+1} = \tilde{X}_{n+1}$.

Suppose that $\tilde{X}_{n+1}$ does not satisfy Condition (K). Consider the set of all cycles based at vertices in $\tilde{X}_{n+1}$, $\mu_1^1, \ldots, \mu_1^k \subseteq \tilde{X}_{n+1}$ such that: (i) $\mu_1^i \nsubseteq X_n$ for any $1 \leqslant i \leqslant k$; (ii) for every $1 \leqslant i \leqslant k$ and some $v \in \mu_1^i$, $\mathrm{card}(CSP_{\tilde{X}_{n+1}}(v)) = 1$. Since $\tilde{X}_{n+1} \subseteq E$ and E satisfies Condition (K), there exist closed simple paths $\mu_2^1, \ldots, \mu_2^k \subseteq E$ such that, for each $1 \leqslant i \leqslant k$, $\mu_1^i \neq \mu_2^i$ and $\mu_1^i \cap \mu_2^i \neq \emptyset$. For each $1 \leqslant i \leqslant k$, let $\mu_2^i = (e_1^i, \ldots, e_{j_i}^i)$.

We consider the finite subgraph $\tilde{Y}_{n+1}$ of E such that: (a) $\tilde{Y}_{n+1}^1 = \tilde{X}_{n+1}^1 \cup \{e_l^i \mid 1 \leqslant i \leqslant k, 1 \leqslant l \leqslant j_i\}$; (b) $\tilde{Y}_{n+1}^0 = \tilde{X}_{n+1}^0 \cup \{s(e_l^i), r(e_l^i) \mid 1 \leqslant i \leqslant k,\ 1 \leqslant l \leqslant j_i\}$. Clearly, $\tilde{Y}_{n+1}$ satisfies (1).

Now, let X_{n+1} be the finite subgraph of E such that: (a) $X_{n+1}^1 = \tilde{Y}_{n+1}^1 \cup \{f \in E^1 \mid s(f) \in (\mu_2^i)^0$ for some $1 \leqslant i \leqslant k\}$; (b) $X_{n+1}^0 = \tilde{Y}_{n+1}^0 \cup \{r(e) \mid e \in X_{n+1}^1\}$. If $\mu \subseteq X_{n+1}$ is a closed simple path such that $\mu \nsubseteq \tilde{Y}_{n+1}$, then either it appears because one of the $e \in X_{n+1}^1 \setminus \tilde{Y}_{n+1}^1$ is a single loop (i.e., a cycle with an edge only) based at some vertex in one μ_2^i, or $s(e) \in (\mu_2^i)^0$ and $r(e)$ connects to a path that comes back to $s(e)$. In any case, the (potential) new closed simple paths are based at vertices of μ_2^i for some i, whence X_{n+1} satisfies (1). Also, since the step from $\tilde{Y}_{n+1}$ to X_{n+1} adds all the exits of all the vertices in the cycles μ_2^i, we conclude that for any vertex $v \in X_{n+1}^0$, v is either a sink, or every $e \in E^1$ with $s(e) \in X_{n+1}^0$ belongs to X_{n+1}^1. Hence, $X_{n+1} \subseteq E$ is a complete graph homomorphism. This completes the recurrence argument.

Finally, since $v_n \in X_n$ for every $n \geqslant 0$, we conclude that $E^0 = \bigcup_{n \geqslant 0} X_n^0$ and by the construction, $E^1 = \bigcup_{n \geqslant 0} X_n^1$. □

The following definitions can be found in [13, Definition 3.2]. Let F be a subgraph of a graph E. Then:

(1) The *loop completion* $\ell_E(F)$ of F in E is the subgraph of E obtained as the union of F with every closed path based at an element of F^0.
(2) The *exit completion* F_e of F is a subgraph obtained by adding to F the edges $V = \{e \in E^1 \mid s(e) = s(f)$ for some $f \in F^1\}$, and the vertices $\{r(e) \mid e \in V\}$. We say that F is *exit complete* if $F = F_e$.

Lemma 3.8. *If F is an exit complete subgraph of a graph E, then $L(F)$ is isomorphic to a subalgebra of $L(E)$.*

Proof. Since $F = F_e$, for every vertex $v \in F^0$ we have that either v is a sink or $s_F^{-1}(v) = s_E^{-1}(v)$. Thus, the relations defining $L(F)$ and $L(F') \subseteq L(E)$ are exactly the same, so that there is a natural injective morphism form $L(F)$ to $L(E)$, as desired. □

Lemma 3.9. *If F is a subgraph of a graph E then*:

(1) *F_e is exit complete.*
(2) *If F is finite then so is F_e whereas $l_E(F)$ need not be.*

Proof. (1) Clearly $F_e^1 \subseteq (F_e)_e^1$. Let us see the other inclusion. Take $g \in (F_e)_e^1$. If $g \in F_e^1$ we have finished. If not, there exists $f \in F_e^1$ with $s(g) = s(f)$. We have two possibilities: If $f \in F^1$ then, by definition, $g \in F_e^1$. If $f \notin F^1$ we can find $h \in F^1$ for which $s(f) = s(h)$. Therefore $s(g) = s(h)$ and again $g \in F_e^1$. Now it easily follows $F_e^0 = (F_e)_e^0$.

(2) Since F is finite (and row-finite) then F^1 is finite. Now, for each $f \in F^1$ there are finitely many edges $e \in E^1$ with $s(e) = s(f)$ (because E is row-finite), and therefore we are adding a finite number of edges and consequently of vertices. Thus, F_e is finite. To show that $l_E(F)$ can be infinite, consider the infinite graph E

Then $F = (\{v\}, \emptyset)$ is finite while $l_E(F) = E$ is not. □

Lemma 3.10. *Let E be a graph and T be any subgraph of E. Define $F = l_E(T)$, $G = l_E(T)_e$, S the set of sinks of G and $J = G/\bar{S}$. Then*:

(1) $CSP_F(v) = CSP_E(v)$ *for every* $v \in F^0$.
(2) $CSP_G(v) = CSP_E(v)$ *for every* $v \in G^0$ *such that* $CSP_G(v) \neq \emptyset$.
(3) $CSP_J(v) = CSP_E(v)$ *for every* $v \in J^0$ *such that* $CSP_J(v) \neq \emptyset$.
(4) *If E satisfies Condition* (K) *then so do F, G and J.*

Proof. (1) is evident from the definition of the loop completion.

(2) Consider $v \in G^0$ such that $CSP_G(v) \neq \emptyset$ and take $p = (p_1, \ldots, p_n) \in CSP_G(v)$. Suppose $p^0 \cap F^0 = \emptyset$; then for an arbitrary edge p_i we have $r(p_i) \notin F^0$. The construction of the exit completion yields that p_i is a new added edge and consequently there exists $f \in F^1$ with $s(f) = s(p_i)$ and hence $s(p_i) \in p^0 \cap F^0$, a contradiction. Therefore, $p^0 \cap F^0 \neq \emptyset$. Take w in the previous intersection. Then $CSP_G(v) =$ (v and w are in the same closed path) $CSP_G(w) \supseteq$ (because F is a subgraph of G) $CSP_F(w) =$ (by (1)) $CSP_E(w) =$ (v and w are in the same closed path) $CSP_E(v) \supseteq CSP_G(v)$. Hence, (2) holds. Note that the result may fail for $CSP_G(v) = \emptyset$.

(3) Let $v \in J^0$ such that $CSP_J(v) \neq \emptyset$. Obviously $CSP_J(v) \subseteq CSP_G(v)$. Now consider $p = (p_1, \ldots, p_k) \in CSP_G(v)$. We claim that $p^0 \cap \bar{S} = \emptyset$. If not, there exists $m = \min\{n \in \mathbb{N} \mid p^0 \cap \Lambda_n(S) \neq \emptyset\}$. Take $v \in p^0 \cap \Lambda_m(S)$. If $m > 0$ then by minimality we have that $r(s^{-1}(v)) \subseteq \Lambda_{m-1}(S)$. In particular, if $v = s(p_i)$ then $r(p_i) \in p^0 \cap \Lambda_{m-1}(S)$, which contradicts the minimality of m. If $m = 0$ then $v \in p^0 \cap \Lambda_0(S) =$ (S is the set of sinks) $p^0 \cap S$. This is absurd since p has no sinks. Any possibility leads to a contradiction so $p^0 \subseteq J^0$ and, consequently, $p_1, \ldots, p_k \in J^1$. Thus, $p \in CSP_J(v)$. Now (2) gives the result.

(4) follows directly from (1)–(3). □

4. Exchange Leavitt path algebras

A (not necessarily unital) ring R is called an *exchange ring* (see [3]) if for every element $x \in R$ the equivalent conditions in the next lemma are satisfied.

Lemma 4.1. [3, Lemma 1.1] *Let R be a ring and let R' be a unital ring containing R as a two-sided ideal. Then the following conditions are equivalent for an element $x \in R$:*

(1) *there exists $e^2 = e \in R$ with $e - x \in R'(x - x^2)$,*
(2) *there exist $e^2 = e \in Rx$ and $c \in R'$ such that $(1 - e) - c(1 - x) \in J(R')$,*
(3) *there exists $e^2 = e \in Rx$ such that $R' = Re + R'(1 - x)$,*
(4) *there exists $e^2 = e \in Rx$ such that $1 - e \in R'(1 - x)$,*
(5) *there exist $r, s \in R$, $e^2 = e \in R$ such that $e = rx = s + x - sx$.*

(*Here $J(R')$ denotes the Jacobson radical of R'.*)

Observe that R being an exchange ring does not depend on the particular unital ring where R is embedded as an ideal (look at condition (5) in the previous lemma). Other characterizations of the exchange property for not necessarily unital rings can be found in [3].

Remark 4.2. Since any K-matricial algebra is an exchange ring, then so is any K-locally matricial algebra (apply [5, Theorem 3.2]).

Theorem 4.3. *Let E be a graph. If $L(E)$ is an exchange ring, then E satisfies Condition* (K).

Proof. We claim that E satisfies Condition (L). Suppose that there exist a vertex v and a cycle α with $s(\alpha) = v$ such that α has no exits. Denote by H the hereditary saturated subset of E^0 generated by α^0. By Lemma 2.4, $I(H)$ is Morita equivalent to $L(E_H)$. If M is the graph having only a vertex w and an edge e such that $r(e) = s(e) = w$, then $L(M) \cong K[x, x^{-1}]$ by [1, Example 1.4(ii)]. Consider the map $f : L(M) \to L(E_H)$ given by $f(w) = v$, $f(e) = \alpha$, $f(e^*) = \alpha^*$. It is well defined because the relations in M are consistent with those in $L(E_H)$ (the only nontrivial one being $\alpha\alpha^* = v$, which holds due to the absence of exits for α, as in [1, p. 330]). It is a (nonunital) monomorphism of K-algebras; clearly, $\operatorname{Im} f \subseteq vL(E_H)v$. Now, we prove $vL(E_H)v \subseteq \operatorname{Im} f$. To this end, it is enough to see $vpq^*v \in \operatorname{Im} f$ for every $p = e'_1 \dots e'_r$, $q = e_1 \dots e_s$, with $e'_1, \dots, e'_r, e_1, \dots, e_s \in E^1_H$. Reasoning as in [1, proof of Theorem 3.11] we get that vpq^*v has the form: v, $v\alpha^n v$ or $v(\alpha^*)^m v$, with $m, n \in \mathbb{N}$. Hence our claim follows.

By [3, Theorem 2.3], the ring $I(H)$ is an exchange ring; moreover, $L(E_H)$ is an exchange ring by Lemma 2.4 and [5, Theorem 2.3], and the same can be said about the corner $vL(E_H)v$ by [5, Corollary 1.5]. But $vL(E_H)v \cong L(H) \cong K[x, x^{-1}]$ is not an exchange ring, which leads to a contradiction.

Now, we will prove that E satisfies Condition (K). Suppose on the contrary that there exists a vertex v and $\alpha = (\alpha_1, \dots, \alpha_n) \in CSP(v)$, with $\operatorname{card}(CSP(v)) = 1$ (in fact, α must be a cycle). Consider $A = \{e \in E^1 \mid e \text{ exit of } \alpha\}$, $B = \{r(e) \mid e \in A\}$, and let H be the hereditary saturated closure of B. With a similar argument to that used in [2, p. 6] we get that $H \cap \alpha^0 = \emptyset$, so that, H is a proper subset of E^0. Then, $\alpha^0 \subseteq (E/H)^0$ and $\{\alpha_1, \dots, \alpha_n\} \subseteq (E/H)^1$, whence α is a cycle in E/H with no exits.

Since $L(E/H) \cong L(E)/I(H)$ (Lemma 2.3(1)), $L(E/H)$ is an exchange ring [3, Theorem 2.2] and, by the previous step, E/H satisfies Condition (L), a contradiction. □

Recall that an idempotent e in a ring R is called *infinite* if eR is isomorphic as a right R-module to a proper direct summand of itself. The ring R is called *purely infinite* in case every nonzero right ideal of R contains an infinite idempotent.

Proposition 4.4. *If E is a graph satisfying Condition* (K) *and $\mathcal{L}(L(E))$ is finite, then $L(E)$ is an exchange ring.*

Proof. Since $\mathcal{L}(L(E))$ is finite, we can construct an ascending chain of ideals

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = L(E)$$

such that, for every $0 \leqslant i \leqslant n-1$, I_i is maximal among the ideals of $L(E)$ contained in I_{i+1}. Now, let us prove the result by induction on n.

If $n = 1$, then $L(E)$ is a simple ring and then E is cofinal by Lemma 2.8 and [1, Theorem 3.11]. Since E satisfies Condition (K), exactly two possibilities can occur:

(1) E has no closed simple paths, whence it is acyclic and thus, by Corollary 3.6, $L(E)$ is a locally matricial algebra, and so an exchange ring by Remark 4.2.
(2) E has at least one closed simple path, whence $L(E)$ is a purely infinite simple ring by cofinality [1, Theorem 3.11] and [2, Theorem 11]. Thus, $L(E)$ is an exchange ring by [4, Corollary 1.2].

In any case, $L(E)$ turns out to be an exchange ring.

Now, suppose that the result holds for $k < n$. By Proposition 3.3 and [7, Theorem 4.3], there exist hereditary saturated sets H_i $(1 \leqslant i \leqslant n)$ such that:

(i) $I_i = I(H_i)$ for every $0 \leqslant i \leqslant n$; in particular, $H_i \subsetneq H_{i+1}$ for every $0 \leqslant i \leqslant n-1$.
(ii) For any $0 \leqslant i \leqslant n-1$, there does not exist an hereditary saturated set T such that $H_i \subsetneq T \subsetneq H_{i+1}$.

Consider the restriction graph $E_{H_{n-1}}$. By Lemma 3.2, $E_{H_{n-1}}$ satisfies Condition (K), so that $\mathcal{L}_{\mathrm{gr}}(L(E_{H_{n-1}})) = \mathcal{L}(L(E_{H_{n-1}}))$ by Proposition 3.3. If for each $0 \leqslant i \leqslant n-1$, $J_i \lhd L(E_{H_{n-1}})$ is the ideal generated by H_i, then the previous remarks imply that

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{n-1} = L(E_{H_{n-1}}),$$

where, for every $0 \leqslant i \leqslant n-2$, J_i is maximal among the ideals of $L(E)$ contained in J_{i+1}; otherwise, since $\mathcal{L}_{\mathrm{gr}}(L(E_{H_{n-1}})) = \mathcal{L}(L(E_{H_{n-1}}))$, Lemma 2.5 would contradict property (ii) satisfied by the set H_i. Thus, by induction hypothesis, $L(E_{H_{n-1}})$ is an exchange ring. Since $I(H_{n-1})$ is Morita equivalent to $L(E_{H_{n-1}})$ by Lemma 2.4, $I(H_{n-1})$ is an exchange ideal by [5, Theorem 2.3]. Now, by Lemma 2.3(1), $L(E)/I(H_{n-1}) \cong L(E/H_{n-1})$.

Hence, E/H_{n-1} is a graph satisfying Condition (K) by Lemma 3.2, and $L(E/H_{n-1})$ is simple by construction. Following the same dichotomy for E/H_{n-1} as in (1) and (2) above, we get that $L(E/H_{n-1})$ is an exchange ring. Then, by using Lemma 2.6 and [3, Theorem 3.5], we conclude that $L(E)$ is an exchange ring, as desired. □

We would like to thank Gene Abrams for showing (4) ⇒ (3) in the following theorem.

Theorem 4.5. *For a graph E, the following conditions are equivalent*:

(1) $L(E)$ *is an exchange ring.*
(2) E/H *satisfies Condition* (L) *for every hereditary saturated subset H of E^0.*
(3) E *satisfies Condition* (K).
(4) $\mathcal{L}_{\mathrm{gr}}(L(E)) = \mathcal{L}(L(E))$.
(5) E_H *and* E/H *satisfy Condition* (K) *for every hereditary saturated subset H of E^0.*
(6) E_H *and* E/H *satisfy Condition* (K) *for some hereditary saturated subset H of E^0.*

Proof. (1) ⇒ (2). By Lemma 2.3(1), $L(E)/I(H) \cong L(E/H)$. Then, by [3, Theorem 2.2], $L(E/H)$ is an exchange ring. Apply Theorem 4.3 and Remark 3.1(1) to obtain (2).

(2) ⇒ (3) is just the third paragraph in the proof of Theorem 4.3.

(3) ⇒ (4) is Proposition 3.3.

(4) ⇒ (3). Suppose on the contrary that E does not satisfy Condition (K). Apply (2) ⇒ (3) to find a hereditary saturated subset H of E^0 such that E/H does not satisfy Condition (L), that is, there exists a cycle p in E/H based at v without an exit. Now [1, Theorem 3.11, pp. 330, 331] shows that in this situation we have $v \notin J := I(v+p)$, meaning in particular that the ideal J is not graded. Now if $H \neq \emptyset$, Lemma 2.3 shows that there exists a graded isomorphism $\Phi : L(E)/I(H) \to L(E/H)$ so that we can lift $\Phi^{-1}(J)$ to an ideal $\mathcal{J}$ of $L(E)$ which cannot be graded (a quotient of a graded ideal is again graded). If $H = \emptyset$ then clearly J is an ideal of $L(E/H) = L(E)$ which is not graded. In any case we get a contradiction.

(3) ⇒ (1). We have two different proofs of this fact. The first one is inspired in the results of [7], while the second one follows the style of [13, proof of Theorem 4.1]:

(i) By Lemma 3.7, there exists a family $\{X_n\}_{n\geqslant 0}$ of finite subgraphs such that, for every $n \geqslant 0$, X_n satisfies Condition (K), $E = \bigcup_{n\geqslant 0} X_n$ and the natural inclusion maps $f_n : X_n \hookrightarrow E$ are complete graph homomorphisms (therefore so are the inclusions $f_{n,n+1} : X_n \hookrightarrow X_{n+1}$). By [7, Lemma 2.2], we have induced maps $L(f_{n,n+1}) : L(X_n) \to L(X_{n+1})$ and $L(f_n) : L(X_n) \to L(E)$ such that $L(E) \cong \varinjlim(L(X_n), L(f_{n,n+1}))$.

Fix $n \geqslant 0$. Since X_n satisfies Condition (K), by Proposition 3.3 and [7, Theorem 4.3], $\mathcal{L}(L(X_n))$ is isomorphic to the lattice of hereditary saturated subsets of X_n^0. Hence, $\mathcal{L}(L(X_n))$ is finite. Thus, $L(X_n)$ is an exchange ring by Proposition 4.4. Since $L(E)$ is a direct limit of exchange rings, it is itself an exchange ring, as desired.

(ii) Take an element $x \in L(E)$. By [1, Lemma 1.5], there exist a finite family of vertices $V = \{v_1, \ldots, v_m\}$, and a finite family of edges $W = \{e_1, \ldots, e_n\}$, such that x is in the linear span of V and the set consisting of expressions $e_{i_1} \ldots e_{i_r} e_{j_1}^* \ldots e_{j_s}^*$, with e_{i_l}

and e_{j_k} edges in W. For $T^1 = W$ and $T^0 = V \cup \{r(e_i), s(e_i),\ i = 1, \dots, n\}$, let T be the graph $T = (T^0, T^1, r|_{T^1}, s|_{T^1})$ and consider G, J and S as in Lemma 3.10. It can be proved, as in [13, p. 224], that the number of hereditary subsets of J^0 is finite. Apply Lemma 3.10(4), Proposition 3.3 and [7, Theorem 4.3] to obtain that $\mathcal{L}(L(J))$ is finite. Now, Proposition 4.4 shows that $L(J)$ is an exchange ring. By Lemma 2.3(1), $L(J) \cong L(G)/I(\bar{S})$, and by Lemma 2.6 and [3, Theorem 3.5], $L(G)$ is an exchange ring. This means (use condition (5) in Lemma 4.1) that given x there exist $e^2 = e, r, s \in L(G) \subseteq L(E)$ (Lemmas 3.9(1) and 3.8) such that $e = rx = s + x - sx$. Whence, $L(E)$ is an exchange ring.

(3) $\Rightarrow$ (5) is Lemma 3.2.

(5) $\Rightarrow$ (6) is a tautology.

(6) $\Rightarrow$ (1). By (3) $\Rightarrow$ (1), $L(E_H)$ and $L(E/H)$ are exchange rings. Since $L(E_H)$ is Morita equivalent to $I(H)$ by Lemma 2.4 then $I(H)$ is an exchange ring because both are idempotent rings and we may apply [5, Theorem 2.3]. By Lemma 2.3(1), $L(E/H) \cong L(E)/I(H)$. Now, $L(E)/I(H)$ and $I(H)$ exchange rings, Lemma 2.6 and [3, Theorem 3.5] imply that $L(E)$ is an exchange ring. □

5. Some special facts

The following definitions are particular cases of those appearing in [11, Definition 1.3]. Let E be a graph and let $\emptyset \neq H \in \mathcal{H}_E$. Define

$$F_E(H) = \big\{\alpha = (\alpha_1, \dots, \alpha_n) \mid \alpha_i \in E^1,\ s(\alpha_1), r(\alpha_i) \in E^0 \setminus H \text{ for } i < n,\ r(\alpha_n) \in H\big\}.$$

Denote by $\bar{F}_E(H)$ another copy of $F_E(H)$. For $\alpha \in F_E(H)$, we write $\bar{\alpha}$ to denote a copy of α in $\bar{F}_E(H)$. Then, we define the graph ${}_HE = ({}_HE^0, {}_HE^1, s', r')$ as follows:

(1) ${}_HE^0 = ({}_HE)^0 = H \cup F_E(H)$.
(2) ${}_HE^1 = ({}_HE)^1 = \{e \in E^1 \mid s(e) \in H\} \cup \bar{F}_E(H)$.
(3) For every $e \in E^1$ with $s(e) \in H$, $s'(e) = s(e)$ and $r'(e) = r(e)$.
(4) For every $\bar{\alpha} \in \bar{F}_E(H)$, $s'(\bar{\alpha}) = \alpha$ and $r'(\bar{\alpha}) = r(\alpha)$.

Lemma 5.1. *Let E be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. Then*:

(1) *If E_H satisfies Condition* (L)*, then so does ${}_HE$.*
(2) *If E_H satisfies Condition* (K)*, then so does ${}_HE$.*

Proof. Notice that each vertex $\alpha \in F_E(H)$ is a source emitting exactly one edge $\bar{\alpha} \in \bar{F}_E(H)$ which ends in H. Thus, every closed simple path in the graph ${}_HE$ comes from a closed simple path in E_H, hence, the result follows. □

The class of Leavitt path algebras is closed under quotients (Lemma 2.3(1)). A direct consequence of the next result is that under Condition (L), this class is also closed for ideals.

Lemma 5.2. (Cf. [11, Lemma 1.5].) *Let E be a graph, and let $\emptyset \neq H \in \mathcal{H}_E$. If E_H satisfies Condition* (L)*, then $I(H)$ and $L({}_H E)$ are isomorphic as not necessarily unital rings.*

Proof. We define a map $\phi : L({}_H E) \to I(H)$ as follows: (i) For every $v \in H$, $\phi(v) = v$; (ii) for every $\alpha \in F_E(H)$, $\phi(\alpha) = \alpha\alpha^*$; (iii) for every $e \in E^1$ with $s(e) \in H$, $\phi(e) = e$ and $\phi(e^*) = e^*$; (iv) for every $\bar{\alpha} \in \bar{F}_E(H)$, $\phi(\bar{\alpha}) = \alpha$ and $\phi(\bar{\alpha}^*) = \alpha^*$.

By definition, it is tedious but straightforward to check that the images of the relations in $L({}_H E)$ satisfy the relations defining $L(E)$. Thus, ϕ is a well-defined K-algebra morphism.

Since for any $v \in H$, $\phi(v) = v$, to see that ϕ is surjective, by [1, Lemma 1.5], it is enough to show that every finite path α of E with $r(\alpha)$ or $s(\alpha)$ in H is in the image of ϕ. So let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be with $\alpha_i \in E^1$. If $s(\alpha) \in H$, then $s(\alpha_i) \in H$ for every i because H is hereditary and thus $\alpha = \phi(\alpha_1) \cdots \phi(\alpha_n) = \phi(\alpha)$.

Suppose that $s(\alpha_1) \in E^0 \setminus H$ and $r(\alpha_n) \in H$. Then, there exists $1 \leqslant j \leqslant n-1$ such that $r(\alpha_j) \in E^0 \setminus H$ and $r(\alpha_{j+1}) \in H$. Thus, $\alpha = (\alpha_1, \ldots, \alpha_{j+1})(\alpha_{j+2}, \ldots, \alpha_n)$, where $\beta = (\alpha_1, \ldots, \alpha_{j+1}) \in F_E(H)$. Hence, $\alpha = \phi(\bar{\beta})\phi(\alpha_{j+2}) \cdots \phi(\alpha_n)$.

Similarly it can be proved that $\alpha^* \in \operatorname{Im} \phi$.

Finally, if $0 \neq \operatorname{Ker}(\phi)$, then $\operatorname{Ker}(\phi) \cap ({}_H E)^0 \neq \emptyset$ by [2, Proposition 6] and Lemma 5.1(1), contradicting the definition of ϕ. □

Note that the isomorphism above is not $\mathbb{Z}$-graded because while $\bar{\alpha}$ has degree 1 in ${}_H E$ for every $\alpha \in F_E(H)$, $\phi(\bar{\alpha}) = \alpha$ does not necessarily have degree 1.

Lemma 5.3. *Let E be a graph satisfying Condition* (K)*. Then*:

(1) *If $J \lhd I \lhd L(E)$, then $J \lhd L(E)$.*
(2) *In particular, if $H \in \mathcal{H}_E$ and $J \lhd I(H)$, then there exists $X \in \mathcal{H}_E$ such that $X \subset H$ and $J = I(X)$.*

Proof. (1) By Proposition 3.3, $I = I(H)$ for $H = I \cap E^0 \in \mathcal{H}_E$, and by Proposition 5.2 (and Remark 3.1(1)) $I(H)$ is isomorphic to the Leavitt path algebra $L({}_H E)$ and therefore it has a set of local units. Take $x \in J$ and $z \in L(E)$, then there exists $y \in I$ such that $x = xy = yx$. Now $zx = (zy)x \in IJ \subseteq J$ and similarly $xz \in J$.

(2) Again Proposition 3.3 gives that $J = I(X)$ for $X = J \cap E^0$, and therefore $X = J \cap E^0 \subseteq I \cap E^0 = H$. □

Proposition 5.4. *Let E be a graph satisfying Condition* (K)*, let*

$$X_0 = \{ v \in E^0 \mid \exists e \neq f \in E^1 \text{ with } s(e) = s(f) = v,\ r(e) \geqslant v,\ r(f) \geqslant v \},$$

and let X be the hereditary saturated closure of X_0. If $L(E)$ has no unital purely infinite simple quotients, then neither does $I(X)$.

Proof. We will suppose that $X_0 \neq \emptyset$, because otherwise there is nothing to prove.

Case 1. We will begin by proving that if $L(E)$ has no unital purely infinite simple quotients, then $I(X)$ cannot be a unital purely infinite simple ring. Suppose that this statement is false. By Lemma 5.2 and Remark 3.1(1), $I(X) \cong L({}_XE)$, thus, since $I(X)$ is unital, ${}_XE$ is a finite graph; in particular, both X and $F_E(X)$ are finite, and so are

$$X_1 = \{v \in E^0 \mid v = s(\alpha_i) \text{ for some } \alpha = (\alpha_1, \dots, \alpha_n) \in F_E(X)\}$$

and $Y = X \cup X_1$. We claim that $K = E^0 \setminus Y$ belongs to $\mathcal{H}_E$. Let $v \in K$, $w \in E^0$, $e \in E^1$ be such that $s(e) = v$ and $r(e) = w$. We want to prove $w \in K$. Suppose on the contrary that $w \in Y$. If $w \in X$, then $e \in F_E(X)$ and so $v = s(e) \in X_1 \subseteq Y$, a contradiction, hence $w \in X_1 \setminus X$. In this case there exists a path $\alpha = (\alpha_1, \dots, \alpha_n) \in F_E(X)$ such that $w = s(\alpha_i)$, for some $i \in \{1, \dots, n-1\}$. Then $\beta = (e, \alpha_i, \dots, \alpha_n) \in F_E(X)$ and $v = s(\beta) \in X_1 \subseteq Y$, a contradiction. This shows that K is hereditary. Now we prove that it is saturated. Consider $v \in E^0$ and $\emptyset \neq r(s^{-1}(v)) \subseteq K$. Suppose $v \notin K$. Then $v \in X$ or $v \in X_1 \setminus X$. In the first case, since X is hereditary, $\emptyset \neq r(s^{-1}(v)) \subseteq X$, a contradiction. In the second one, there exists $\alpha = (\alpha_1, \dots, \alpha_n) \in F_E(X)$ such that $v = s(\alpha_i)$ for some $i \in \{1, \dots, n-1\}$. Then $r(\alpha_i) \in r(s^{-1}(v)) \subseteq K$, a contradiction because $r(\alpha_i) \in Y$, by the definition of Y.

The following step consists of showing that $L(E/K)$, which is isomorphic to $L(E)/I(K)$ by Lemma 2.3, is a unital purely infinite simple ring. First note that $(E/K)^0 = Y$ is finite and therefore $L(E/K)$ is a unital ring.

Now, since X is finite, $L(E_X)$ is unital. As $L(E_X)$ is Morita equivalent to the unital purely infinite simple ring $I(X)$ by Lemma 2.4, $L(E_X)$ is purely infinite simple. By [2, Theorem 11] and Lemma 28, E_X is cofinal, satisfies Condition (L), and every vertex in E_X^0 connects to a cycle. As E satisfies Condition (K), so does E/K by Lemma 3.2, whence E/K satisfies Condition (L). Observe that E/K contains at least a cycle; moreover, since every vertex in $F_E(X)$ connects to a vertex in X, then every vertex in E/K connects to a cycle. Finally, notice that E/K has no sinks, as otherwise, since any sink would be in X because $(E/K)^0 = X \cup X_1$ and $X_1 \setminus X$ clearly does not have sinks, ${}_XE$ would have a sink, which is not possible because $L({}_XE) \cong I(X)$ is a unital purely infinite simple ring. Then, $(E/K)^{\leqslant\infty} = (E/K)^{\infty}$. Hence, if $v \in (E/K)^0$ and α is in $(E/K)^{\infty}$, then $\mathrm{card}(\alpha^0) < \infty$ because Y is finite, so α contains a cycle β (note that for a cycle β' in E/K, $\beta'^0 \cap X_0 \neq \emptyset$ because E satisfies Condition (K), and then by hereditariness $\beta'^0 \subseteq X$), thus E/K is cofinal. By [2, Theorem 11] and Lemma 2.8, $L(E/K)$ is a unital purely infinite simple ring, a contradiction.

Case 2. $I(X)$ has no unital purely infinite simple quotients. Suppose that $I(X)/J$ is a unital purely infinite simple ring for some ideal J of $I(X)$. By Lemma 3.2, E_X satisfies Condition (K), whence so does ${}_XE$ by Lemma 5.1(2). Lemma 5.3 implies that there exists $H \in \mathcal{H}_E$ such that $H \subseteq X$ and $J = I(H)$. By Lemma 2.3 $L(E)/I(H) \cong L(E/H)$, and by Lemma 3.2, E/H satisfies Condition (K). This isomorphism shows that $L(E/H)$ has no unital purely infinite simple quotients because neither does $L(E)$. If Ψ is the isomorphism in Lemma 2.3, and $Z_0 = \Psi(X_0)$, then $Z = \overline{Z_0} = \Psi(X)$ by Lemma 2.3(4), and, in particular, $I(Z) = \Psi(I(X))$. Thus, by Case 1, applied to $E/H, Z_0$ and Z, we get a contradiction. $\square$

The rest of this section is devoted to characterizing the primeness of an ideal of the form $I(H)$, for H hereditary and saturated, in terms of the so-called maximal tails.

The following definition is a particular case of that of [10]: Let E be a graph. A non-empty subset $M \subseteq E^0$ is a *maximal tail* if it satisfies the following properties:

(MT1) If $v \in E^0$, $w \in M$ and $v \geqslant w$, then $v \in M$.
(MT2) If $v \in M$ with $s^{-1}(v) \neq \emptyset$, then there exists $e \in E^1$ with $s(e) = v$ and $r(e) \in M$.
(MT3) For every $v, w \in M$ there exists $y \in M$ such that $v \geqslant y$ and $w \geqslant y$.

Remark 5.5. Let E be a graph. If $J, K \in \mathcal{H}_E$, then $I(J)I(K) = I(J \cap K)$. To see this, notice that by Remark 2.2, $I(J) \cap I(K) = I(J \cap K)$. It is clear that $I(J)I(K) \subseteq I(J \cap K)$. Since every vertex is an idempotent, the reverse inclusion is clear.

Recall that a graded ideal I of a graded ring R is said to be *graded prime* if for every pair of graded ideals J, K of R such that $JK \subseteq I$, it is necessary that either $J \subseteq I$ or $K \subseteq I$. The definition of prime ideal is analogous to the previous one by eliminating the condition of being graded. It follows by [16, Proposition II.1.4] that for an ordered group (as it is our case), a graded ideal is graded prime if and only if it is prime.

Proposition 5.6. *Let E be a graph, and let $H \in \mathcal{H}_E$. Then, the following are equivalent*:

(1) *The ideal $I(H)$ is prime.*
(2) *$M = E^0 \setminus H$ is a maximal tail.*

Proof. (1) $\Rightarrow$ (2). It is not difficult to see that M satisfies (MT1) and (MT2). Suppose that there exist $v, w \in M$ such that no $y \in M$ satisfies:

$$(*)\ v \geqslant y \text{ and } w \geqslant y.$$

Fix such v, w. We will prove that $\overline{\{v\}} \cap \overline{\{w\}} \cap M = \emptyset$. Suppose that this is false. Let m be the smallest number such that $\Lambda_m(T(v)) \cap \overline{\{w\}} \cap M \neq \emptyset$ and take $y \in \Lambda_m(T(v)) \cap \overline{\{w\}} \cap M$. If $m > 0$, then $s^{-1}(y) \neq \emptyset$ and $\emptyset \neq r(s^{-1}(y)) \subseteq \Lambda_{m-1}(T(v)) \cap \overline{\{w\}}$ because $\overline{\{w\}}$ is hereditary. By the minimality of m, $\Lambda_{m-1}(T(v)) \cap \overline{\{w\}} \cap M = \emptyset$, hence $r(s^{-1}(y)) \subseteq M$. Since M is saturated, this implies $y \in M$, a contradiction. Similarly it can be proved that 0 is the smallest number n such that $T(v) \cap \Lambda_n(T(w)) \cap M \neq \emptyset$, that is, $T(v) \cap T(w) \cap M \neq \emptyset$, but this is a contradiction by $(*)$. Now, $I(v)I(w) =$ (by Lemma 2.1)$I(\overline{\{v\}})I(\overline{\{w\}}) =$ (by Remark 5.5)$I(\overline{\{v\}} \cap \overline{\{w\}}) \subseteq$ (as we have just proved) $I(H)$. By (1), and taking into account Lemma 2.1, this implies $v \in I(H)$ or $w \in I(H)$, a contradiction.

(2) $\Rightarrow$ (1). Consider two ideals J_1 and J_2 in $L(E)$ such that $J_1 J_2 \subseteq I(H)$. By Remark 2.2 there exist $H_1, H_2 \in \mathcal{H}_E$ such that $J_1 = I(H_1)$ and $J_2 = I(H_2)$. By Remark 5.5, $H_1 \cap H_2 \subseteq H$. If $H_i \subsetneq H$ for $i = 1, 2$, then there exist $v_i \in H_i \setminus H$ $(i = 1, 2)$. In particular, $v_1, v_2 \in M$, so that there exists $x \in M$ such that $v_i \geqslant x$ $(i = 1, 2)$. Hence, $x \in H_1 \cap H_2 \subseteq H$,

which contradicts $x \in M$. Thus, either $H_1 \subseteq H$ or $H_2 \subseteq H$, and thus either $I(H_1) \subseteq I(H)$ or $I(H_2) \subseteq I(H)$, as desired. □

Corollary 5.7. *If E is a graph satisfying Condition* (K)*, then there is a bijection between maximal tails and prime ideals. In particular, if E has no proper maximal tails, then $L(E)$ is simple.*

Proof. The first statement is a consequence of Proposition 5.6 and Proposition 3.3. This implies the second statement because the absence of proper maximal tails is equivalent to the absence of nonzero prime ideals. □

6. Stable rank for quasi stable rings

Let S be any unital ring containing an associative ring R as a two-sided ideal. The following definitions can be found in [20]. A column vector $b = (b_i)_{i=1}^n$ is called *R-unimodular* if $b_1 - 1, b_i \in R$ for $i > 1$ and there exist $a_1 - 1, a_i \in R$ $(i > 1)$ such that $\sum_{i=1}^n a_i b_i = 1$. The *stable rank* of R (denoted by $\mathrm{sr}(R)$) is the least natural number m for which for any R-unimodular vector $b = (b_i)_{i=1}^{m+1}$ there exist $v_i \in R$ such that the vector $(b_i + v_i b_{m+1})_{i=1}^m$ is R-unimodular. If such a natural m does not exist we say that the stable rank of R is infinite.

Recall that a ring R is said to be *stable* if $R \cong M_\infty(R)$. In this section, we cover the final step of the proof of Lemma 7.4. To this end, we need to compute the stable rank of some rings with local units whose behavior is similar to that of stable rings with local units. It is not known whether the property we consider is equivalent to stability of the ring.

Lemma 6.1. *Let R be a ring with ascending local unit $\{p_n\}_{n \geqslant 1}$. If for every $n \geqslant 1$ there exists $m > n$ such that $p_n \lesssim p_m - p_n$, then* $\mathrm{sr}(R) \leqslant 2$.

Proof. Fix S a unital ring containing R as two-sided ideal. Let $a_1, a_2, a_3, b_1, b_2, b_3 \in S$ such that $a_1 - 1, a_2, a_3, b_1 - 1, b_2, b_3 \in R$, while $a_1 b_1 + a_2 b_2 + a_3 b_3 = 1$. By hypothesis, there exists $n \in \mathbb{N}$ such that $a_1 - 1, a_2, a_3, b_1 - 1, b_2, b_3 \in p_n R p_n$. Let $m > n$ such that $p_n \lesssim p_m - p_n$. Then, there exists $q_n \sim p_n$, $q_n \leqslant p_m - p_n$. In particular, $q_n p_n = p_n q_n = 0$. Now, there exist $u \in p_n R q_n$, $v \in q_n R p_n$ such that $uv = p_n$, $vu = q_n$, $u = p_n u = u q_n$ and $v = q_n v = v p_n$.

Fix $v_1 = 0$, $v_2 = u$, $c_1 = b_1$, and $c_2 = b_2 + v b_3$. Notice that $(a_1 + a_3 v_1) - 1$, $c_1 - 1$, $(a_2 + a_3 v_2)$, $c_2 \in R$. Also, $a_3 u v b_3 = a_3 p_n b_3 = a_3 b_3$, $a_3 u b_2 = a_3 u q_n p_n b_2 = 0$, and $a_2 v b_3 = a_2 p_n q_n v b_3 = 0$. Hence,

$$(a_1 + a_3 v_1) c_1 + (a_2 + a_3 v_2) c_2 = a_1 b_1 + a_2 b_2 + a_3 b_3 = 1.$$

Thus, any unimodular 3-row is reducible, whence the result holds. □

A monoid M is *cancellative* if whenever $x + z = y + z$, for $x, y, z \in M$, then $x = y$. And M is said to be *unperforated* in case for all elements $x, y \in M$ and all positive integers n, we have $nx \leqslant ny$ implies $x \leqslant y$. A monoid M is *conical* if for every $x, y \in M$ such that $x + y = 0$, we have $x = 0 = y$. In what follows, all the considered monoids will be conical.

Given an abelian monoid M, and an element $x \in M$, we define

$$S(M, x) = \{f : M \to [0, \infty] \mid f \text{ is a monoid morphism such that } f(x) = 1\}.$$

Standard arguments show that, when M is a cancellative monoid, then $S(M, x)$ is nonempty for every nonzero element $x \in M$.

Lemma 6.2. *Let R be a nonunital ring with ascending local unit $\{p_n\}_{n \geqslant 1}$ such that $V(R)$ is cancellative and unperforated, and let $S_R = \{s : V(R) \to \mathbb{R}^+ \mid s$ is a morphism of monoids$\}$. If for every $s \in S_R$, $\sup_{n \geqslant 1}\{s([p_n])\} = \infty$, then for every $n \geqslant 1$ there exists $m > n$ such that $p_n \lesssim p_m - p_n$.*

Proof. Fix $n \in \mathbb{N}$, and consider $S_n = S(V(R), 2[p_n])$. For every $t \in S_n$, $\sup_{m \geqslant 1} t([p_m]) = \infty$. Otherwise, there exists $t \in S_n$ such that $\sup_{m \geqslant 1} t([p_m]) = \alpha \in \mathbb{R}^+$. Since $\{p_n\}_{n \geqslant 1}$ is a local unit, we conclude that $t(x) < \infty$ for every $x \in V(R)$, so that $t \in S_R$, contradicting the hypothesis. Thus, the maps $\widehat{p_k} : S_n \to [0, \infty]$, defined by evaluation, satisfy that the (pointwise) supremum $\sup_{k \geqslant 1} \widehat{p_k} = \infty$. Since S_n is compact, there exists $m > n$ such that $1 < \widehat{p_m}$, i.e., for every $s \in S_n$, $s(2[p_n]) < s([p_m])$.

Now, take $t \in S(V(R), [p_m])$. Since $p_n < p_m$, $0 \leqslant t(2[p_n]) = a \leqslant 2$. If $a = 0$, then clearly $0 = t(2[p_n]) < t([p_m]) = 1$. If $a \neq 0$, then $t'(-) := a^{-1} \cdot t(-)$ belongs to S_n, whence $1 = t'(2[p_n]) < t'([p_m])$ by the argument above. So, $t(2[p_n]) < t([p_m]) = 1$. Thus, for every $t \in S(V(R), [p_m])$, we have $t(2[p_n]) < t([p_m]) = 1$. By [18, Proposition 3.2], $2p_n \lesssim p_m = p_n + (p_m - p_n)$. Then, since $V(R)$ is cancellative, we get $p_n \lesssim p_m - p_n$, as desired. □

Definition 6.3. Let E be a graph. For every $v \in E^0$, we define $L(v) = \{w \in E^0 \mid w \geqslant v\}$. We say that $v \in E^0$ is *left infinite* if $\operatorname{card}(L(v)) = \infty$.

Definition 6.4. Let E be a graph. A *graph trace* on E is a function $g : E^0 \to \mathbb{R}^+$ such that, for every $v \in E^0$ with $s^{-1}(v) \neq \emptyset$, $g(v) = \sum_{s(e)=v} g(r(e))$. We define the *norm* of g to be the (possibly infinite) value $\|g\| = \sum_{v \in E^0} g(v)$. We say that g is *bounded* if $\|g\| < \infty$.

Remark 6.5. Let E be a graph, let $E^0 = \{v_i \mid i \geqslant 1\}$, let $p_n = \sum_{i=1}^n v_i$, and let

$$S_E = \{s : V(L(E)) \to \mathbb{R}^+ \mid s \text{ is a morphism of monoids}\}.$$

By [7, Theorem 2.5], any element $s \in S_E$ induces a graph trace by the rule $g_s(v) = s([v])$. Moreover, g_s is bounded if and only if $\sup_{n \in \mathbb{N}}\{s([p_n])\} < \infty$.

Conversely, by [7, Theorem 2.5] and [7, Lemma 3.3], if g is a graph trace on E, and $v, w \in E^0$ with $[v] = [w] \in V(L(E))$, then $g(v) = g(w)$. So, the rule $s_g([v]) = g(v)$ is well defined and extends by additivity to an element $s_g \in S_E$. Certainly, g is bounded if and only if $\sup_{n \in \mathbb{N}}\{s_g([p_n])\} < \infty$.

The next result in the context of C^*-algebras is [19, Lemma 3.8]. Here, we follow a different approach to prove it.

Lemma 6.6. *Let E be a graph, let $H \in \mathcal{H}_E$, and let $\pi : L(E) \to L(E)/I(H)$ be the natural projection map. If $e \in L(E)$ is an idempotent, $W \subseteq E^0 \setminus H$ is a finite set, and $\pi(e) \lesssim \sum_{w\in W} \pi(w)$ in $L(E)/I(H)$, then there exists a finite set $X \subseteq H$ such that $e \lesssim \sum_{w\in W} w + \sum_{x\in X} x$.*

Proof. By [7, Theorem 2.5 and Lemma 5.6], $V(L(E))/V(I(H)) \cong V(L(E/H))$. Thus, $[\pi(e)] \leqslant \sum_{w\in W}[\pi(w)] \in V(L(E/H))$ implies that there exist $a, b \in V(I(H))$ such that $[e] + a \leqslant \sum_{w\in W}[w] + b \in V(L(E))$. Since $V(I(H)) = \langle [v] \mid v \in H\rangle$, there exists a finite set $X \subseteq H$ such that $b = \sum_{x\in X} x$. Then, $[e] \leqslant \sum_{w\in W}[w] + \sum_{x\in X}[x]$, as desired. □

Proposition 6.7. (Cf. [19, Theorem 3.2].) *Let E be a graph. If every vertex of E lying on a closed simple path is left infinite and E has no nonzero bounded graph traces, then for every finite set $V \subseteq E^0$ there exists a finite set $W \subseteq E^0$ with $V \cap W = \emptyset$ and $\sum_{v\in V} v \lesssim \sum_{w\in W} w$.*

Proof. The proof of this result corresponds to (d) ⇒ (e) ⇒ (f) of [19, Theorem 3.2], with suitable adaptation of the arguments except for the Case 2 in (d) ⇒ (e), in which the way to prove the following statement is different: If $F \subseteq E^0$ is a finite set, and $n = \max\{i \in \mathbb{N} \mid w_i \in F\}$, there exists $m > n$ such that $p_n \lesssim p_m - p_n$.

Suppose then $v \notin \bar{H}$. List the vertices of $E/\bar{H} = \{w_i \mid i \geqslant 1\}$ in such a way that $w_1 = v$. Let $\pi : L(E) \to L(E)/I(H)$ be the natural projection map. For every $n \geqslant 1$, set $p_n = \sum_{i=1}^{n} \pi(w_i)$. Clearly, $\{p_n\}_{n\geqslant 1}$ is an ascending local unit for $L(E/\bar{H})$. Since every vertex on a closed simple path is left infinite, no vertex on $E/\bar{H}$ lies on a closed simple path. Thus, $E/\bar{H}$ is acyclic, whence $L(E/\bar{H})$ is locally matricial by Corollary 3.6. In particular, $V(L(E/\bar{H}))$ is cancellative and unperforated. Moreover, since E has no nonzero bounded graph traces, neither does $E/\bar{H}$. Otherwise, by Remark 6.5, there exists a monoid morphism $s : V(L(E/\bar{H})) \to \mathbb{R}^+$ with $\sup_{n\in\mathbb{N}}\{s([p_n])\} < \infty$. Hence, s induces a monoid morphism $s \circ \pi : V(L(E)) \to \mathbb{R}^+$ such that $\sum_{v\in E^0}(s \circ \pi)([v]) = \sum_{v\in E^0\setminus\bar{H}} s([v]) < \infty$, consequently there exists a bounded graph trace on E, contradicting the assumption. By Remark 6.5 and Lemma 6.2, for every $n \geqslant 1$ there exists $m > n$ such that $p_n \lesssim p_m - p_n$. □

Corollary 6.8. *Let E be a graph. If every vertex of E lying on a closed simple path is left infinite and E has no nonzero bounded graph traces, then* $\mathrm{sr}(L(E)) \leqslant 2$.

Proof. Let $E^0 = \{v_i \mid i \geqslant 1\}$, and for each $n \in \mathbb{N}$ consider $p_n = \sum_{i=1}^{n} v_i$. Then, $\{p_n\}_{n\geqslant 1}$ is an ascending local unit for $L(E)$. Fix $n \geqslant 1$ and set $V = \{v_1, \dots, v_n\}$. By Proposition 6.7, there exists a finite subset $W \subseteq E^0$ such that $V \cap W = \emptyset$ and $p_n = \sum_{v\in V} v \lesssim \sum_{w\in W} w$. If m is the largest subindex of $w \in W$, notice that $m > n$ and that $\sum_{w\in W} w \leqslant p_m - p_n$. Hence, the result holds because $L(E)$ satisfies the hypotheses of Lemma 6.1. □

7. Stable rank for exchange Leavitt path algebras

In this section, we characterize the stable rank of exchange Leavitt path algebras in terms of intrinsic properties of the graph.

Lemma 7.1. *Let E be an acyclic graph. Then the stable rank of $L(E)$ is* 1.

Proof. If E is finite, then $L(E)$ is a K-matricial algebra by Corollary 3.5, whence $\mathrm{sr}(L(E)) = 1$. Now suppose that E is infinite. By Corollary 3.6, there exists a family $\{X_n\}_{n \geqslant 0}$ of finite subgraphs of E such that $L(E) \cong \varinjlim L(X_n)$. By the definitions of direct limit and stable rank,

$$\mathrm{sr}\big(L(E)\big) \leqslant \liminf_{n\to\infty} \mathrm{sr}\big(L(X_n)\big). \qquad (*)$$

If E is acyclic, then so are the X_n's, whence $\mathrm{sr}(L(E)) = 1$ by the result above and $(*)$. □

Lemma 7.2. *Let E be a graph satisfying Condition* (K)*. Then, $L(E)$ has a unital purely infinite simple quotient if and only if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks.*

Proof. First, suppose that J is an ideal of $L(E)$ such that $L(E)/J$ is a unital purely infinite simple ring. By Proposition 3.3, there exists $H \in \mathcal{H}_E$ such that $J = I(H)$. By Lemma 2.3(1), $L(E)/J \cong L(E/H)$. Moreover, E/H satisfies Condition (K) by Lemma 3.2. Hence, since $L(E/H)$ is unital, E/H is finite. Since $L(E)/J$ is purely infinite simple, E/H is cofinal and every vertex connects to a closed simple path by [2, Theorem 11] and Lemma 28, whence E/H has no sinks.

Conversely, suppose that there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks. Thus, it contains a closed simple path and every vertex connects to a closed simple path. Then, since E/H satisfies Condition (K) by Lemma 3.2, $L(E/H)$ is unital, purely infinite and simple by [2, Theorem 11] and Lemma 2.8. By Lemma 2.3(1), $L(E)/M \cong L(E/H)$ and the proof is complete. □

Corollary 7.3. *Let E be a graph satisfying Condition* (K)*. If there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks, then the stable rank of $L(E)$ is ∞.*

Proof. By Lemma 7.2, there exists a maximal ideal $M \lhd L(E)$ such that $L(E)/M$ is a unital purely infinite simple ring. Thus, $\mathrm{sr}(L(E)/M) = \infty$ (see [8]). Since $\mathrm{sr}(L(E)/M) \leqslant \mathrm{sr}(L(E))$ (see [20, Theorem 4]), we conclude that $\mathrm{sr}(L(E)) = \infty$. □

The proof of the following result closely follows that of [11, Lemma 3.2].

Lemma 7.4. *Let E be a nonacyclic graph satisfying Condition* (K)*. If $L(E)$ does not have any unital purely infinite simple quotient, then there exists a graded ideal $J \lhd L(E)$ with* $\mathrm{sr}(J) = 2$ *such that $L(E)/J$ is a locally matricial K-algebra.*

Proof. Let

$$X_0 = \{v \in E^0 \mid \exists e \neq f \in E^1 \text{ with } s(e) = s(f) = v,\ r(e) \geqslant v,\ r(f) \geqslant v\},$$

and let X be the hereditary saturated closure of X_0. Consider $J = I(X)$, and notice that $L(E)/J \cong L(E/X)$ by Lemma 2.3(1). Moreover, since E satisfies Condition (K), then so does E/X by Lemma 3.2. If there is a closed simple path α in E/X, then every $v \in \alpha^0$ satisfies $\operatorname{card}(CSP_{E/X}(v)) \geqslant 2$, therefore, there exists a vertex $v_0 \in \alpha^0 \cap X_0 \subseteq X$, contradicting the assumption. So, E/X contains no closed simple paths, whence it is an acyclic graph and thus $L(E)/J$ is locally matricial by Corollary 3.6.

Now, by Remark 3.1(1), Lemma 3.2 and Lemma 5.2, $J \cong L({}_XE)$. We will show that every vertex lying in a closed simple path of ${}_XE$ is left infinite, and that ${}_XE$ has no nonzero bounded graph traces, as a way of contradiction.

Suppose that there exists a closed simple path α in ${}_XE$ such that the set Y of vertices of ${}_XE$ connecting to the vertices of α^0 is finite. It is not difficult to see that $\alpha^0 \cup Y$ is a maximal tail in ${}_XE$. Let M be a maximal tail of the smallest cardinal contained in $\alpha^0 \cup Y$. Observe that $M \cap X_0 \neq \emptyset$; otherwise $X \setminus M$, which is a hereditary saturated proper subset of X, would contain X_0, which is impossible. Denote by $\tilde{M}$ the quotient graph of ${}_XE$ by the hereditary saturated set $H = {}_XE^0 \setminus M$, i.e., $\tilde{M} = {}_XE/H$. Then, since M is finite, $L(\tilde{M})$ is a unital ring. As E satisfies Condition (K), so does ${}_XE$ (by Lemmas 3.2 and 5.1(2)) and thus so does $\tilde{M}$ (by Lemma 3.2 again). Then, since M does not contain smaller maximal tails, $L(\tilde{M})$ is simple by Corollary 5.7. As $M \cap X_0 \neq \emptyset$, $\tilde{M}$ is nonacyclic. Thus, $L(\tilde{M}) \cong L({}_XE)/I(H)$ (by Lemma 2.3(1)) is a unital purely infinite simple ring by [1, Theorem 3.11], Lemma 2.8 and [2, Theorem 11]. By Proposition 5.4 $L(E)$ has a unital purely infinite simple quotient, contradicting the hypothesis. Hence, every vertex lying in a closed simple path in ${}_XE$ is left infinite.

Now, suppose that there exists a nonzero bounded graph trace g on ${}_XE$. By Remark 6.5, $s_g : V(L({}_XE)) \to \mathbb{R}^+$ is a nonzero morphism such that $\sum_{v \in {}_XE^0} s_g([v]) < \infty$. But for any $v \in X_0$ we have $2s_g([v]) \leqslant s_g([v])$, so that $g(v) = 0$. Hence, $X_0 \subseteq \{w \in {}_XE^0 \mid g(w) = 0\}$, which is a hereditary saturated subset of ${}_XE$ by [19, Lemma 3.7]. Thus, since ${}_XE = \overline{X_0}^{({}_XE)}$, we conclude that $g \equiv 0$, contradicting the assumption. Hence, there exist no nonzero bounded graph traces on ${}_XE$.

Thus, $\operatorname{sr}(J) = \operatorname{sr}(L({}_XE)) \leqslant 2$ by Corollary 6.8. Since every vertex in X_0 is properly infinite as an idempotent of $L({}_XE)$, $\operatorname{sr}(L({}_XE)) \neq 1$, so that $\operatorname{sr}(J) = 2$, as desired. □

Corollary 7.5. *Let E be a nonacyclic graph satisfying Condition* (K). *If $L(E)$ does not have any unital purely infinite simple quotient, then* $\operatorname{sr}(L(E)) = 2$.

Proof. Consider J the graded ideal obtained in the previous lemma. By [20, Theorem 4],

$$2 = \max\{\operatorname{sr}(J), \operatorname{sr}(L(E)/J)\} \leqslant \operatorname{sr}(L(E)) \leqslant \max\{\operatorname{sr}(J), \operatorname{sr}(L(E)/J) + 1\} = 2.$$

Then, $\operatorname{sr}(L(E)) = 2$, as desired. □

Theorem 7.6. *Let E be a graph satisfying Condition* (K). *Then the values of the stable rank of $L(E)$ are*:

(1) $\mathrm{sr}(L(E)) = 1$ *if E is acyclic.*
(2) $\mathrm{sr}(L(E)) = \infty$ *if there exists $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks.*
(3) $\mathrm{sr}(L(E)) = 2$ *otherwise.*

Proof. Statement (1) holds by Lemma 7.1, and statement (2) by Corollary 7.3. If E is nonacyclic and there does not exist $H \in \mathcal{H}_E$ such that the quotient graph E/H is nonempty, finite, cofinal and contains no sinks, then $L(E)$ does not have any unital purely infinite simple quotient by Lemma 7.2. Hence, statement (3) holds by Corollary 7.5. □

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