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Graphs without minor complete subgraphs

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Abstract

The extremal number $ex(n; MK^p)$ denotes the maximum number of edges of a graph of order n containing no complete graph K^p as a minor. In this paper we give the exact value of the extremal number $ex(n; MK^p)$ for $\lceil(5n+9)/8\rceil \leqslant p \leqslant \lfloor(2n-1)/3\rfloor$ provided that $n-p \geqslant 24$. Indeed we show that this number is the size of the Turán Graph $T_{2p-n-1}(n)$ and this graph is the only extremal graph.

Keywords: Extremal problems; Minor; Complete graphs

1. Introduction

In this paper we consider the extremal problem of determining the maximum number of edges $ex(n; MK^p)$ of a graph with n vertices that it does not contain the complete graph K^p as a minor, where n and p are two positive integers, with $n \geqslant p$.

As Thomason says in [13], there is some interest in knowing the maximum size of a graph not containing the complete graph K^p as a minor, not least because of the relationship between this extremal problem and Hadwiger's Conjecture [6], asserting that if the chromatic number of a graph is $\chi(G) \geqslant p$, then G contains K^p as a minor.

There are many works devoted to study the extremal function asymptotically, finding the minimum number $c(p)$ such that every graph G of order n and size, at least, $c(p)n$ contains K^p as a minor. Mader [11] proved that $c(p) \leqslant 8p \log_2 p$. Kostochka [9,10] and also Fernandez de la Vega [5], based on Bollobás et al. [2] noticed that $c(p)$ is not just a linear function of p, by considering random graphs with average degree of order $p\sqrt{\log p}$. Kostochka [9,10] and Thomason [12] independently showed that $p\sqrt{\log p}$ is the correct order of $c(p)$. Finally, Thomason [13] proved that $c(p) = (\alpha + o(1))p\sqrt{\log p}$, where $\alpha = 0.319\ldots$ is an explicit constant. But exact values for the function $ex(n; MK^p)$ are only known for small values of p. By taking the graph $K^{p-2} + \overline{K^{n-p+2}}$ it is easy to check that

$$ex(n; MK^p) \geqslant (p-2)n - \binom{p-1}{2}.$$

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Dirac [4] proved that inequality holds for $p \leqslant 5$ and Mader [11] showed it for $p \leqslant 7$ and noticed that it does not hold for $p = 8$ and $n = 10$. The case $p = 8$ was solved by Jorgensen [8], showing that

$$ex(n; MK^8) = \begin{cases} 6n - 20 & \text{if } 5 \text{ divides } n, \\ 6n - 21 & \text{otherwise.} \end{cases}$$

In a recent work, the authors [3] have solved this extremal problem finding the exact value for that function and characterizing the corresponding family of extremal graphs for every pair of n and p satisfying $\lceil (2n+3)/3 \rceil \leqslant p < n$.

Our aim in this paper is to compute the exact value for the function $ex(n; MK^p)$ when $\lceil (5n+9)/8 \rceil \leqslant p \leqslant \lfloor (2n-1)/3 \rfloor$ and $n - p \geqslant 24$. In fact, we show that in the aforementioned sector of pairs (n, p), the solution is given by the size of the Turán graph $T_{2p-n-1}(n)$. Further, we show that this graph is the only extremal graph for this problem.

2. Definitions and notations

As usual, we say that a graph G contains K^p as a minor or is contractible to K^p if the complete graph K^p may be obtained from G by a sequence of vertex and edge deletions and edge contractions.

We denote by $T_r(n)$ the Turán graph of order n (see [14]), that is, the complete r-partite graph of order n whose vertex classes are as equal as possible. The number of edges of $T_r(n)$ is known by the *Turán number* and denoted by $t_r(n)$.

For a graph G, $V = V(G)$ and $E = E(G)$ stand, respectively, for its sets of vertices and edges, $v(G)$ being the order of G and $e(G)$ being the size of G. For any vertex v belonging to $V(G)$, we denote by $N_G(v)$ the set of neighbors of v in G, being $\delta_G(v) = |N_G(v)|$ the degree of v.

A subset W of vertices of G is a vertex cover of G if every edge of G is incident with at least one vertex of W. We denote by $vc(G)$ the minimum cardinality of a vertex cover of G.

Given a graph H, we say that $\{v_1, \ldots, v_r\}$ is a decreasing sequence of vertices in H if

$$\delta_{H_{j-1}}(v_j) = \max_{v \in V(H_{j-1})} \{\delta_{H_{j-1}}(v)\} \quad \text{for each } j = 1, \ldots, r,$$

where $H_0 = H$ and H_j is the resultant graph from H by removing the set $\{v_1, \ldots, v_j\}$. Furthermore, we denote by $\mathscr{V}_r^t$ the family of graphs H for which there exists a decreasing sequence of vertices $\{v_1, \ldots, v_r\}$ in H such that the minimum cardinality of a vertex cover in the resultant graph H_r by removing these vertices of H is at most t, i.e., $vc(H_r) \leqslant t$.

Notations and terminologies not explicitly given here can be found in [1].

3. Exact values for the function $ex(n; MK^p)$

In this section we compute the exact value of the extremal function $ex(n; MK^p)$ for $\lceil (5n+9)/8 \rceil \leqslant p \leqslant \lfloor (2n-1)/3 \rfloor$ provided that $n - p \geqslant 24$. In fact, we prove that this exact value is the Turán number $t_{2p-n-1}(n)$. First, we need to state various prior results.

3.1. Preliminaries

Let G be a graph of order n whose complement graph is formed by $4n - 6p + 3$ disjoint copies of K^4 and $8p - 5n - 4$ disjoint copies of K^3. It is not difficult to check that G is the Turán graph $T_{2p-n-1}(n)$ and does not contain K^p as a minor, hence the Turán number $t_{2p-n-1}(n)$ is a lowerbound for the extremal function $ex(n; MK^p)$, that is,

$$ex(n; MK^p) \geqslant t_{2p-n-1}(n).$$

So, in order to prove that $t_{2p-n-1}(n)$ is the exact value we only must show the other inequality. Observe that the size of the Turán graph $T_r(n)$ may be expressed as follows:

$$\begin{aligned} t_r(n) &= \binom{n}{2} - \left(r \sum_{i=0}^{\lfloor n/r \rfloor - 1} i + \left(n - r \left\lfloor \frac{n}{r} \right\rfloor \right) \left\lfloor \frac{n}{r} \right\rfloor \right) \\ &= \binom{n}{2} - \left(\left(n - \frac{r}{2} \right) \left\lfloor \frac{n}{r} \right\rfloor - \frac{r}{2} \left\lfloor \frac{n}{r} \right\rfloor^2 \right). \end{aligned}$$

Since $\lceil (5n+9)/8 \rceil \leqslant p \leqslant \lfloor (2n-1)/3 \rfloor$, we have $\lfloor n/(2p-n-1) \rfloor = 3$ and therefore

$$t_{2p-n-1}(n) = \binom{n}{2} - (9n - 12p + 6).$$

So, taking $q = n - p$, to show that $ex(n; MK^p) \leqslant t_{2p-n-1}(n)$ is equivalent to guarantee the inequality

$$ex(n; MK^{n-q}) \leqslant \binom{n}{2} - (3q + 3k + 6),$$

for $n = 3q - k$, with $q \geqslant 24$ and $1 \leqslant k \leqslant \lfloor (q-9)/3 \rfloor$.

To do that, we must prove that every graph G with $3q - k$ vertices and $\binom{n}{2} - (3q + 3k + 5)$ edges, being $q \geqslant 24$ and $1 \leqslant k \leqslant \lfloor (q-9)/3 \rfloor$, is contractible to K^p.

Before that, we need to state some prior results. We will use Hall's condition for complete matching in bipartite graph.

Theorem 3.1 (*See [7]*). *Given a bipartite graph B with classes X and Y, if $|N_B(A)| \geqslant |A|$ for all $A \subseteq X$, then there exists a complete matching in B, where $N_B(A) = \bigcup_{v \in A} N_B(v)$.*

In the next result we compute the minimum cardinality of a vertex cover of a graph with maximum degree 2.

Lemma 3.2. *Let H be a graph with maximum degree 2 and let us denote by r the maximum number of independent vertices with degree 2 in H. Then*

$$vc(H) = e(H) - r.$$

Proof. By definition, we have $vc(H) \leqslant r + e(H_r) = e(H) - r$. So, it is sufficient to prove the other inequality. For that, let $U = \{w_1, \ldots, w_s\}$ be a vertex cover of H with minimum cardinality and let us denote by n_2 the number of independent vertices of U with degree 2 in H. Then

$$\begin{aligned} vc(H) = n_2 + (|U| - n_2) &= n_2 + (e(H) - 2n_2) \\ &\geqslant e(H) - r, \end{aligned}$$

so the result holds. □

In the next two lemmas, we give a sufficient condition under which a graph H belongs to the family $\mathscr{V}_q^q$.

Lemma 3.3. *Let k and q be two positive integers, with $1 \leqslant k \leqslant \lfloor (q-9)/3 \rfloor$. Let H be a graph with $3q - k - i$ vertices and at most $3q + 3k + 5 - 4i$ edges, with $0 \leqslant i \leqslant q$. If the maximum degree of H is at most 3, then $H \in \mathscr{V}_{q-i}^q$.*

Proof. Let $m \geqslant 0$ be an integer such that $e(H) = 3q + 3k + 5 - 4i - m$ and let us consider any decreasing sequence of vertices $\{v_1, \ldots, v_{q-i}\}$ of H. If $e(H_{q-i}) \leqslant q$ then we are done, hence assume that $e(H_{q-i}) \geqslant q + 1$. Observe that $\Delta(H_{q-i}) \geqslant 2$, because otherwise

$$q + 1 \leqslant e(H_{q-i}) \leqslant \frac{1}{2} v(H_{q-i}) = q - \frac{k}{2} < q + 1,$$

arriving to a contradiction. Indeed, $\Delta(H_{q-i}) = 2$, because if $\Delta(H_{q-i}) \geqslant 3$, then

$$e(H) \geqslant 3(q-i) + e(H_{q-i}) \geqslant 3q - 3i + q + 1 = 4q + 1 - 3i > 3q + 3k + 5 - 4i \geqslant e(H),$$

and this is impossible.

Let us denote by j the maximum number of independent vertices of degree 3 in H and by r the maximum number of independent vertices of degree 2 in H_{q-i}.

Since $\Delta(H_j) \leqslant 2$, we have $e(H_j) \leqslant v(H_j)$, which implies that

$$j \geqslant 2k + \frac{5}{2} - \frac{3}{2} i - \frac{m}{2}. \tag{1}$$

Since $\Delta(H_{q-i+r})\leqslant 1$, we have $2e(H_{q-i+r})\leqslant v(H_{q-i+r})$ and consequently,

$$r\geqslant \frac{7}{3}k+\frac{10}{3}-\frac{4}{3}i-\frac{2m}{3}-\frac{2}{3}j. \tag{2}$$

Finally, since $\Delta(H_{q-i})=2$, by applying Lemma 3.2, we have $vc(H_{q-i})=e(H_{q-i})-r$. But in this case, applying first the inequality (2) and second the inequality (1), it is verified that

$$\begin{aligned} vc(H_{q-i}) = e(H_{q-i}) - r &= 3q+3k+5-4i-m-3j-2(q-i-j)-r \\ &\leqslant q+\frac{2}{3}k+\frac{5}{3}-\frac{2}{3}i-\frac{1}{3}j-\frac{m}{3} \\ &\leqslant q+\frac{5}{6}-\frac{1}{6}i-\frac{m}{6} \\ &< q+1 \end{aligned}$$

and hence, $H\in \mathscr{V}^q_{q-i}$. $\square$

Lemma 3.4. *Let k and q be two positive integers, with* $1\leqslant k\leqslant \lfloor (q-9)/3\rfloor$. *If H is a graph with* $v(H)=3q-k$ *vertices and* $e(H)=3q+5+3k$ *edges, then* $H\in\mathscr{V}^q_q$.

Proof. If the maximum degree of H is at most 3, the result is immediate by applying Lemma 3.3 for $i=0$. So, assume that the maximum degree of H is at least 4. Let $\{v_1,\dots,v_q\}$ be a decreasing sequence of vertices of H. We may suppose that there exists $i\in\{1,\dots,q\}$ such that $\Delta(H_{i-1})\geqslant 4$ and $\Delta(H_i)\leqslant 3$, because on the contrary, $e(H_q)\leqslant e(H)-4q<0$ and this is a contradiction.

Thus, the graph H_i has order $3q-k-i$ and size at most $3q+5+3k-4i$. Then, by applying Lemma 3.3, there exists a decreasing sequence of vertices in H_i, $\{w_1,\dots,w_{q-i}\}$, such that $vc((H_i)_{q-i})\leqslant q$. Thus, $\{v_1,\dots,v_i,w_1,\dots,w_{q-i}\}$ is a decreasing sequence of vertices in H in such a way that $vc(H_q)\leqslant q$, which implies that $H\in\mathscr{V}^q_q$. $\square$

The next result provides a sufficient condition for a graph being contractible to a complete graph.

Lemma 3.5. *Let n, q and k be three positive integers, with* $n=3q-k$, $q\geqslant 24$ *and* $1\leqslant k\leqslant\lfloor (q-9)/3\rfloor$. *Let H be a graph with n vertices and at most* $3q+3k+6$ *edges. If* $H\in\mathscr{V}^q_q$ *then* $\overline{H}$ *is contractible to* K^{n-q}.

Proof. Since $H\in\mathscr{V}^q_q$, there exists a decreasing sequence of vertices $\{v_1,\dots,v_q\}$ in H such that $vc(H_q)=s\leqslant q$. We denote by $\{w_1,\dots,w_s\}$, with $s\leqslant q$, a vertex cover of H_q with minimum cardinality. We may suppose that $\{w_1,\dots,w_s\}$ is chosen in such a way that $\sum_{i=1}^{s}\delta_{H_q}(w_i)\geqslant\sum_{i=1}^{s}\delta_{H_q}(\widetilde{w}_i)$, for all $\{\widetilde{w}_1,\dots,\widetilde{w}_s\}$ vertex cover of H_q with minimum cardinality. Clearly, $\Delta(H_q)\leqslant 3$, because if not we have $e(H)\geqslant 4q+e(H_q)>3q+3k+6\geqslant e(H)$ and this is not possible.

Let us see that we may contract some edges of $\overline{H}$ in such a way that the resultant graph contains the complete graph K^{n-q} as a subgraph, whose vertices are the vertices of H_q.

For that, let us consider the bipartite graph B whose classes are $X=\{w_1,\dots,w_s\}$ and $Y=\{v_1,\dots,v_q\}$, defined in such a way that a vertex w_i is adjacent to v_j in B, iff

$$N_{H_q}(w_i)\cup\{w_i\}\subseteq N_{\overline{H}}(v_j).$$

If we prove the existence of a complete matching in B, then we are done, since it would be sufficient to contract in G the edges of the complete matching in order to obtain a complete graph K^p. We use Hall's condition on complete matching. So, we have to study $|N_B(A)|$ for each $A\subseteq X$. Given $A\subseteq X$, denoting by $m=|A|$, let us distinguish three cases according to the maximum degree of H_q.

(a) Suppose that $\Delta(H_q)=1$.

For $1 \leqslant m \leqslant 4$, if $|N_B(A)| < m$, then at least nine vertices of the set $\{v_{q-(m+8)}, \ldots, v_q\}$ are nonadjacent in B to each of the vertices of A. Since $\{v_1, \ldots, v_q\}$ is a decreasing sequence of vertices of H, we have

$$\begin{cases} \delta_{H_{q-m}}(v_{q-(m-1)}) \geqslant 2, \\ \delta_{H_{q-(m+2)}}(v_{q-(m+1)}) \geqslant 3, \\ \delta_{H_{q-(m+4)}}(v_{q-(m+3)}) \geqslant 4, \\ \delta_{H_{q-(m+6)}}(v_{q-(m+5)}) \geqslant 5, \\ \delta_{H_{q-(m+8)}}(v_{q-(m+7)}) \geqslant 6. \end{cases}$$

Hence,

$$\begin{aligned} e(H) &\geqslant (q-m-7)6 + 2\cdot 5 + 2\cdot 4 + 2\cdot 3 + 2\cdot 2 + m - 1 + m \\ &= 6q - 4m - 15 \\ &\geqslant 6q - 31 \\ &> 3q + 6 + 3k \end{aligned}$$

and this is not possible because we suppose that $e(H) \leqslant 3q + 3k + 6$.

For $5 \leqslant m \leqslant q-2$, if $|N_B(A)| < m$, then there exists at least one vertex nonadjacent in B to any vertex of A in the set of vertices $\{v_{q-(m-1)}, \ldots, v_q\}$. Thus,

$$\delta_{H_{q-m}}(v_{q-(m-1)}) \geqslant m$$

and hence,

$$e(H) \geqslant (q-m+1)m + m - 1 + m \geqslant 5q - 11 > 3q + 6 + 3k,$$

a contradiction.

For $m = q-1$, if $|N_B(A)| < q-1$, then, on the one hand, for $1 \leqslant k \leqslant 2$, reasoning as in the case $5 \leqslant m \leqslant q-2$, we have

$$\delta_{H_1}(v_2) \geqslant q - 1$$

and hence, $e(H) \geqslant 2(q-1) + (q-2)1 + q - 1 = 4q - 5 > 3q + 6 + 3k$ for $q \geqslant 18$. On the other hand, if $k > 2$, the number of vertices of H is $v(H) \geqslant q + 2(q-1) = 3q - 2 > 3q - k$ and this is an absurdity.

For $m = q$, we deduce that the order of H is

$$v(H) \geqslant q + 2q = 3q > 3q - k$$

against our assumptions.

(b) Suppose that $\Delta(H_q) = 2$.

For $1 \leqslant m \leqslant 7$, if $|N_B(A)| < m$, then at least 12 vertices of the set $\{v_{q-(m+11)}, \ldots, v_q\}$ are nonadjacent in B to each of the vertices of A. Thus,

$$\begin{cases} \delta_{H_{q-1}}(v_q) \geqslant 2, \\ \delta_{H_{q-(m+2)}}(v_{q-(m+1)}) \geqslant 3, \\ \delta_{H_{q-(m+5)}}(v_{q-(m+4)}) \geqslant 4, \\ \delta_{H_{q-(m+8)}}(v_{q-(m+7)}) \geqslant 5, \\ \delta_{H_{q-(m+11)}}(v_{q-(m+10)}) \geqslant 6 \end{cases}$$

which implies that

$$\begin{aligned} e(H) &\geqslant (q-m-10)\cdot 6 + 3\cdot 5 + 3\cdot 4 + 3\cdot 3 + 2(m+1) + m \\ &= 6q - 3m - 22 \\ &\geqslant 6q - 43 \\ &> 3q + 6 + 3k, \end{aligned}$$

again a contradiction.

For $8 \leqslant m \leqslant q$, denoting by r the number of vertices of degree 2 in the set A, if $|N_B(A)| < m$, then there exists at least one vertex nonadjacent to any vertex of A in the set of vertices $\{v_{q-(m-1)}, \ldots, v_q\}$. This implies,

$$\delta_{H_{q-m}}(v_{q-(m-1)}) \geqslant m - \frac{r}{2} \quad \text{and} \quad e(H_q) \geqslant m + \frac{r}{2}$$

and as a consequence,

$$\begin{aligned} e(H) &\geqslant (q-m+1)\left(m - \frac{r}{2}\right) + (m-1)2 + m + \frac{r}{2} \\ &\geqslant (q-m)\frac{m}{2} + m - \frac{r}{2} + 3m - 2 + \frac{r}{2} \\ &= -\frac{m^2}{2} + \frac{q+8}{2}m - 2 \\ &\geqslant 4q - 2 \\ &> 3q + 6 + 3k \end{aligned}$$

but this is not possible.

(c) Suppose that $\Delta(H_q) = 3$.

For $1 \leqslant m \leqslant 11$, if $|N_B(A)| < m$, then at least 15 vertices of the set $\{v_{q-(m+14)}, \ldots, v_q\}$ are nonadjacent in B to each of the vertices of A. Then,

$$\begin{cases} \delta_{H_{q-1}}(v_q) \geqslant 3, \\ \delta_{H_{q-(m+6)}}(v_{q-(m+5)}) \geqslant 4, \\ \delta_{H_{q-(m+10)}}(v_{q-(m+9)}) \geqslant 5, \\ \delta_{H_{q-(m+14)}}(v_{q-(m+13)}) \geqslant 6 \end{cases}$$

and hence,

$$\begin{aligned} e(H) &\geqslant (q-m-13)\cdot 6 + 20 + 16 + 3(m+5) + m \\ &= 6q - 2m - 27 \\ &\geqslant 6q - 49 \\ &> 3q + 6 + 3k, \end{aligned}$$

but this is an absurdity.

For $12 \leqslant m \leqslant q$, if $|N_B(A)| < m$, then there exists at least one vertex nonadjacent to any vertex of A in the set of vertices $\{v_{q-(m-1)}, \ldots, v_q\}$. This implies that

$$\delta_{H_{q-m}}(v_{q-(m-1)}) \geqslant \frac{m}{3}$$

and therefore,

$$\begin{aligned} e(H) &\geqslant (q-m+1)\frac{m}{3} + 3(m-1) + m \\ &= -\frac{m^2}{3} + \frac{q+13}{3}m - 3 \\ &\geqslant 4q + 1 \\ &> 3q + 6 + 3k, \end{aligned}$$

a contradiction.

Hence, $|N_B(A)| \geqslant |A|$, for each $A \subseteq X$. So, by applying Hall's condition, there exists a complete matching in the bipartite graph B, and so the result follows. □

3.2. Main result

As we noted before, our aim is to find out the exact value of the function $ex(n; MK^p)$ for infinitely many related values of n and p. In the following theorem we deduce that this extremal number is the Turán number $t_{2p-n-1}(n)$.

Theorem 3.6. *Let n and p be two positive integers, such that* $\lceil(5n+9)/8\rceil \leqslant p \leqslant \lfloor(2n-1)/3\rfloor$ *and* $n-p \geqslant 24$. *Then*

$$ex(n; MK^p) = t_{2p-n-1}(n).$$

Proof. Observe that $n < 2p-3$, so, as we commented in Section 3.1, in order to show the result we only must prove that $ex(n; MK^{n-q}) \leqslant \binom{n}{2} - (3q+3k+6)$, for $q \geqslant 24$ and $n = 3q-k$, with $1 \leqslant k \leqslant \lfloor(q-9)/3\rfloor$.

Let G be a graph with n vertices and at least $\binom{n}{2} - (3q+3k+5)$ edges and let us denote by $H = \overline{G}$ the complement graph of G. H has n vertices and at most $3q+3k+5$ edges. So, by applying Lemma 3.4 we have that $H \in \mathscr{V}_q^q$, and by applying Lemma 3.5 we deduce that $\overline{H} = G$ is contractible to K^p. So the result holds. □

4. Characterization of the family $EX(n; MK^p)$

In the previous section we have determined the exact value of the function $ex(n; MK^p)$, being the Turán number $t_{2p-n-1}(n)$. Now we complete this study by showing that the only graph of order n and size $t_{2p-n-1}(n)$ not containing K^p as a minor is the Turán graph $T_{2p-n-1}(n)$.

In order to prove the characterization theorem we will make use of some results. The first one relates the vertices of maximum degree in a graph H with the number of them being independent.

Lemma 4.1. *Let m be a positive integer and H a graph with maximum degree* Δ *and at least m vertices of maximum degree. Then at least* $\lfloor(m+\Delta)/(\Delta+1)\rfloor$ *of them are independent.*

Proof. Let $k+1 = \lfloor(m+\Delta)/(\Delta+1)\rfloor$. Then H has at least $(\Delta+1)k+1$ vertices of degree Δ. We will show that at least $k+1$ of them are independent.

We apply induction on k. For $k=0$ the result is clear because $m \geqslant 1$. Hence, let us suppose that $k \geqslant 1$ and that the result holds for $k-1$. Let w be a vertex of H with degree Δ and we consider the graph $H^* = H - w$. It is clear that H^* has at least $(\Delta+1)k+1-|N_H(w) \cup \{w\}| = (\Delta+1)k+1-(\Delta+1) = (\Delta+1)(k-1)+1$ vertices of degree Δ. So, by the induction hypothesis, at least k of these vertices, $\{w_1, \ldots, w_k\}$, are independent. Thus, the $k+1$ vertices $w, w_1, \ldots, w_k$ have degree Δ and are independent. □

The following result relates the vertices of maximum degree Δ in a graph H with the number of disjoint copies of $K^{\Delta+1}$ contained in H.

Lemma 4.2. *Let r be a positive integer and H a graph with maximum degree* Δ. *If H has* $r(\Delta+1)$ *vertices of degree* Δ *and exactly r of them are independent, then H contains r disjoint copies of* $K^{\Delta+1}$.

Proof. Let H be a graph and let Δ_H be the subset of vertices of degree Δ. Let $w_1, \ldots, w_r$ be the r independent vertices of degree Δ belonging to Δ_H. Hence $\Delta_H \subseteq \bigcup_{i=1}^r N(w_i) \cup \{w_i\}$. Since $|\Delta_H| = r(\Delta+1)$ it follows:

$$r(\Delta+1) = |\Delta_H| \leqslant \left|\bigcup_{i=1}^r N_H(w_i) \cup \{w_i\}\right| \leqslant r(\Delta+1)$$

which implies that every vertex of $N_H(w_i)$ has degree Δ, for any $i \in 1, \ldots, r$ and further, $(N_H(w_i) \cup \{w_i\}) \cap (N_H(w_j) \cup \{w_j\}) = \emptyset$ for all $i \neq j, i, j = 1, \ldots, r$. So it is enough to show that for each $i = 1, \ldots, r$ the subgraph $H[\{w_i\} \cup N_H(w_i)]$ is $K^{\Delta+1}$. Otherwise, there must exist two vertices a and b of $H[\{w_i\} \cup N_H(w_i)]$ such that the edge ab does not belong to this graph. Hence, $\{w_1, \ldots, w_r, a, b\} \backslash \{w_i\}$ is a set of $r+1$ independent vertices of degree Δ in H against the hypothesis. Hence, H contains r disjoint copies of $K^{\Delta+1}$ and the result follows. □

Let us see that the family of graphs of order n and size $ex(n; MK^p)$ not containing K^p as a minor is only formed by the Turán graph $T_{2p-n-1}(n)$, if $\lceil(5n+9)/8\rceil \leqslant p \leqslant \lfloor(2n-1)/3\rfloor$ and $n-p \geqslant 24$.

Theorem 4.3. *Let n and p be two positive integers, such that* $\lceil(5n+9)/8\rceil \leqslant p \leqslant \lfloor(2n-1)/3\rfloor$, *with* $n-p \geqslant 24$. *Then*

$$EX(n; MK^p) = \{T_{2p-n-1}(n)\}$$

Proof. We know by Theorem 3.6 that the exact value of the function $ex(n; MK^p)$ is the Turán number $t_{2p-n-1}(n)$. Moreover, as we noted in Section 3.1, it is not difficult to check that the Turán graph T_{2p-n-1} is not contractible to K^p. Hence, it is sufficient to show that this graph is the only extremal graph.

For that, we reason as in the previous section changing p by $n-q$. So, we will prove that if G is a graph belonging to the family $EX(n, MK^p)$, then, its complement graph H is the graph formed by $2k+3$ disjoint copies of K^4 and $q-3k-4$ disjoint copies of K^3.

Let us consider $G \in EX(n; MK^{n-q})$ and denote by H its complement graph. By Theorem 3.6, we know that H has $v(H) = 3q - k$ vertices and $e(H) = 3q + 3k + 6$ edges. So, by applying Lemma 3.5, we deduce that $H \notin \mathscr{V}_q^q$, i.e., for all decreasing sequence of vertices of H, $\{v_1, \ldots, v_q\}$, the minimum cardinality of a vertex cover of H_q is at least $q+1$.

Let $\{v_1, \ldots, v_q\}$ be a decreasing sequence of vertices of H. First, let us see that $\Delta(H_q) = 2$. On the one hand, if $\Delta(H_q) \leqslant 1$, then $vc(H_q) \leqslant \frac{1}{2} v(H_q) = q - k/2 < q+1$, a contradiction. On the other hand, if $\Delta(H_q) \geqslant 3$, then $vc(H_q) \leqslant e(H) - 3q \leqslant q - 3 < q+1$, being also an absurd, hence $\Delta(H_q) = 2$.

Now, let us see that $\Delta(H) = 3$. Since $e(H) > v(H)$, it is clear that $\Delta(H) \geqslant 3$. So, it is sufficient to prove the other inequality. For that, assume that $\Delta(H) \geqslant 4$ and let us denote by $i \geqslant 1$ the maximum number of independent vertices with degree greater than or equal to 3 in H, and by r the maximum number of independent vertices with degree 2 in H_q.

On the one hand, since $\Delta(H) \geqslant 4$ and $\Delta(H_i) = 2$, by applying Lemma 3.2, we have

$$2(q-i) + vc(H_q) + r \leqslant e(H_i) \leqslant e(H) - (3i+1).$$

Further, since $e(H_i) \leqslant v(H_i)$, we deduce that

$$i \geqslant r + k + 1. \tag{3}$$

On the other hand, $\Delta(H_{q+r}) \leqslant 1$ and hence, $2e(H_{q+r}) \leqslant v(H_{q+r})$. Besides, by applying Lemma 3.2, we have

$$e(H_{q+r}) = e(H_q) - 2r = vc(H_q) - r \geqslant q + 1 - r$$

which implies that

$$r \geqslant k + 2. \tag{4}$$

So, if $\Delta(H) \geqslant 4$, taking into account (3) and (4), we have

$$\begin{aligned} e(H) \geqslant 3i + 1 + 2(q-i) + vc(H_q) + r &\geqslant 3q + i + 2 + r \\ &\geqslant 3q + 2r + k + 3 \\ &\geqslant 3q + 3k + 7, \end{aligned}$$

but this is a contradiction. Then $\Delta(H) = 3$.

Finally, let us see that H is the graph formed by $(2k+3)$ disjoint copies of K^4 and $q-3k-4$ disjoint copies of K^3.

Let us denote by n_i the number of vertices of H with degree i. Since $\Delta(H) = 3$ then $n_3 \geqslant 2e(H) - 2v(H) = 8k + 12$, that is, H has at least $8k+12$ vertices of degree 3. So, by applying Lemma 4.1, at least $\lfloor (8k+12+3)/4 \rfloor = 2k+3$ of them are independent.

In fact, H has, exactly, $2k+3$ independent vertices of degree 3, because if not,

$$\begin{aligned} e(H) \geqslant 3(2k+4) + 2(q-2k-4) + e(H_q) &= 2q + 2k + 4 + vc(H_q) + r \\ &\geqslant 2q + 2k + 4 + q + 1 + r \\ &\geqslant 3q + 3k + 7 \end{aligned}$$

against our assumptions.

Thus, H has $8k+12 = 4(2k+3)$ vertices of degree 3 and, exactly, $2k+3$ of them are independent. Then, by applying Lemma 4.2, we deduce that H is formed by $2k+3$ disjoint copies of K^4 and a graph H^* with maximum degree 2.

Finally, we only must show that H^* is formed by $q-3k-4$ disjoint copies of K^3. The graph H satisfies that

$$\begin{cases} 3n_3+2n_2+n_1=2(3q+3k+6), \\ n_3+n_2+n_1+n_0=3q-k \end{cases}$$

and we have proved that $n_3=8k+12$.

Therefore,

$$\begin{cases} 2n_2+n_1=6q-18k-24, \\ n_2+n_1+n_0=3q-9k-12 \end{cases}$$

and hence, $n_0=n_1=0$ and $n_2=3q-9k-12$. So, H^* has maximum degree 2 and $3q-9k-12$ vertices of degree 2 and then, by applying Lemma 4.1, at least

$$\left\lfloor \frac{3q-9k-12+2}{3} \right\rfloor = q-3k-4$$

of them are independent.

But indeed, this is the exact number of independent vertices of degree 2 in H^*. If not, H_{2k+3} would have at least $2k+3+(q-3k-3)=q-k$ independent vertices of degree 2, which implies that $r \geqslant q-k-(q-(2k+3))=k+3$. But then

$$\begin{aligned} e(H) \geqslant 3(2k+3)+2(q-2k-3)+e(H_q) &= 2q+2k+3+vc(H_q)+r \\ &\geqslant 2q+2k+3+q+1+k+3 \\ &\geqslant 3q+3k+7, \end{aligned}$$

and this is a contradiction. Then, by applying Lemma 4.2 H^* is formed by $q-3k-4$ disjoint copies of K^3, and the result holds. □

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