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THE HAHN-SCHUR THEOREM ON EFFECT ALGEBRAS

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In this paper we obtain new results on the uniform convergence on matrices and a new version of the matrix theorem of the Hahn-Schur summation theorem in effect algebras.

1. INTRODUCTION

In 1936 Birkhoff and Von Neumann ([**10**]) began to consider the lattice of closed subspaces of a separable infinite-dimensional Hilbert space as a mathematical model for a calculus of quantum logic. From 1980, with Gudder's work ([**11**]), the quantum mechanical system is given by a probability measure on its set of events that, in general, fails to form a Boolean algebra. To model unsharp quantum logics Foulis and Bennett in 1994 ([**7**]) introduced effect algebras as follows:

A structure $(L, \oplus, 0, 1)$ is called an effect algebra if $0, 1$ are two distinguished elements and $\oplus$ is a partial binary operation on L which satisfies the following conditions for any $a, b, c \in L$.

(1) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined.
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined.
(3) For every $a \in L$ there exists a unique $b \in L$ such that $a \oplus b = 1$.
(4) If $1 \oplus a$ is defined, then $\mathbf{a} = 0$.

In [**8**], the authors study the relation between partially ordered Abelian groups and effect algebras.

Let $(L, \oplus, 0, 1)$ be an effect algebra. If $a \oplus b$ is defined we say that a and b are orthogonal. If $a \oplus b = 1$ we say that b is the orthocomplement of a, and write $b = a'$. It is clear that $1' = 0$, $(a')' = a$, $a \oplus 0$ is defined for all $a \in L$ and $a \oplus 0 = a$.

We define now a partial order: $a \leqslant b$ if there exists $c \in L$ such that $a \oplus c = b$. In this case the element c is unique and it will be denoted by $c = b \ominus a$. If $a \leqslant b$ but $a \neq b$ we write $a < b$. We can prove that $a \leqslant b$ if and only if $a \oplus b'$ is defined, also $0 \leqslant a \leqslant 1$ if $a \in L$.

The effect algebra $(L, \oplus, 0, 1)$ is called a lattice effect algebra if $(L, \leqslant)$ is a lattice. If for $a, b \in L$, $a \leqslant b$ or $b \leqslant a$ then $(L, \oplus, 0, 1)$ is said to be a totally ordered effect algebra;

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if for all $a, b \in L$, $a < b$, there exists $c \in L$ such that $a < c < b$, then L is said to be connected.

Let $F = \{a_i : 1 \leqslant i \leqslant n\}$ be a finite subset of L. If $a_1 \oplus a_2$ is defined and $(a_1 \oplus a_2) \oplus a_3, \ldots, (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ too, we say that F is orthogonal and we define $\oplus F = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$. (by the commutative and associative laws, this sum does not depend on any permutation of elements).

An arbitrary subset of L, G is called orthogonal if F is orthogonal for each F, finite subset of G. In this case, if the supremum $\bigvee\{\oplus F : F \subset G, F \text{ is finite}\}$ exists it is called $\oplus$-sum of G and we shall denote it by $\oplus G$.

We say that L is complete if for each orthogonal subset G of L, the $\oplus$-sum, $\oplus G$ exists; if for each countable orthogonal subset G of L, the $\oplus$-sum exists then we say that L is σ-complete.

If $(L, \oplus, 0, 1)$ is an effect algebra we can consider in L the order topology. Birkhoff, 1948 [**9**] and Riecanova [**13**] proved the continuity of $\oplus$ and $\ominus$.

The following is proved in [**14**]:

(i) If L is totally ordered and $A = (a_i)_i$ is orthogonal $\oplus$-summable, then $\{a_n\}_{n\in\mathbb{N}}$ is order convergent to 0.

(ii) If L is σ-complete, $(a_i)_i$ and $(b_i)_i$ are two orthogonal $\oplus$-summable sequences of L and for each $i \in \mathbb{N}$, $b_i \leqslant a_i$, then we have:

$$\bigvee_{n\in\mathbb{N}}\Big\{\bigoplus_{i=1}^{n}(a_i \ominus b_i)\Big\} = \bigvee_{n\in\mathbb{N}}\Big\{\bigoplus_{i=1}^{n} a_i\Big\} \ominus \bigvee_{n\in\mathbb{N}}\Big\{\bigoplus_{i=1}^{n} b_i\Big\}$$

(iii) If L is σ-complete, totally ordered and connected, then for each $h \in L$, $0 < h$, there exists an orthogonal $\oplus$-summable sequence $(h_i)_i$ of L such that

$$\bigvee_{n}\Big\{\bigoplus_{i=1}^{n} h_i\Big\} < h.$$

Some interesting results about matrix convergence theorem in quantum logics are studied in [**14, 16, 15, 12**]. These results generalise the results in [**6**]. In this paper, we obtain a new version of the Hahn-Schur summation theorem in effect algebras. This version can have direct applications to measure theory on effect algebras ([**3, 4**]). Some similar results for normed spaces are obtained in [**1**].

2. Main theorem and its proof

Let $(L, \oplus, 0, 1)$ be a totally ordered effect algebra.

We say that ([**14**]) the sequence $(a_n)_n$ of L is a Cauchy sequence if for each $h \in L$, $0 < h$, there exists $n_0 \in \mathbb{N}$ such that when $n_0 \leqslant n$, $n_0 \leqslant m$, if $a_n \leqslant a_m$ then $a_m \ominus a_n < h$ and if $a_m \leqslant a_n$, then $a_n \ominus a_m < h$.

In the next remark we obtain results that will be useful later.

If L is an effect algebra, then L is a D-algebra ([**13**]) and the following is satisfied: if $b \ominus a$ and $c \ominus b$ are defined then also $c \ominus a$ and $(c \ominus a) \ominus (c \ominus b)$ are defined and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Let $(L, \oplus, 0, 1)$ be a totally ordered effect algebra.

If $a, b \in L$, by $a - b$ we mean $a \ominus b$ if $b \leqslant a$ and $b \ominus a$ if $a \leqslant b$.

If $\{a, b, c\} \subset L$ then $c - a \leqslant (b - a) \oplus (c - b)$ ([**5**]).

DEFINITION 0.1: Let $(L, \oplus, 0, 1)$ be a totally ordered effect algebra. An orthogonal sequence $(a_i)_i$ is called an unconditionally Cauchy series if for each $h > 0$ there exists n_0 such that when $n \geqslant n_0$ and $B \subset \{n, n+1, \ldots\}$ is finite then $\bigoplus_{i \in B} a_i < h$.

We shall denote by $\phi_0(\mathbb{N})$ the set of finite subsets of $\mathbb{N}$. Every subset $\mathcal{F}$ of $\mathcal{P}(\mathbb{N})$ such that $\phi_0(\mathbb{N}) \subset \mathcal{F}$ will be called a natural family.

A natural family $\mathcal{F}$ is subsequentially complete (SC) if for every sequence (A_i) of disjoints sets of $\mathcal{F}$ there exists $M \subset \mathbb{N}$ infinite such that $\bigcup_{i \in M} A_i \in \mathcal{F}$.

The following result is a new version of the Hahn-Schur theorem and it is proved in [**14**]:

Let $(L, \oplus, 0, 1)$ be a σ-complete totally ordered connected effect algebra, $a_{ij} \in L$ for $i, j \in \mathbb{N}$ such that $(a_{ij})_j$ is an orthogonal sequence of L for each $i \in \mathbb{N}$. If for each subset A of $\mathbb{N}$, the $\oplus$-sum sequence $\left(\bigoplus_{j \in A} a_{ij}\right)_i$ is convergent, then $(a_{ij})_{j \in \mathbb{N}}$ are uniformly $\oplus$-summable with respect to $i \in \mathbb{N}$.

We shall generalise this result using natural families, which can be non-SC.

In [**1**] the following definition is introduced:

DEFINITION 0.2: Let $\mathcal{F}$ be a natural family, we shall say that $\mathcal{F}$ is S if for every pair $[(A_i)_i, (B_i)_i]$ of disjoint sequences of mutually disjoint sets of $\phi_0(\mathbb{N})$ there exists $M \subset \mathbb{N}$ infinite and $B \in \mathcal{F}$ such that if $i \in M$ then $A_i \subset B$ and $B \cap B_i = \varnothing$.

REMARK. In [**2**] it is proved that there exist natural families being S but not SC.

REMARK. Let L be a totally ordered effect algebra and let $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$ be two orthogonal sets of n elements of L. In [**14**] it is proved that

$$\left(\bigoplus_{i=1}^{n} a_i\right) - \left(\bigoplus_{i=1}^{n} b_i\right) = \left(\bigoplus_{i \in A_+} (a_i \ominus b_i)\right) \ominus \left(\bigoplus_{i \in A_-} (b_i \ominus a_i)\right)$$

where $A_+ = \{i \leqslant n : b_i \leqslant a_i\}$, $A_- = \{1, \ldots, n\} \backslash A_+$.

If L is σ-complete and $(a_i)_i$, $(b_i)_i$ are orthogonal sequences we have that:

$$\left(\bigoplus_{i=1}^{\infty} a_i\right) - \left(\bigoplus_{i=1}^{\infty} b_i\right) = \left(\bigoplus_{i \in A_+} (a_i \ominus b_i)\right) \ominus \left(\bigoplus_{i \in A_-} (b_i \ominus a_i)\right)$$

where $A_+ = \{i \in \mathbb{N} : b_i \leqslant a_i\}$, $A_- = \{i \in \mathbb{N} : a_i < b_i\}$

If $(L, \oplus, 0, 1)$ is a totally ordered effect algebra and $(a_i)_i$ is an unconditional Cauchy series then $(a_i)_i$ converges to zero. If $\bigoplus_{i\in\mathbb{N}} a_i$ exists then it is easy to prove that $(a_i)_i$ is an unconditional Cauchy series.

The next theorem is similar to a result of [**1**] valid in normed spaces:

THEOREM 1. (Hahn-Schur Summation) *Let $(L, \oplus, 0, 1)$ be a σ-complete totally ordered connected effect algebra. Let (a_{ij}) be a matrix in L such that the rows are orthogonal sequences and the columns are Cauchy sequences. The following assertions are equivalent:*

1. *There exists a natural family $\mathcal{F}$ with the property S such that $\left(\bigoplus_{j\in B} a_{ij}\right)_i$ has the Cauchy property if $B \in \mathcal{F}$.*
2. *If $(A_n)_n$ is a sequence of disjoint subsets of $\phi_0(\mathbb{N})$ then $\left(\bigoplus_{j\in A_n} a_{ij}\right)_i$ has the Cauchy property uniformly in $n \in \mathbb{N}$.*
3. *$\bigoplus_{j\in\mathbb{N}} a_{ij}$ is uniformly and unconditional Cauchy series in $i \in \mathbb{N}$.*
4. *$\bigoplus_{j\in A} a_{ij}$ is uniformly an unconditional Cauchy series in $i \in \mathbb{N}$ and $A \in \mathcal{P}(\mathbb{N})$*
5. *$\left(\bigoplus_{j\in A} a_{ij}\right)_i$ has the Cauchy property if $A \in \mathcal{P}(\mathbb{N})$.*

PROOF: We suppose that 1 is true and 2 is false. Then, there exists $h > 0$ such that for each k there exist $i > k$ and n_k such that $\bigoplus_{j\in A_{n_k}} a_{kj} - \bigoplus_{j\in A_{n_k}} a_{ij} > h$. Also, it is easy to show that for each $k \in \mathbb{N}$ and each $m \in \mathbb{N}$ there exist $i > k$ and n_k such that $\bigoplus_{j\in A_{n_k}} a_{kj} - \bigoplus_{j\in A_{n_k}} a_{ij} > h$ and $A_{n_k} \cap \{1, \ldots, m\} = \emptyset$.

Let $\{h_1, h_2, h_3, h_4\} \subset L\backslash\{0\}$ be such that $h_1 \oplus h_2 \oplus h_3 \oplus h_4 < h$.

For $k_1 = 1$, there exist i_1 and k_1 and n_1 such that $\bigoplus_{j\in A_{n_1}} a_{k_1 j} - \bigoplus_{j\in A_{n_2}} a_{i_2 j} > h$.

Since $\bigoplus_j a_{k_1 j}$ and $\bigoplus_j a_{i_1 j}$ are an unconditional Cauchy series, we have that there exists m_1 such that $\bigoplus_{j\in B} a_{k_1 j} < h_1$ and $\bigoplus_{j\in B} a_{i_1 j} < h_2$ if $B \subset \{m_1 + 1, \ldots\}$ is finite.

We observe that if $B \subset \{m_1 + 1, \ldots\}$ and $B^+ \subset \{j \in B : a_{i_1 j} \leqslant a_{k_1 j}\}$ and $B^- \subset \{j \in B : a_{k_1 j} \leqslant a_{i_1 j}\}$ then

$$\bigoplus_{j\in B} a_{k_1 j} - \bigoplus_{j\in B} a_{i_1 j} = \Big[\bigoplus_{j\in B^+} (a_{k_1 j} - a_{i_1 j})\Big] - \Big[\bigoplus_{j\in B^-} a_{i_1 j} - a_{k_1 j}\Big] < h_1 + h_2$$

because

$$\bigoplus_{j\in B^+} (a_{k_1 j} - a_{i_1 j}) \leqslant \bigoplus_{j\in B^+} a_{k_1 j} < h_1$$
$$\bigoplus_{j\in B^-} (a_{i_1 j} - a_{k_1 j}) \leqslant \bigoplus_{j\in B^-} a_{i_1 j} < h_2$$

As for $j \in \{1, \dots, m_1\}$, $(a_{ij})_i$ has the Cauchy property, we can consider $\{h_{31}, \dots, h_{3m_1}\} \subset L\backslash\{0\}$ such that if $j \in \{1, \dots, m_1\}$ and $p, q \geqslant i_0$ is $x_{pj} - x_{qj} < h_{1j}$. So if $C \subset \{1, \dots, m_1\}$ and $C^+ = \{j \in C : a_{pj} < a_{qj}\}$ we have that $\bigoplus_{j \in c^+} (a_{pj} - a_{qj}) < h_3$, $\bigoplus_{j \in C^-} (a_{qj} - a_{pj}) < h_3$ and it will be also $\left(\bigoplus_{j \in C} a_{pj}\right) - \left(\bigoplus_{j \in C} a_{qj}\right) < h_3$.

Take $k_2 > i_{01}$ and $i_2 > k_2$ and n_2 such that $\left(\bigoplus_{j \in A_{n_2}} a_{k_2 j}\right) - \left(\bigoplus_{j \in A_{n_2}} a_{i_2 j}\right) > h$ and $A_{n_2} \cap \{1, \dots, m_1\} = \emptyset$.

We obtain, inductively, the sequences of integers $k_1 < i_1 < k_2 < i_2 < \cdots < k_r < i_r < \cdots$ and $m_1 < m_2 < \cdots < m_r < \cdots$ such that, if $r \in \mathbb{N}$ we have:

i.- $\bigoplus_{j \in C^+} (a_{k_r j} - a_{i_r j}) < h_3$, $\bigoplus_{j \in C^-} (a_{i_r j} - a_{k_r j}) < h_3$ if $C \subset \{1, \dots, m_{r-1}\}$ and $C^+ = \{j \in C : a_{i_r j} \leqslant a_{k_r j}\}$, $C^- = \{j \in C : a_{k_r j} < a_{i_r j}\}$.

ii.- $\bigoplus_{j \in A_{n_r}} a_{k_r j} - \bigoplus_{j \in A_{n_r}} a_{i_r j} > h$ and $m_{r-1} < \inf A_{n_r} \leqslant \sup A_{n_r} < m_r$.

iii.- If $B \subset \{m_{r+1}, \dots\}$ is finite and $B^+ = \{j \in B : a_{i_r j} \leqslant a_{k_r j}\}$ and $B^- = \{j \in B : a_{k_r j} < a_{i_r j}\}$ then $\bigoplus_{j \in B^+} (a_{k_r j} - a_{i_r j}) < h_1$, $\bigoplus_{j \in B^-} (a_{i_r j} - a_{k_r j}) < h_2$ and $\left(\bigoplus_{j \in B} a_{k_r j}\right) - \left(\bigoplus_{j \in B} a_{i_r j}\right) < h_1 + h_2$.

For each $r \in \mathbb{N}$, let $B_{n_r} = (m_{r-1}, m_r]\backslash A_{n_r}$, we have a pair of disjoint sequences of disjoint sets of $\phi_0(\mathbb{N})$: $\left[(A_{n_r}), (B_{n_r})\right]_r$; since the family $\mathcal{F}$ has the S property, there exists $M \subset \mathbb{N}$ infinite and $A \in \mathcal{F}$ such that if $r \in M$ then $A_{n_r} \subset A$ and $A \cap B_{n_r} = \emptyset$. We have that $\left(\bigoplus_{j \in B} a_{ij}\right)_i$ has the Cauchy property. Then there exists i_0 such that

$$\left(\bigoplus_{j \in A} a_{pj}\right) - \left(\bigoplus_{j \in A} a_{qj}\right) < h_4,$$

if $p, q \geqslant i_0$.

Let r be such that $i_0 < k_r$, we have that $a = \left(\bigoplus_{j \in A} a_{k_r j}\right) - \left(\bigoplus_{j \in A} a_{i_r j}\right)$.

Let $A^+ = \{j \in A : a_{i_r j} \leqslant a_{k_r j}\}$ and $A^- = \{j \in A : a_{k_r j} < a_{i_r j}\}$. We have that

$$\begin{aligned} a^+ &= \bigoplus_{j \in A^+} (a_{k_r j} - a_{i_r j}) \\ &= \left[\bigoplus_{\substack{j \in A^+ \\ j \leqslant m_{r-1}}} (a_{k_r j} - a_{i_r j})\right] \oplus \left[\bigoplus_{j \in A_{n_r} \cap A^+} (a_{k_r j} - a_{i_r j})\right] \oplus \left[\bigoplus_{\substack{j \in A^+ \\ j > m_r}} (a_{k_r j} - a_{i_r j})\right] = b^+ \oplus c^+ \end{aligned}$$

where $c^+ = \bigoplus_{j \in A_{n_r} \cap A^+} (a_{k_r j} - a_{i_r j})$ and

$$b^+ = \left[\bigoplus_{\substack{j \in A^+ \\ j \leqslant m_{r-1}}} (a_{k_r j} - a_{i_r j})\right] \oplus \left[\bigoplus_{\substack{j \in A^+ \\ j > m_r}} (a_{k_r j} - a_{i_r j})\right] < h_1 + h_3.$$

Analogously, $a^- = b^- \oplus c^-$, where $a^- = \bigoplus_{j\in A^-} (a_{i_rj} - a_{k_rj})$ and $c^- = \bigoplus_{j\in A^-\cap A_{n_r}} (a_{i_rj} - a_{k_rj})$ and

$$b^- = \Big[\bigoplus_{\substack{j\in A^-\\ j\leqslant m_{r-1}}} (a_{i_rj} - a_{k_rj})\Big] \oplus \Big[\bigoplus_{\substack{j\in A^-\\ j> m_r}} (a_{i_rj} - a_{k_rj})\Big].$$

We also have that $b^- < h_2 + h_3$.

If we consider c^+, a^+, c^- we deduce that $c^+ \ominus c^-$ is less or equal than $b^+ \oplus (a^+ - c^-)$.

If we consider a^+, a^-, c^- we deduce that $a^+ \ominus c^-$ is less or equal than $b^- \oplus (a^+ - a^-)$.

So we deduce that $\bigoplus_{j\in A_{n_r}} a_{k_rj} - \bigoplus_{j\in A_{n_r}} a_{i_rj} = c^+ - c^-$ is less or equal than h, that is a contradiction.

Now we are going to prove that $2 \Rightarrow 3$. If 2 is true and 3 is false, we have that there exists $h > 0$ such that for each n there exists $B \subset \{n+1, \ldots\}$ finite and $i_n \in \mathbb{N}$ such that $\bigoplus_{j\in B} a_{i_nj} > h$.

We consider $h_1 \in L\backslash\{0\}$ such that $h_1 < h$. For $n_1 = 1$ there exists i_1 and $B_1 \subset \{1, 2, \ldots\}$ finite such that $\bigoplus_{j\in B_1} a_{ij} > h$. For i_1 there exists $m_1 > \sup B_1$ such that $\bigoplus_{j\in B} a_{ij} < h_1$ if $B \subset \{m_1+1, \ldots\}$. For m_1 there exists i_2 and $B_2 \subset \{m_1+1, \ldots\}$ finite such that $\bigoplus_{j\in B_2} a_{i_2j} > h$.

Inductively we determine two sequences of integers $i_1 < i_2 < \cdots < i_r < \cdots$, $m_1 < m_2 < \cdots < m_r < \cdots$ and a sequence $(B_r)_r$ of pairwise disjoint sets of $\phi_0(\mathbb{N})$ such that, if $r > 1$:

i.- $B_r \subset \{m_{r-1}+1, \ldots\}$ and $\bigoplus_{j\in B_r} a_{i_rj} > h$.

ii.- $\bigoplus_{j\in B} a_{i_qj} < h$ if $B \subset \{m_{r-1}+1, \ldots\}$, so for each $r > 1$ we have $\bigoplus_{j\in B_{r-1}} a_{i_{r+1}j} - \bigoplus_{j\in B_{r+1}} a_{i_rj} > h - h_1$ and this contradicts that $\Big(\bigoplus_{j\in B_r} a_{ij}\Big)_i$ has Cauchy property uniformly in $r \in \mathbb{N}$.

$3 \Rightarrow 4$ and $5 \Rightarrow 1$ are trivial. We are going to prove $4 \Rightarrow 5$.

We suppose that there exists $A \in \mathcal{P}(\mathbb{N})$ such that $\Big(\bigoplus_{j\in A} a_{ij}\Big)_i$ is not a Cauchy sequence. We have that there exists $h > 0$ such that for each n there exist i, k such that $n < i < k$ and $\bigoplus_{j\in A} a_{kj} - \bigoplus_{j\in A} a_{ij} > h$.

For $n = 1$ there exist: i_1, k_1 such that $1 < i_1 < k_1$ and $\bigoplus_{j\in A} a_{kj} - \bigoplus_{j\in A} a_{ij} > h$.

Inductively, we obtain sequences $k_1 < i_1 < k_2 < i_2 < \cdots < k_r < i_r < \cdots$ such that $\bigoplus_{j\in A} a_{k_rj} - \bigoplus_{j\in A} a_{i_rj}$, if $r \in \mathbb{N}$.

Let $h_1, h_2 \in L\backslash\{0\}$ be such that $h_1 + h_2 < h_3$.

By hypothesis, there exists $m \in \mathbb{N}$ such that if $B \subset \{m+1, \ldots\} \cap A$ then $\bigoplus_{j\in B} a_{ij} < h_1$ for each $i \in \mathbb{N}$.

We have that, for $j \in \{1, \dots, m\} \cap A$, $(a_{ij})_i$ is a Cauchy sequence and it will exist i_0 such that, if $p, q \geqslant i_0$ then $a_{pj} - a_{qj} \leqslant h_{2j}$, where $\{h_{21}, \dots, h_{2m}\} \subset L\backslash\{0\}$ and $h_{21} \oplus \cdots \oplus h_{2m} < h_2$. So, if $c \subset \{1, \dots, m\} \cap A$ and $p, q \geqslant i_0$, then $\bigoplus_{j \in c^+} (a_{pj} - a_{qj}) < h_1$, $\bigoplus_{j \in c^-} (a_{qj} - a_{pj}) < h_1$ and $\bigoplus_{j \in c} (a_{pj} - a_{qj}) < h_1$, where $c^+ = \{j \in c : a_{qj} \leqslant a_{pj}\}$ and $c^- = \{j \in c : a_{pj} \leqslant a_{qj}\}$.

Let r be such that $i_0 < k_r < i_r$. If $a^+ = \bigoplus_{j \in A^+} (a_{k_r j} - a_{i_r j})$ and $a^- = \bigoplus_{j \in A^-} (a_{i_r j} - a_{k_r j})$, where $A^+ = \{j \in A : a_{i_r j} \leqslant a_{k_r}\}$ and $A^- = \{j \in A : a_{k_r j} \leqslant a_{i_r}\}$, we deduce

$$a^+ = \left[\bigoplus_{\substack{j \in A^+ \\ j \leqslant m}} (a_{k_r j} - a_{i_r j})\right] \oplus \left[\bigoplus_{\substack{j \in A^+ \\ j > m}} a_{k_r j} - a_{i_r j}\right] < h_1 + h_2.$$

Analogously, we have that $a^- < h_1 + h_2$. So $\bigoplus_{j \in A} (a_{k_r j} - a_{i_r j}) = a^+ \ominus a^- < h$ and this is a contradiction. □

REMARK. In the same conditions of the last theorem, we shall have that the rows $(a_{ij})_j$ have uniformly the Cauchy property and we can deduce from [5] that the columns also have uniformly the Cauchy property, from this it is easy to deduce that given $h \in L \backslash \{0\}$ there exist i_0, j_0 such that $a_{ij} < h$ if $i \geqslant i_0$ and $j \geqslant j_0$. In other words, $\lim a_{ij} = 0$ in the sense of Pringsheim.

REFERENCES

[1] A. Aizpuru and A. Gutiérrez-Dávila, 'Unconditionally Cauchy series and uniform convergence on Matrices', *Chinese Ann. Math. Ser. B* **25** (2004), 335–346.

[2] A. Aizpuru and A. Gutiérrez-Dávila, 'On the interchange of series and applications', *Bull. Belg. Math. Soc.* **11** (2004), 409–430.

[3] A. Aizpuru, A. Gutiérrez-Dávila and J. Wu, 'Measures defined on quantum logics of sets', *Internat. J. Theoret. Phys.* **44** (2005), 1451–1458.

[4] A. Aizpuru and M. Tamayo, 'Classical properties of measure theory on effect algebras', *Fuzzy Sets and Systems* **157** (2006), 2139–2143.

[5] A. Aizpuru and M. Tamayo, 'Matrix convergence theorems in quantum logics', (preprint).

[6] P. Antosik and C. Swartz, *Matrix methods in analysis*, Lecture Notes in Mathematics **1113** (Springer–Verlag, Berlin, 1985).

[7] M.K. Bennett and D.J. Foulis, 'Effect algebras and unsharp quantum logics', *Found. Phys.* **24** (1994), 1331–1352.

[8] M.K. Bennett and D.J. Foulis, 'Interval and scale effect algebras', *Adv. in Appl. Math.* **19** (1997), 200–215.

[9] M.K. Birkhoff, *Lattice theory* **25**, Colloquium New York (American Mathematical Society, 1948).

[10] M.K. Birkhoff and J. Von Neumann, 'The logic of quantum mechanics', *Ann. Math.* **37** (1936), 823–843.

[11] S. Gudder, *Quantum probability* (Academic Press, London, New York, 1988).

[12] F.G. Mazario, 'Convergence theorems for topological group valued measures on effect algebras', *Bull. Austral. Math. Soc.* **64** (2001), 213–231.

[13] Z. Riecanova, 'Subalgebras, intervals and Central elements of generalized effect algebras', *Internat. J. Theoret. Phys.* **38** (1994), 3204–3220.

[14] J. Wu, S. Lu and D. Kim, 'Antosik-Mikusinski Matrix convergence theorem in quantum logics', *Internat. J. Theoret. Phys.* **42** (2003), 1905–1911.

[15] J. Wu, S. Lu and M. Chu, 'Quantum-logics-valued measure convergence theorem', *Internat. J. Theoret. Phys* **42** (2003), 2603–2608.

[16] J. Wu and Z. Ma, 'The Brooks-Jewett theorem on effect algebras with the sequential completeness property', *Czechoslovak J. Phys.* **53** (2003), 379–383.

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