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Linear bijections which preserve the diameter of vector-valued maps

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Abstract

We study diameter preserving linear bijections from $\mathscr{C}(X, V)$ onto $\mathscr{C}(Y, Z)$, where X, Y are compact Hausdorff spaces and V, Z are Banach spaces. In particular, assuming that Z is rotund and the extreme points of B_{V^*} satisfy a certain geometric condition, we prove that there exists a diameter preserving linear bijection from $\mathscr{C}(X, V)$ onto $\mathscr{C}(Y, Z)$ if and only if X is homeomorphic to Y and Z is linearly isometric to V. We also consider the case when X and Y are locally compact, noncompact spaces.

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1. Introduction

A problem related to Banach–Stone theorem is the study of linear bijections between function spaces which preserve the diameter of the range, that is, the seminorm $\rho(f) = \sup\{\|f(x) - f(y)\| : x, y \in X\}$ for $f \in \mathscr{C}(X, E)$, being X a compact space and E a Banach space. Such maps are called diameter preserving linear bijections.

Gÿory and Molnár began the study of this problem in [5], followed by González and Uspenkij in [4]. Cabello, in [3] obtains the next result:

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Let X be a compact Hausdorff space, then there exists a diameter preserving linear bijection $T : \mathscr{C}(X) \to \mathscr{C}(X)$ if and only if there exists a homeomorphism $\varphi : X \to X$, a linear functional $\mu : \mathscr{C}(X) \to \mathbb{K}$, where $\mathbb{K}$ is the scalar field, and a scalar z with $|z| = 1$ and $\mu(1_X) + z \neq 0$ such that $Tf = zf \circ \varphi + \mu(f)1_X$ for each $f \in \mathscr{C}(X)$.

Recently there have been new results about diameter preserving linear bijections [1,3,4,6]. The linear injections which preserve the diameter of scalar maps are studied in [2].

In this paper we study diameter preserving linear bijections $T : \mathscr{C}(X, V) \to \mathscr{C}(Y, Z)$, where X, Y are compact Hausdorff spaces and V, Z are Banach spaces, being Z rotund and satisfying the next geometric condition: for $z_1^*, z_2^* \in Z^*$, if $\|z_1^*\| = \|z_2^*\| = 1$ and $\|z_1^* + z_1^*\| = 2$ then $\|z_1^* - z_1^*\| < 1$. There is a large list of Banach spaces satisfying the conditions required on Z, for instance every rotund Banach space whose dual is rotund as well (the L^p spaces with $1 < p < \infty$, etc.).

In this situation we prove that there exist a homeomorphism $\varphi : Y \to X$, a linear, surjective isometry $G : V \to Z$ and a linear map $L : \mathscr{C}(X, V) \to Z$ such that if $f \in \mathscr{C}(X, V)$ and $y \in Y$ then $Tf(y) = G(f(\varphi(y))) + L(f)$.

2. Main result

Let X, Y be compact Hausdorff spaces. Let V, Z be Banach spaces. We denote by $Ex(V^*)$ the extreme points of V^* and $EN(V^*)$ the set of extreme points of B_{V^*} which attain the norm.

It is easily seen that $w^*Cl(Co(EN(V^*))) = B_{V^*}$ so that, if $v \in V$ then $\|v\| = \sup\{|v^*(v)| : v^* \in EN(V^*)\}$.

We will denote by $EN(Z^*)$ the corresponding set of extreme points which attain the norm and by $\mathscr{C}(X, V)$ and $\mathscr{C}(Y, Z)$ the spaces of continuous functions with the supremum norm.

In this paper we suppose that Z is rotund and if $z_1^*, z_2^* \in EN(Z^*)$ and $\|z_1^* + z_2^*\| = 2$ then $\|z_1^* - z_2^*\| < 1$.

Let $T : \mathscr{C}(X, V) \to \mathscr{C}(Y, Z)$ be a diameter preserving linear bijection. Let us denote by $\mathscr{C}_V$ the subspace of constant maps of $\mathscr{C}(X, V)$ and by $\mathscr{C}_Z$ the subspace of constant maps of $\mathscr{C}(Y, Z)$.

It can be easily seen that $\rho([f]) = \sup\{\|f(x_1) - f(x_2)\| : x_1, x_2 \in X\}$ is a well defined, complete norm in $\mathscr{C}(X, V)/\mathscr{C}_V$, and we have that $[f_1] = [f_2]$ in $\mathscr{C}(X, V)/\mathscr{C}_V$ if and only if $\rho(f_1 - f_2) = 0$.

Let us consider the map:

$$\widehat{T} : (\mathscr{C}(X, V)/\mathscr{C}_V, \rho) \to (\mathscr{C}(Y, Z)/\mathscr{C}_Z, \rho)$$

defined by $\widehat{T}[f] = [Tf]$.

$\widehat{T}$ is a linear, surjective isometry.

In [1] it is proved that the extreme points of the unit ball of $(\mathscr{C}(Y, Z)/\mathscr{C}_Z, \rho)^*$ are precisely $z^*(\delta_{y_1} - \delta_{y_2})$ where z^* is and extreme point of B_{Z^*} and $y_1, y_2 \in Y (y_1 \neq y_2)$. So that, for each, z^* and $y_1, y_2 \in Y$ there exist $x_1, x_2 \in X$ and $v^* \in Ex(V^*)$ such that $\widehat{T}^*z^*(\delta_{y_1} - \delta_{y_2}) = v_*((\delta_{x_1} - \delta_{x_2}))$. It is easily seen that $z^* \in EN(Z^*)$ if and only if $v^* \in EN(V^*)$.

Given $x_1 \in X$ we can consider $x, x' \in X$ such that $x_1 \notin \{x, x'\}$ and given $v^* \in EN(V^*)$ we have that:

$$\widehat{T}^{*^{-1}} v^*(\delta_x - \delta_{x_1}) = z_1^*(\delta_y - \delta_{y_1}),$$
$$\widehat{T}^{*^{-1}} v^*(\delta_{x'} - \delta_{x_1}) = z_2^*(\delta_{y_2} - \delta_{y_3})$$

for $y, y_1, y_2, y_3 \in Y$ and $z_1^*, z_2^* \in EN(Z^*)$.

We obtain that $\widehat{T}^{*^{-1}} v^*(\delta_x - \delta_{x'}) = z_1^*(\delta_y - \delta_{y_1}) - z_1^*(\delta_{y_2} - \delta_{y_3})$ and necessarily $z_1^*(\delta_y - \delta_{y_1}) - z_2^*(\delta_{y_2} - \delta_{y_3})$ must be an extreme point of the unit ball of $(\mathscr{C}(X, Z)/\mathscr{C}_Z, \rho)^*$, so we can have the next cases:

(a) $\#\big(\{y, y_1\} \cap \{y_2, y_3\}\big) = 2$, being $\#\big(\{y, y_1\} \cap \{y_2, y_3\}\big)$ the cardinal of this set. In this case we have that:

$$\widehat{T}^{*^{-1}} v^*\big(\delta_x - \delta_{x_1}\big) = z_1^*\big(\delta_y - \delta_{y_1}\big),$$
$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x'} - \delta_{x_1}\big) = z_2^*\big(\delta_y - \delta_{y_1}\big).$$

If we consider a map $f \in \mathscr{C}(X, V)$ such that $f(x) = f(x') = v$ where v is an element of S_V satisfying that $v^*(v) = 1$ and $f(x_1) = 0$, $\|f\| = \rho(f) = 1$ we will have that $z_1^*(Tf(y) - Tf(y_1)) = 1$ and $z_2^*(Tf(y) - Tf(y_1)) = 1$ so that $\|(Tf(y) - Tf(y_1))\| = 1$ and $\|z_1^* + z_2^*\| = 2$, but since $\widehat{T}^{*^{-1}} v^*(\delta x - \delta x') = z_1^*(\delta y - \delta y_1)$ we deduce that $\|z_1^* - z_2^*\| = 1$ and this is a contradiction.

(b) $\#\big(\{y, y_1\} \cap \{y_2, y_3\}\big) = 0$. In this case we have that $\widehat{T}^{*^{-1}} v^*(\delta_x - \delta_{x'}) = z_1^*(\delta_y - \delta_{y_1}) - z_2^*(\delta_{y_2} - \delta_{y_3})$ but this element cannot be an extreme point of the unit ball of $(\mathscr{C}(Y, Z)/\mathscr{C}_Z, \rho)^*$.

(c) $\#\big(\{y, y_1\} \cap \{y_2, y_3\}\big) = 1$. In this case we have different possibilities:
(c-1) $y = y_2$. We have that $y \neq y_1$, $y_2 \neq y_1$, $y \neq y_3$ and

$$\widehat{T}^{*^{-1}}\big(\delta_x - \delta_{x'}\big) = z_1^*\big(\delta_y - \delta_{y_1}\big) - z_2^*\big(\delta_y - \delta_{y_3}\big).$$

This element will be an extreme point if and only if $z_1^* = z_2^*$ so we will have

$$\widehat{T}^{*^{-1}} v^*\big(\delta_x - \delta_{x'}\big) = z_1^*\big(\delta_{y_3} - \delta_{y_1}\big),$$
$$\widehat{T}^{*^{-1}} v^*\big(\delta_x - \delta_{x_1}\big) = z_1^*\big(\delta_y - \delta_{y_1}\big),$$
$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x'} - \delta_{x_1}\big) = z_1^*\big(\delta_y - \delta_{y_3}\big).$$

Let $x'' \notin \{x, x', x_1\}$. There exist $z_3^* \in EN(Z^*)$ and $y_0, y_0' \in Y$ such that:

$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x''} - \delta_{x_1}\big) = z_3^*\big(\delta_{y_0} - \delta'_{y_0}\big).$$

Necessarily, $\#\big(\{y_0, y_0'\} \cap \{y, y_1\}\big) = 1$ and $\#\big(\{y_0, y_0'\} \cap \{y, y_3\}\big) = 1$. It is not possible that $y_0 = y_1$ and $y_0' = y_3$ because in this case we will have:

$$\widehat{T}^{*^{-1}} v^*\big(\delta_x - \delta_{x'}\big) = z_1^*\big(\delta_{y_3} - \delta_{y_1}\big),$$
$$\widehat{T}^{*^{-1}} v^*\big(\delta_x - \delta_{x''}\big) = z_1^*\big(\delta_y - \delta_{y_1}\big) - z_3^*\big(\delta_{y_1} - \delta_{y_3}\big),$$

and we deduce that: $z_3^* = -z_1^*$ and

$$\widehat{T}^{*^{-1}} v^*\big(\delta_x - \delta_{x''}\big) = z_1^*\big(\delta_{y_3} - \delta_{y_1}\big).$$

and this contradicts the injectivity of $\widehat{T}^*$.

If $y_0 = y$ then

$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x''} - \delta_x\big) = z_3^*\big(\delta_{y_0} - \delta_y\big) - z_1^*\big(\delta_y - \delta_{y_1}\big),$$

and we deduce that $z_1^* = z_3^*$ and

$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x''} - \delta_{x_1}\big) = z_1^*\big(\delta_y - \delta_{y_0'}\big).$$

If $y_0' = y_0$, then

$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x''} - \delta_{x_1}\big) = z_3^*\big(\delta_{y_0} - \delta_y\big),$$

and we obtain that

$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x''} - \delta_{x_1}\big) = z_3^*\big(\delta_{y_0} - \delta_y\big) - z_1^*\big(\delta_y - \delta_{y1}\big),$$

and we deduce that $z_1^* = -z_3^*$ and

$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x''} - \delta_{x_1}\big) = z_1^*\big(\delta_y - \delta_{y_0}\big).$$

So that, given $v^* \in EN(V^*)$ and $x_1 \in X$, if $y = y_2$, we can deduce that there exists $z^* \in EN(Z^*)$ such that for each $x'' \in X\backslash\{x_1\}$, there exists $y'' \in Y\backslash\{y\}$ such that $\widehat{T}^{*^{-1}} v^*\big(\delta_{x''} - \delta_{x_1}\big) = z^*\big(\delta_y - \delta_{y''}\big) = (-z)^*\big(\delta_{y''} - \delta_y\big)$.
In other cases: $y = y_3$, $y_1 = y_2$, $y_1 = y_3$ we obtain analogously the same result.

That is, we have that given $v^* \in EN(V^*)$ and $x_1 \in X$ there exists $z^* \in EN(Z^*)$ and $y_1 \in Y$ such that, for each $x \in X\backslash\{x_1\}$ there exists $y \in Y\backslash\{y_1\}$ (we will denote it by $t(x)$, and $y_1 = t(x_1)$) such that:

$$\widehat{T}^{*^{-1}} v^*\left(\delta_x - \delta_{x_1}\right) = z^*\left(\delta_y - \delta_{y_1}\right).$$

If $x' \in X\backslash\{x, x_1\}$, since

$$\widehat{T}^{*^{-1}} v^*\big(\delta_x - \delta_{x_1}\big) = z^*\big(\delta_{t(x)} - \delta_{t(x_1)}\big),$$

and

$$\widehat{T}^{*^{-1}} v^*\big(\delta_{x'} - \delta_{x_1}\big) = z^*\big(\delta_{t(x')} - \delta_{t(x_1)}\big),$$

we obtain that

$$\widehat{T}^{*^{-1}} v^*\big(\delta_x - \delta_{x'}\big) = z^*\big(\delta_{t(x)} - \delta_{t(x')}\big).$$

We have that t is a bijection and we will denote by φ the inverse map, $\varphi : Y \to X$.
We will prove that the map t does not depend on v^*.
Suppose that:

$$\widehat{T}^{*^{-1}} z_1^*\big(\delta_{y_1} - \delta_{y_2}\big) = v_1^*\big(\delta_{x_1} - \delta_{x_2}\big),$$
$$\widehat{T}^{*^{-1}} z_2^*\big(\delta_{y_1} - \delta_{y_2}\big) = v_2^*\big(\delta_{x_3} - \delta_{x_4}\big).$$

We have the next cases:

(i) $\{x_1, x_2\} \cap \{x_3, x_4\} = \varnothing$, where $z_1^* \neq z_2^*$.
In this case $v_1^* \neq v_2^*$.
Let $v_1, v_2 \in S_X$ such that $v_1^*(v_1) = v_2^*(v_2) = 1$ and let $f_1, f_2 \in \mathscr{C}(X, V)$ such that support $f_1 \cap \text{support} f_2 = \varnothing$, $\|f_1\| = \|f_2\| = 1$; $f_1(x_1) = v_1$, $f_1(x_2) = -v_1$, $f_2(x_3) = v_2$, $f_2(x_4) = -v_2$, $\rho(f_1) = \rho(f_2) = 2$, $\rho(f_1 + f_2) \leqslant 2$ and $\rho(f_1 - f_2) \leqslant 2$.
We have that $z_1^*(Tf_1(y_1) - Tf_1(y_2)) = 2$ and $\|Tf_1(y_1) - Tf_1(y_2)\| = 2$.
Analogously $\|Tf_2(y_1) - Tf_2(y_2)\| = 2$ and we have that:

$$\begin{aligned} 4 &= 2\|Tf_1(y_1) - Tf_1(y_2)\| \\ &= \|T(f_1 + f_2)(y_1) - T(f_1 + f_2)(y_2) + T(f_1 - f_2)(y_1) - T(f_1 - f_2)(y_2)\| \\ &\leqslant \|T(f_1 + f_2)(y_1) - T(f_1 + f_2)(y_2)\| + \|T(f_1 - f_2)(y_1) - T(f_1 - f_2)(y_2)\| \leqslant 4. \end{aligned}$$

Since $\rho(f_1 + f_2) \leqslant 2$ and $\rho(f_1 - f_2) \leqslant 2$ we deduce that, if

$$\begin{aligned} z_1 &= Tf_1(y_1) + Tf_2(y_1) - Tf_1(y_2) - Tf_2(y_2), \\ z_2 &= Tf_1(y_1) - Tf_2(y_2) - Tf_1(y_2) + Tf(y_2), \end{aligned}$$

then $\|z_1\| = \|z_2\| = 2$ and $\|z_1 + z_2\| = 4$, $\|z_1 - z_2\| = 4$ which contradicts the fact that Z is rotund.

(ii) $\#\big(\{x_1, x_2\} \cap \{x_3, x_4\}\big) = 1$. Suppose that

$$\begin{aligned} \widehat{T}^* z_1^*\big(\delta_{y_1} - \delta_{y_2}\big) &= v_1^*\big(\delta_{x_2} - \delta_{x_1}\big), \\ \widehat{T}^* z_2^*\big(\delta_{y_1} - \delta_{y_2}\big) &= v_2^*\big(\delta_{x_3} - \delta_{x_1}\big). \end{aligned}$$

Let $g_1 : X \to [-1, 1]$ be a continuous map such that $g_1(x_2) = 1$ and $g_1(x_1) = -1$ and let $f_1 : X \to V$ be a map defined by $f_1(x) = g_1(x)v_1$ where $v_1 \in \mathscr{S}_V$ and $v_1^*(v_1) = 1$.
Let $g_2 : X \to [0, 1]$ a continuous map such that $g_2(x_3) = 1$, $g_2(x_1) = 0$ and $\text{support} g_1 \cap \text{support} g_2 = \varnothing$.
Let $f_2 : X \to V$ a map defined by $f_2(x) = g_2(x)v_2$ where $v_2 \in \mathscr{S}_V$ and $v_2^*(v_2) = 1$.
We have that $\|f_1\| = \|f_2\| = 1$, $\rho(f_1) = 2$ and $\rho(f_2) = 1$.
It is easily seen that $\rho(f_1 + f_2) \leqslant 2$ and $\rho(f_1 - f_2) \leqslant 2$.
We have that

$$z_1^*(Tf_1(y_1) - Tf_1(y_2)) = v_1^*(f_1(x_2) - f_1(x_1)) = 2,$$

and since $\rho(Tf_1) = 2$ we obtain that

$$\begin{aligned} &\|Tf_1(y_1) - Tf_1(y_2)\| = 2, \text{ and} \\ &z_2^*(Tf_2(y_1) - Tf_2(y_2)) = v_2^*(f_2(x_3) - f_2(x_1)) = 1, \end{aligned}$$

so that $\|Tf_2(y_1) - Tf_2(y_2)\| \geqslant 1$ and we have that

$$4 = 2\|Tf_1(y_1) - Tf_2(y_2)\|$$

$$= \|T(f_1+f_2)(y_1) - T(f_1-f_2)(y_2)\|$$
$$\leqslant \|T(f_1+f_2)(y_1) - T(f_1+f_2)(y_2)\| + \|T(f_1-f_2)(y_1) - T(f_1-f_2)(y_2)\| \leqslant 4,$$

so that if

$$z_1 = T(f_1+f_2)(y_1) - T(f_1+f_2)(y_2),$$
$$z_2 = T(f_1-f_2)(y_1) - T(f_1-f_2)(y_2),$$

then $\|z_1\| = \|z_2\| = 2$, $\|z_1+z_2\| = 4$ and $\|z_1 - z_2\| \geqslant 2$ and this is a contradiction with the fact that Z is rotund.

So, necessarily $\{x_1, x_2\} = \{x_3, x_4\}$.

We have proved that there exist a bijection $F : EN(Z^*) \to EN(V^*)$ and a bijection $\varphi : Y \to X$ such that if $z^* \in EN(Z)$ and $y_1, y_2 \in Y$ then:

$$T^*z^*(\delta_{y_1} - \delta_{y_2}) = F(z^*)(\delta_{\varphi(y_1)} - \delta_{\varphi(y_2)}).$$

We will prove now that φ is a homeomorphism.

We suppose that $(y_\alpha)_{\alpha\in I}$ is a net in Y such that $\lim_{\alpha\in I} y_\alpha = y_0$.

Given $z^* \in EN(Z^*)$ and $y_1 \in Y$, $y_1 \neq y_0$ we have that

$$w^* \lim z^*(\delta_{y_\alpha} - \delta_{y_1}) = z^*(\delta_{y_0} - \delta_{y_1}),$$

and we obtain that

$$T^*z^*(\delta_{y_0} - \delta_{y_1}) = w^* \lim T^*z^*(\delta_{y_\alpha} - \delta_{y_1}) = w^* \lim F(z^*)(\delta_{\varphi(y_\alpha)} - \delta_{\varphi(y_1)})$$
$$= F(z^*)(\delta_{\varphi(y_0)} - \delta_{\varphi(y_1)})$$

for each $z^* \in EN(Z^*)$, so that $\lim \varphi(y_\alpha) = \varphi(y_0)$.

Let $f, f' \in \mathscr{C}(X, V)$ and let $y_1, y_2 \in Y$. If $f(\varphi(y_1)) - f(\varphi(y_2)) = f'(\varphi(y_1)) - f'(\varphi(y_2))$ then for each $z^* \in EN(z^*)$ we have

$$T^*z^*(\delta_{y_1} - \delta_{y_2})f = F(z^*)(\delta_{\varphi(y_1)} - \delta_{\varphi(y_2)})f = F(Z^*)(\delta_{\varphi(y_1)} - \delta_{\varphi(y_2)})f'$$
$$= Tz^*(\delta_{y_1} - \delta_{y_2})f',$$

and we deduce that

$$Tf(y_1) - Tf(y_2) = Tf'(y_1) - Tf'(y_2).$$

Given $y, y_1 \in Y$ we define $G : V \to Z$ by $G(v) = Tf(y) - Tf(y_1)$ if $f(y) - f(y_1) = v$.

For each $z^* \in EN(Z^*)$ we have that $z^*(G(v)) = z^*(Tf(y) - Tf(y_1)) = F(z^*)(f(y) - f(y_1))$ so that G is a linear, surjective isometry such that $G^*(z^*) = F(z^*)$ if $z^* \in EN(Z^*)$.

It is easily seen that G does not depend on $y, y_1 \in Y$, so that, for each $f \in \mathscr{C}(X, V)$ and each $y, y_1 \in Y$ we have that $Tf(y) - Tf(y_1) = G(f(\varphi(y)) - f(\varphi(y_1)))$.

We can define the map $\phi : \mathscr{C}(X, V) \to \mathscr{C}(Y, Z)$ by $\phi(f)(y) = G(f(\varphi(y)))$ and we have that ϕ is a diameter preserving, linear, bijective isometry.

Now we consider the map $\widehat{\phi} : \mathscr{C}(X, V)/\mathscr{C}_V \to \mathscr{C}(Y, Z)/\mathscr{C}_Z$ defined by $\widehat{\phi}[f] = [\phi f]$.

If $f \in \mathscr{C}(X, V)$ and $\widehat{\phi}[f] \neq \widehat{T}[f]$, then $\rho(\phi f - Tf) \neq 0$, so that, there exist $y_1, y_2 \in Y$ such that $(\phi f - Tf)(y_1) - (\phi f - Tf)(y_2) \neq 0$ and this is not possible, because $\phi f(y_1) - \phi f(y_2) = Tf(y_1) - Tf(y_2) = G(f(\varphi(t_1)) - f(\varphi(t_2)))$, so that, there exists $z \in Z$ which we denote by Lf such that, if $f \in \mathscr{C}(X, V)$ and $y \in Y$ then: $Tf(y) = G(f(\varphi(y))) + L(f)$.

Evidently L is a linear map from $\mathscr{C}(X, V)$ onto Z which is continuous if and only if T is. With the last arguments we have proved the next.

Theorem 1. *Let X, Y be two compact Hausdorff spaces and let V, Z be two Banach spaces such that Z is rotund and satisfies that if $z_1^*, z_2^* \in EN(Z^*)$ and $\|z_1^* + z_2^*\| = 2$ then $\|z_1^* - z_2^*\| < 1$. There exists a diameter preserving linear bijection $T : \mathscr{C}(X, V) \to \mathscr{C}(Y, Z)$ if and only if X is homeomorphic to Y and V is linearly isometric to Z. In this situation there exist a homeomorphism $\varphi : Y \to X$, a linear, surjective isometry $G : V \to Z$ and a linear map $L : \mathscr{C}(X, V) \to Z$ such that if $f \in \mathscr{C}(X, V)$ and $y \in Y$ then $Tf(y) = G(f(\varphi(y))) + L(f)$.*

Remark 1. If X, Y are locally compact, non-compact spaces and we consider the spaces $\mathscr{C}_0(X, V), \mathscr{C}_0(Y, Z)$, we can deduce a similar result because $\mathscr{C}_0(X, V)$ is linearly isometric to $\mathscr{C}(\gamma X, V)/C$ (where γX is the Alexandroff compactification of X and C is the subspace of constant functions of $\mathscr{C}(\gamma X, V)$).

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