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NEW SOLUTIONS OF THE SCHWARZIAN KORTEWEG–DE VRIES EQUATION IN 2+1 DIMENSIONS BASED ON WEAK SYMMETRIES

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We consider the (2+1)*-dimensional integrable Schwarzian Korteweg–de Vries equation. Using weak symmetries, we obtain a system of partial differential equations in* 1+1 *dimensions. Further reductions lead to second-order ordinary differential equations that provide new solutions expressible in terms of known functions. These solutions depend on two arbitrary functions and one arbitrary solution of the Riemann wave equation and cannot be obtained by classical or nonclassical symmetries. Some of the obtained solutions of the Schwarzian Korteweg–de Vries equation exhibit a wide variety of qualitative behaviors; traveling waves and soliton solutions are among the most interesting.*

Keywords: weak symmetry, partial differential equation, solitary wave

1. Introduction

It is well known that the Schwarzian Korteweg–de Vries (SKdV) equation

$$\frac{\Phi_t}{\Phi_x} + \{\widehat{\Phi}; x\} = 0,$$

where

$$\{\widehat{\Phi}; x\} = \left(\frac{\Phi_{xx}}{\Phi_x}\right)_x - \frac{1}{2}\left(\frac{\Phi_{xx}}{\Phi_x}\right)^2$$

is the Schwarzian derivative [1], is of great interest both in physics and mathematics. This equation was introduced by Krichever and Novikov [2] and Weiss [3] and is a specialization of the KdV equation that is invariant under the Möbius transformations, i.e., under the group $PSL(2)$.

In the context of integrable (2+1)-dimensional equations, i.e., integrable equations with two spatial variables and one temporal variable, Kudryashov and Pickering [4] and also Toda and Yu [5] used the Calogero method to develop the model

$$W_t + \frac{1}{4}W_{xxz} - \frac{W_x W_{xz}}{2W} - \frac{W_{xx} W_z}{4W} + \frac{W_x^2 W_z}{2W^2} - \frac{W_x}{8}\left(\partial_x^{-1}\left(\frac{W_x^2}{W^2}\right)\right)_z = 0, \tag{1}$$

where we set $\partial_x^{-1} f = \int f\,dx$. Equation (1) is invariant under the Möbius transformation and reduces to the SKdV equation for solutions of the form $W(x,z,t) = \overline{W}(x+z,t)$. In [5], the corresponding Lax pair was presented, and it was proved that it passes the Painlevé test in the sense of the Weiss–Tabor–Carnevale method [6].

In the last few years, there has been a continuous interest in the topic of nonclassical and weak symmetries. Olver and Vorob'ev [7] and Clarkson [8] surveyed this research.

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Olver and Rosenau's key question in [9] seems to be whether the reduction methods can be unified by the concept of a differential equation with a *side condition*. They introduced what they called "weak symmetry groups" in [10]. A weak symmetry group G for Δ is a group that no longer transforms solutions into solutions but instead has the property that G-invariant solutions of Δ are found from a reduced system of differential equations involving fewer independent variables than the original system Δ. Bluman and Cole's nonclassical method was found to be a special case of this notion of a weak symmetry group. Surprisingly, at least at first glance, it turns out that every solution of Δ can be obtained as a group-invariant solution of some weak symmetry group and that all groups, more or less, are weak symmetry groups.

Vorob'ev [11] analyzed partial symmetries that he introduced and the nonclassical infinitesimal weak symmetries introduced by Olver and Rosenau. For a family of nonlinear heat equations of the form $u_t = \big(k(u)u_x\big)_x + q(u)$, he indicated pairs of functions $\big(k(u), q(u)\big)$ such that the corresponding equations admit nontrivial two-dimensional modules of partial symmetries that yield explicit solutions.

Dzhamay and Vorob'ev [12] analyzed the nonclassical infinitesimal weak symmetries of the partial differential equations (PDEs) introduced by Olver and Rosenau. They demonstrated that in the case of a PDE in two independent variables, obtaining such symmetries is equivalent to obtaining the two-dimensional modules of nonclassical partial symmetries.

The central question in [13] was which side conditions are admissible for providing genuine solutions to the given differential equations, and it was shown that weak symmetries are not only of academic interest but also necessary for recovering all the solutions of the Navier–Stokes equations found by the semi-inverse method.

Various generalizations of the classical Lie symmetry were reviewed in [14]: conditional symmetry, partial symmetry, and λ-symmetry together with the corresponding criteria and procedures. "Weak" and "strong" classical and conditional symmetries were considered, and partial and weak conditional λ-symmetries were suggested. The author also discussed links between λ-symmetries and conditional and classical symmetries. He gave simple examples of each type of symmetry. In particular, he gave an example of a symmetry that can be regarded as both a weak conditional symmetry and a partial symmetry for the same equations with different sets of solutions corresponding to each type of symmetry.

With the transformations

$$W = \phi_x, \qquad \phi = e^{\psi}, \qquad \psi_x = u, \qquad \psi_t = v, \tag{2}$$

Eq. (1) can be transformed into the system

$$\begin{aligned} &4u^2v_x - 4uu_xv + u^2u_{xxz} - uu_{xx}u_z - 3uu_xu_{xz} + 3u_x^2u_z - u^4u_z = 0,\\ &u_t - v_x = 0. \end{aligned} \tag{3}$$

This system was studied by the classical Lie group method of infinitesimal transformations in [15] and by the nonclassical method in [16].

Equation (1) arises in a nonlocal form, but it can also be written in a local form as

$$\begin{aligned} \Delta \equiv 8\frac{u_{tx}}{u_x} - 8\frac{u_tu_{xx}}{u_x^2} + 2\frac{u_{xxxz}}{u_x} - 2\frac{u_{xx}u_{xxz}}{u_x^2} - 4\frac{u_{xxz}}{u} + 6\frac{u_xu_{xz}}{u^2} - {}\\ - 2\frac{u_zu_{xxx}}{uu_x} - 2\frac{u_{xx}u_{xz}}{uu_x} + 2\frac{u_zu_{xx}^2}{uu_x^2} + 6\frac{u_zu_{xx}}{u^2} - 6\frac{u_zu_x^2}{u^3} = 0. \end{aligned} \tag{4}$$

In this paper, we focus on obtaining solutions of (1) that are obtainable by neither the classical Lie method of group transformations [15] nor Bluman and Cole's nonclassical method [17]. We use the method of weak symmetries introduced by Olver and Rosenau [9], [10]. This approach is to calculate the symmetries of Eq. (4) supplemented by certain differential constraints chosen to weaken the invariance criterion of the basic system and to provide larger Lie point symmetry groups for the augmented system.

2. Weak symmetries of the SKdV in 2+1 dimensions

To construct new solutions of Eq. (1), we obtain the weak symmetries of SKdV equation (4). The procedure is as follows. We augment the PDE with a side condition. Requiring that both (4) and the side condition be invariant under the one-parameter Lie group of infinitesimal transformations in (x, t, z, u) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x, z, t, u) + O(\varepsilon^2), \\ z^* &= z + \varepsilon\eta(x, z, t, u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x, z, t, u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x, z, t, u) + O(\varepsilon^2), \end{aligned} \tag{5}$$

where ε is the group parameter, we obtain an overdetermined nonlinear system of equations for the infinitesimals $\xi(x, z, t, u)$, $\eta(x, z, t, u)$, $\tau(x, z, t, u)$, and $\phi(x, z, t, u)$.

We use the MACSYMA package `symmgrp.max` [18] to apply the procedure in practice. We apply the procedure with two different side conditions: we consider a first-order side condition in the first case and a second-order side condition in the second case.

Reduction 1. Applying the classical method to (4) and the side condition

$$\alpha(x, z, t, u)\frac{\partial u}{\partial x} + \beta(x, z, t, u)\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} - \Psi(x, z, t, u) = 0 \tag{6}$$

yields a system of 23 equations for the infinitesimals $\xi(x, z, t, u)$, $\eta(x, z, t, u)$, $\tau(x, z, t, u)$, and $\phi(x, z, t, u)$ and the functions $\alpha(x, z, t, u)$, $\beta(x, z, t, u)$, and $\Psi(x, z, t, u)$. From this system, we obtain $\xi = \gamma(t)x + \zeta(t)$, $\eta = \eta(t, z)$, $\tau = \tau(t)$, and $\phi = r(z, t)u$ and find that functions γ, η, τ, ζ, r, α, β, and Ψ satisfy the equations

$$\begin{aligned} &\beta_x\gamma x + \beta_u ru + \beta_x\zeta + 2\beta\gamma + \beta_t\tau + \beta_z\eta = 0, \\ &\beta_{xx}\gamma x + \beta_{ux} ru + \beta_{xx}\zeta + 3\beta_x\gamma + \beta_{tx}\tau + \beta_{zx}\eta = 0, \\ &\beta_{ux}\gamma x + \beta_{uu} ru + \beta_u r + \beta_{ux}\zeta + 2\beta_u\gamma + \beta_{tu}\tau + \beta_{uz}\eta = 0, \\ &\alpha_{ux}\gamma x + \alpha_{uu} ru + \alpha_u r + \alpha_{ux}\zeta + \alpha_u\gamma + \alpha_{tu}\tau + \alpha_u\eta_z + \alpha_{uz}\eta = 0, \\ &\gamma\Psi_x x - \beta r_z u + \Psi_u ru - \psi r + \eta\Psi_z + \zeta\Psi_x + \tau\Psi_t + 2\gamma\psi + \eta_z\psi = 0, \\ &\gamma\Psi_{xx} x - \beta_x r_z u + \Psi_{ux} ru - \Psi_x r + \zeta\Psi_{xx} + \eta\Psi_{xz} + 3\gamma\Psi_x + \eta_z\Psi_x + \tau\Psi_{tx} = 0, \\ &\Psi_{ux} x\gamma - \alpha_{xx} x\gamma + 2\Psi_u\gamma - 2\alpha_x\gamma + \Psi_{ux}\zeta - \alpha_{xx}\zeta + r\Psi_{uu} u - \beta_u r_z u - \alpha_{ux} ru + \\ &\qquad + \eta\Psi_{uz} + \eta_z\Psi_u + \tau\Psi_{tu} - \beta r_z - \alpha_{tx}\tau - \alpha_x\eta_z - \alpha_{xz}\eta = 0. \end{aligned}$$

Solving such a system in general is very complicated. Nevertheless, if $\alpha = \alpha(t)$ and $\psi = 0$, then one solution of the determining equations is

$$\xi = \zeta(t), \qquad \eta = \eta(z, t), \qquad \tau = 1, \qquad \phi = 0, \tag{7}$$

where ζ and η are arbitrary functions and $\beta(z,t)$ must satisfy $\beta_t + \eta\beta_z = 0$.

It is easy to verify from (7) that these infinitesimals do not satisfy the nonclassical determining equations. Solving the corresponding surface condition, we obtain the symmetry reductions

$$w_1 = x - \int \zeta(t)\,dt, \qquad w_2 = w_2(t,z), \qquad u = f(w_1, w_2), \tag{8}$$

where η satisfies $w_{2,t} + \eta w_{2,z} = 0$ and f satisfies the (1+1)-dimensional system of PDEs

$$\begin{aligned}
&- f^2 f_{w_2} f_{w_1} f_{w_1w_1w_1} + f^2 f_{w_2} f_{w_1w_1}^2 + 3 f f_{w_2} f_{w_1}^2 f_{w_1w_1} - f^2 f_{w_2w_1} f_{w_1} f_{w_1w_1} - \\
&\quad - f^3 f_{w_1w_1} f_{w_1w_1} - 3 f_{w_2} f_{w_1}^4 + 3 f f_{w_2w_1} f_{w_1}^3 - 2 f^2 f_{w_2w_1w_1} f_{w_1}^2 + f^3 f_{w_2w_1w_1w_1} f_{w_1} = 0, \\
&4 f^3 f_{w_2w_1} f_{w_1} - 4 f^3 f_{w_2} f_{w_1w_1} = 0.
\end{aligned} \tag{9}$$

Reduction 1.1. Reduced system of PDEs (9) in 1+1 variables is invariant under translations and admits further reductions to a system of ODEs. As a result, we obtain the variables

$$\vartheta = w_1 + w_2, \qquad f = g(\vartheta) \tag{10}$$

and the autonomous ODE

$$-g^3 g' g'''' + g^3 g'' g''' + 3 g^2 {g'}^2 g''' - 6 g {g'}^3 g'' + 3 {g'}^5 = 0. \tag{11}$$

Dividing by $g^2{g'}^2$, integrating once over ϑ, and then multiplying by $g^{-2}g'$, we can reduce Eq. (11) to the second-order ODE

$$g'' = \frac{3}{2}\frac{{g'}^2}{g} - k_2 g + k_1, \tag{12}$$

where k_1 and k_2 are arbitrary constants.

Multiplying (12) by $g^{-3}g'$ and integrating once over ϑ, we obtain

$${g'}^2 = -2k_2 g^2 + k_1 g + 2k_3 g^3. \tag{13}$$

The integration can be completed in terms of elliptic functions.

Reduction 1.2. Reduced system of PDEs (9) in 1+1 variables is invariant under the scaling group and admits further reductions to a system of ODEs. We obtain

$$\vartheta = w_2^n w_1, \qquad f = w_2^m g \tag{14}$$

and the system of ODEs

$$\begin{aligned}
&g^3 g' g'''' \vartheta - g^3 g'' g''' \vartheta - 3 g^2 {g'}^2 g''' \vartheta + 6 g {g'}^3 g'' \vartheta - 3 {g'}^5 \vartheta + 3 g^3 g' g''' - \\
&\qquad\qquad - 2 g^3 {g''}^2 - 5 g^2 {g'}^2 g'' + 3 g {g'}^4 = 0, \\
&m g g'' - (m+n) {g'}^2 = 0.
\end{aligned} \tag{15}$$

Reduction 2. Applying the classical method to (4) and the side conditions

$$\beta\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}=0, \qquad \frac{\partial u}{\partial x\partial z}=0$$

yields a system of ten equations for the infinitesimals $\xi(x,z,t,u)$, $\eta(x,z,t,u)$, $\tau(x,z,t,u)$, and $\phi(x,z,t,u)$ and for the function $\beta(x,z,t,u)$. From this system, we obtain

$$\xi=\gamma(z,t)x+\zeta(z,t), \qquad \eta=\eta(z,t), \qquad \tau=\tau(x,z,t,u), \qquad \phi=\rho(z,t)u^3+\delta(z,t)u,$$

where β, γ, τ, ζ, ρ, δ, and η are functions that must satisfy the equations

$$\begin{aligned} &-\beta_x ux\gamma-2\beta u\gamma-\beta_x u\zeta-\beta_u\rho u^4+2\beta\rho u^3-\beta_u\delta u^2-\beta_t\tau u-\beta_z\eta u=0,\\ &\beta_{ux}ux\gamma+2\beta_u u\gamma+\beta_{ux}u\zeta+\beta_{uu}\rho u^4+\beta_u\rho u^3+6\beta\rho u^2+\beta_{uu}\delta u^2+\beta_{tu}\tau u+\\ &\qquad\qquad\qquad\qquad+\beta_{uz}\eta u+\beta_u\delta u=0,\\ &-\beta_{xx}ux\gamma-3\beta_x u\gamma-\beta_{xx}u\zeta-\beta_{ux}\rho u^4+2\beta_x\rho u^3-\beta_{ux}\delta u^2-\beta_{tx}\tau u-\beta_{xz}\eta u=0. \end{aligned} \tag{16}$$

The complexity of system (16) is why we cannot solve it in general. Nevertheless, if $\beta=\beta(z,t)$, then a solution is

$$\xi=\zeta(z,t), \qquad \eta=\eta(z,t), \qquad \tau=1, \qquad \phi=\delta(z)u.$$

Solving the corresponding surface condition, we obtain

$$w_1=x+b(z,t), \qquad w_2=a(t,z), \qquad u=\rho(z)f(w_1,w_2),$$

which leads to the following system of PDEs in two independent variables with $a_t-k_1a^2a_z=0$ and the arbitrary functions $\rho=\rho(z)$ and $b=b(z,t)$:

$$\begin{aligned} &f^3f_{w_1}f_{w_1w_1w_1w_1}-f^3f_{w_1w_1}f_{w_1w_1w_1}-3f^2f_{w_1}^2f_{w_1w_1w_1}+6ff_{w_1}^3f_{w_1w_1}-3f_{w_1}^5=0,\\ &-4k_1f^3f_{w_1w_1}f_{w_2w_2}^2+4k_1f^3f_{w_1}f_{w_1w_2w_2}^2-f^2f_{w_1}f_{w_1w_1w_1}f_{w_2}+f^2f_{w_1w_1}^2f_{w_2}+\\ &\qquad+3ff_{w_1}^2f_{w_1w_1}f_{w_2}-3f_{w_1}^4f_{w_2}+f^3f_{w_1}f_{w_1w_1w_1w_2}-f^3f_{w_1w_1}f_{w_1w_1w_2}-\\ &\qquad\qquad-2f^2f_{w_1}^2f_{w_1w_1w_2}-f^2f_{w_1}f_{w_1w_2}f_{w_1w_1}+3ff_{w_1}^3f_{w_1w_2}=0. \end{aligned} \tag{17}$$

Reduction 2.1. System (17) is invariant under the scaling group. Hence, solving the characteristic equation, we obtain the symmetry reduction

$$\vartheta=w_1w_2, \qquad f=h(\vartheta),$$

which yields the system of ODEs

$$\begin{aligned} &h^3h'h''''-h^3h''h'''-3h^2h'^2h'''+6hh'^3h''-3h'^5=0,\\ &3h^2h'h'''-2h^2h''^2-5hh'^2h''+3h'^4+4k_1h^2h'^2=0. \end{aligned}$$

Integrating the first equation once over ϑ and changing the variable $h'/h=g$, we obtain the system of second-order ODEs

$$
\begin{aligned}
&gg'' - g'^2 - \frac{g^4}{4} + k_2 = 0, \\
&3gg'' - 2g'^2 - g^4 + 4k_1 g^2 = 0.
\end{aligned}
\tag{18}
$$

Reduction 2.2. System (17) is invariant under the scaling group. Hence, solving the characteristic equation yields the symmetry reduction

$$
\vartheta = w_1 w_2, \qquad f = \frac{1}{w_2} h(\vartheta),
$$

which yields the system of ODEs

$$
\begin{aligned}
&h^3 h' h'''' - h^3 h'' h''' - 3h^2 h'^2 h''' + 6hh'^3 h'' - 3h'^5 = 0, \\
&3h^2 h' h''' - 2h^2 h''^2 + 4k_1 h^3 h'' - 5hh'^2 h'' + 3h'^4 = 0.
\end{aligned}
\tag{19}
$$

Changing the variable $h_\vartheta / h = g$ and integrating once over ϑ, we obtain the system

$$
\begin{aligned}
&gg'' - g'^2 - \frac{g^4}{4} + k_2 = 0, \\
&3gg'' - 2g'^2 - 4k_1 n g' - g^4 + 4k_1 g^2 = 0.
\end{aligned}
$$

3. Some exact solutions

We now present some explicit solutions of the second-order ODEs and also the corresponding solution of the SKdV equation in 2+1 dimensions.

Reduction 1.1. Equation (11) can be integrated in terms of elliptic functions:

$$
g_1 = \mathrm{sn}^{\pm 2}(\vartheta|m), \qquad g_2 = \mathrm{cn}^{\pm 2}(\vartheta|m), \qquad g_3 = \mathrm{dn}^{\pm 2}(\vartheta|m). \tag{20}
$$

Clearly, any of the rational, hyperbolic, or trigonometric degenerations of the elliptic functions also give solutions of (11):

$$
\begin{aligned}
&g = \frac{k_1}{k_2} \tanh^{\pm 2}\left(\frac{\sqrt{-k_2}\,\vartheta}{2}\right), \qquad g = -\frac{k_1}{k_2} \tan^{\pm 2}\left(\frac{\sqrt{k_2}\,\vartheta}{2}\right), \qquad g = -\frac{k_1}{2k_2} \sinh^{\pm 2}\left(\frac{\sqrt{k_2}\,\vartheta}{\sqrt{2}}\right), \\
&g = \frac{k_1}{2k_2} \cosh^{\pm 2}\left(\frac{\sqrt{k_2}\,\vartheta}{\sqrt{2}}\right), \qquad g = \frac{k_1}{2k_2} \sin^{\pm 2}\left(\frac{\sqrt{-k_2}\,\vartheta}{\sqrt{2}}\right), \qquad g = \frac{k_1}{2k_2} \cos^{\pm 2}\left(\frac{\sqrt{-k_2}\,\vartheta}{\sqrt{2}}\right), \\
&g = k_3 \vartheta^{\pm 2}, \qquad g = k_4 e^{\pm\sqrt{2k_2}\,\vartheta}.
\end{aligned}
\tag{21}
$$

Considering the corresponding symmetry reductions (8) and (10), we find that exact solutions of (1) can be written as

$$
\begin{aligned}
&u = \mathrm{sn}^{\pm 2}\big(x + a(z,t) + \delta(t)|m\big), \qquad u = \mathrm{dn}^{\pm 2}\big(x + a(z,t) + \delta(t)|m\big), \\
&u = \mathrm{sn}^{\pm 2}\big(x + a(z,t) + \delta(t)|m\big),
\end{aligned}
\tag{22}
$$

where $a(z,t) = w_2(z,t)$, $\delta(t) = -\int \zeta(t)\,dt$,

$$
\begin{aligned}
u &= \frac{k_1\rho(z)}{k_2}\tanh^{\pm 2}\left(\frac{\sqrt{-k_2}\,\big(x + a(z,t) + \delta(t)\big)}{2}\right),\\
u &= -\frac{k_1\rho(z)}{k_2}\tan^{\pm 2}\left(\frac{\sqrt{k_2}\,\big(x + a(z,t) + \delta(t)\big)}{2}\right),\\
u &= k_3\rho(z)\operatorname{sech}^{\pm 2}\left(\frac{\sqrt{k_2}\,\big(x + a(z,t) + \delta(t)\big)}{\sqrt{2}}\right),\\
u &= k_3\rho(z)\operatorname{csch}^{\pm 2}\left(\frac{\sqrt{k_2}\,\big(x + a(z,t) + \delta(t)\big)}{\sqrt{2}}\right),\\
u &= k_3\rho(z)\sec^{\pm 2}\left(\frac{\sqrt{-k_2}\,\big(x + a(z,t) + \delta(t)\big)}{\sqrt{2}}\right),\\
u &= k_3\rho(z)\csc^{\pm 2}\left(\frac{\sqrt{-k_2}\,\big(x + a(z,t) + \delta(t)\big)}{\sqrt{2}}\right),\\
u &= k_3\rho(z)\big(x + a(z,t) + \delta(t)\big)^{\pm 2},\\
u &= \rho(z)e^{\sqrt{-2k_1}\,(x + a(z,t) + \delta(t))}.
\end{aligned}
\tag{23}
$$

Reduction 1.2. The compatibility of (15) implies that $m = \pm 2n$. Consequently, its solutions are

$$g = k_1(\vartheta + k_2)^{\pm 2},$$

and the solutions of (1) are

$$u = k_1\left(\frac{a^n(x-\delta) + k_2}{a^n}\right)^{\pm 2}.$$

Reduction 2.1. We obtain the following solutions of system (18):

1. If $k_2 = 0$, then $g = 4\sqrt{k_1}\,\csc\big(2\sqrt{k_1}\,\vartheta\big)$ and $g = 4\sqrt{-k_1}\,\operatorname{csch}\big(2\sqrt{-k_1}\,\vartheta\big)$.
2. If $k_2 = 4k_1^2$ and $k_1 = k^2/4$, then $g = k$.

From $h'/h = g$, we respectively obtain

$$h = c\tan^{\pm 2}\big(\sqrt{k_1}\,\vartheta\big), \qquad h = c\tanh^{\pm 2}\big(\sqrt{-k_1}\,\vartheta\big), \qquad h = ce^{\pm k\vartheta}.$$

Consequently, we obtain the solutions of (1)

$$
\begin{aligned}
&u = \rho(z)\tan^{\pm 2}\big(a(z,t)\sqrt{k_1}\,\big(x + b(z,t)\big)\big), \qquad u = \rho(z)\tanh^{\pm 2}\big(a(z,t)\sqrt{-k_1}\,\big(x + b(z,t)\big)\big),\\
&u = e^{\pm 2\sqrt{k_1}a(z,t)(x+b(z,t))},
\end{aligned}
$$

where $a = w_2$, a satisfies the Riemann wave equation $a_t - k_1a^2a_z = 0$, and $\rho = \rho(z)$ and $b = b(z,t)$ are arbitrary functions.

We see that the functions $b = b(z,t)$, $\rho = \rho(z)$, and $a = a(z,t)$ determine the respective wave displacement, wave amplitude, and wave speed.

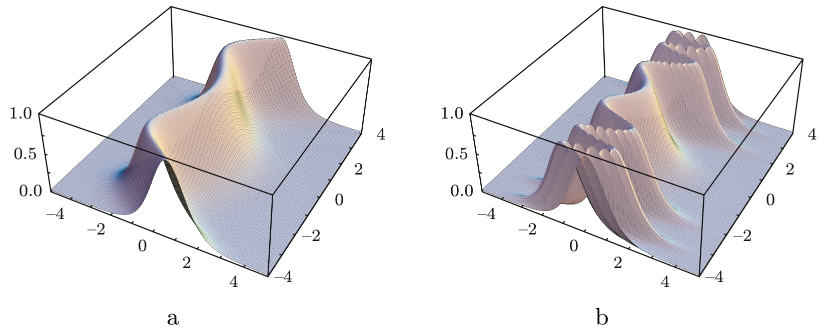

Fig. 1. Solution (24): (a) at $t = 1$ and (b) at $t = 2$.

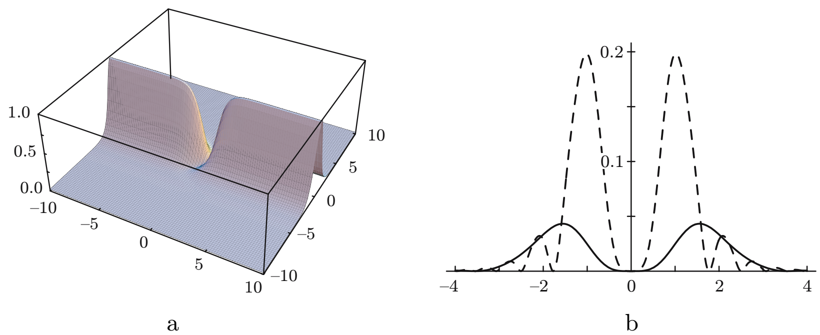

Fig. 2. Solution (25): (a) at $t = 1$ and (b) the corresponding section profiles $x = 0$ at the times $t = 1$ (solid line) and $t = 2$ (dashed line).

Reduction 2.2. We obtain the solutions of system (19)

$$h = \sin\bigl(2\sqrt{-k_1}\,\vartheta\bigr) \pm 1, \qquad h = \cos^{\pm 2}\bigl(\sqrt{-k_1}\,\vartheta\bigr), \qquad h = \cosh^{\pm 2}\bigl(\sqrt{k_1}\,\vartheta\bigr).$$

Consequently, we obtain the solutions of (1)

$$u = \frac{\rho(z)\sin\bigl(2\sqrt{-k_1}\,a(z,t)\bigl(x + b(z,t)\bigr)\pm 1\bigr)}{a(z,t)},$$

$$u = \frac{\rho(z)\cos^{\pm 2}\bigl(\sqrt{-k_1}\,a(z,t)\bigl(x + b(z,t)\bigr)\bigr)}{a(z,t)},$$

$$u = \frac{\rho(z)\cosh^{\pm 2}\bigl(\sqrt{k_1}\,a(z,t)\bigl(x + b(z,t)\bigr)\bigr)}{a(z,t)},$$

where $a = w_2$ and a is an arbitrary solution of the Riemann wave equation $a_t - k_1 a^2 a_z = 0$.

We see that the function $b = b(z,t)$ determines the wave displacement and the function $a = a(z,t)$ modulates the wave amplitude and the wave speed.

In Fig. 1, we present the solution

$$u = \operatorname{sech}^2\left(x + \operatorname{Sin}\left(\frac{z^2 t^2}{4}\right)\right) \tag{24}$$

for $t = 1$ and $t = 2$. We see that this solution evolves without restriction on $x = b(z,t)$. This kind of solution is obtainable by neither the Lie classical method nor the nonclassical method.

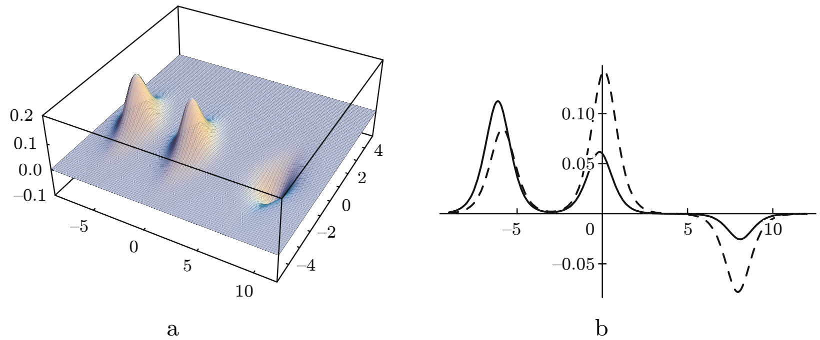

Fig. 3. Solution (26): (a) three dromions at $t = -2$ and (b) the corresponding section profiles $x = 0$ at the times $t = -2$ (dashed line) and $t = 2$ (solid line).

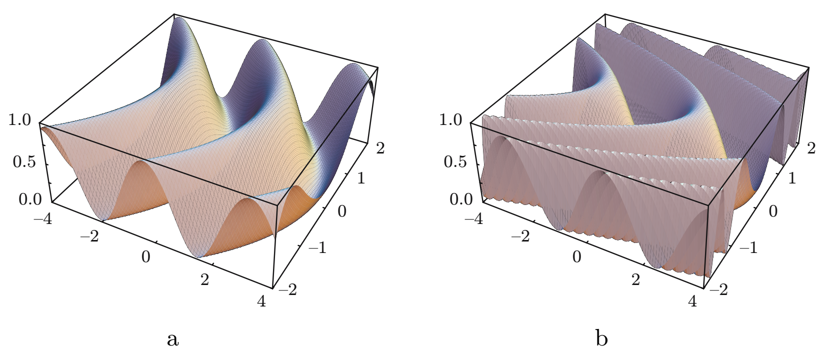

Fig. 4. Jacobi solution (27): (a) at $t = 0.5$ and (b) at $t = 2$.

In Fig. 2, we present the solution

$$u = \operatorname{sech}^2(z)\tanh^2\left(x + \operatorname{Sin}^2\left(\frac{z^2t^2}{4}\right)\right) \tag{25}$$

for $t = 1$ and the corresponding section profiles $x = 0$ at the times $t = 1$ and $t = 2$, reflecting the time evolution.

In Fig. 3, we present the solution

$$u = \rho(z)\operatorname{sech}^2\bigl(x + 0.2\tanh z + 0.2\tanh(z+6) - 0.1\tanh(z-8) + \sin t\bigr) \tag{26}$$

with $\rho(z) = 0.2\operatorname{sech}^2 z + 0.2\operatorname{sech}^2(z+6) - 0.1\operatorname{sech}^2(z-8)$ for $t = -2$ and the corresponding section profiles $x = 0$ at the times $t = -2$ and $t = 2$, which reflect the time evolution.

In Fig. 4, we present the Jacobi solution

$$u = \operatorname{sn}^2\left(x + z^2t \,\middle|\, \frac{1}{3}\right) \tag{27}$$

at $t = 0.5$ and $t = 2$.

4. Conclusions

We have considered the (2+1)-dimensional integrable Schwarzian Korteweg–de Vries equation. Using weak symmetries and further reductions, we derived second-order ordinary differential equations that provide new solutions expressible in terms of known functions. The corresponding solutions of (1) depend on two arbitrary functions $\rho = \rho(z)$ and $b = b(z,t)$ and on an arbitrary solution $a = a(z,t)$ of the Riemann wave equation $a_t - k_1 a^2 a_z = 0$. Some of the found solutions of the Schwarzian Korteweg–de Vries equation exhibit a wide variety of qualitative behaviors: soliton and dromion solutions are among the most interesting.

We saw that in the solitons, the arbitrary function $b = b(z,t)$ determines the wave displacement and the arbitrary solution of the Riemann wave equation modulates the amplitude and speed of the wave. These solutions of Eq. (1) cannot be obtained by the classical Lie method of group transformations [15] or by Bluman and Cole's nonclassical method [16].

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