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INTEGRABLE SEMICLASSICAL DEFORMATIONS OF GENERAL ALGEBRAIC CURVES AND ASSOCIATED CONSERVATION LAWS

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Based on the Lenard relations, we completely classify integrable deformations of general algebraic curves. We construct the general solution of the Lenard relation from the invariance condition with respect to an element of the Galois group of the curve. We give some examples and also some associated conservation laws.

Keywords: algebraic curve, integrable system, Lenard relation, conservation law

1. Introduction

Algebraic curves are fundamental in analyzing integrable nonlinear differential equations [1]–[2]. A particularly interesting problem is characterizing and classifying integrable deformations of algebraic curves. In [2], Krichever formulated a general theory of dispersionless hierarchies of integrable models associated with the deformations of algebraic curves arising in the Whitham averaging method. A different approach was proposed in [3]–[7] for seeking integrable deformations of algebraic curves $\mathcal{C}$ defined by monic polynomial equations

$$F(p,k) := p^N - \sum_{n=1}^{N} u_n(k) p^{N-n} = 0, \quad u_n \in \mathbb{C}[k]. \tag{1}$$

To introduce these deformations, we must consider the N branches $p_j = p_j(k)$, $j = 1,\dots,N$, of the multivalued function defined by (1), i.e.,

$$F(p,k) = \prod_{j=1}^{N}(p - p_j(k)) = 0, \qquad \mathbf{p}(k) = (p_1(k),\dots,p_N(k))^{\mathrm{T}}.$$

The problem can be formulated in terms of these branches as seeking deformations $\mathcal{C}(x,t)$ consistent with the degrees of the polynomials u_n and characterized by the existence of an *action* function $\mathbf{S} = \mathbf{S}(k,x,t)$ satisfying the following conditions:

1. The function $\mathbf{p} = \mathbf{p}(k)$ can be expressed as

$$\mathbf{p} = \mathbf{S}_x.$$

2. The functions $\mathbf{S}_x$ and $\mathbf{S}_t$ are meromorphic functions of p on $\mathcal{C}(x,t)$ with poles only at $k = \infty$.

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As a consequence of these conditions, $\mathbf{p}$ satisfies the equation

$$\partial_t \mathbf{p} = \partial_x \mathbf{Q}, \tag{2}$$

where $\mathbf{Q} := \mathbf{S}_t = (Q_1, \ldots, Q_N)^{\mathrm{T}}$ with $Q_j \in \mathbb{C}[k,p]/\mathcal{C}$ has the form [3], [8]

$$Q_j = \sum_{r=1}^{N} a_r(k,x,t) p_j^{N-r}, \quad a_r \in \mathbb{C}[k], \quad j = 1, \ldots, N.$$

In what follows, we need an important result concerning the branches $p_j(k)$. Let $\mathbb{C}((\lambda))$ denote the field of Laurent series in λ with at most a finite number of terms with positive powers. Then we have the Newton theorem [9].

Theorem 1. *There exists a positive integer l such that the N branches*

$$p_j(z) := (p_j(k))|_{k=z^l}, \quad j = 1, \ldots, N, \tag{3}$$

are elements of $\mathbb{C}((z))$. Furthermore, if $F(p,k)$ is irreducible as a polynomial over the field $\mathbb{C}((k))$, then $l_0 = N$ is the least permissible l, and the branches $p_j(z)$ can be labeled such that

$$p_j(z) = p_N(\epsilon^j z), \qquad \epsilon := e^{2\pi i/N}.$$

Notation convention. Henceforth, given an algebraic curve $\mathcal{C}$, we let z denote the variable associated with the least positive integer l_0 for which the substitution $k = z^{l_0}$ implies $p_j \in \mathbb{C}((z))$ for all j. We call l_0 the Newton exponent of $\mathcal{C}$.

Equation (2) can be rewritten in terms of the potentials u_n, $n = 1, 2, \ldots, N$, as [4], [5]

$$\partial_t \mathbf{u} = J_0 \mathbf{a}, \tag{4}$$

where J_0 is an $N \times N$ matrix differential operator whose elements can be written in terms of the potentials and their derivatives as

$$\begin{aligned}
&(J_0)_{11} = N\partial_x, \\
&(J_0)_{i1} = (i-1)\mathcal{P}_{i-1}\partial_x - \sum_{l=2}^{i-1} u_{i-l}\mathcal{P}_{l-1}\partial_x - N u_{i-1}\partial_x, \quad i \neq 1, \\
&(J_0)_{ij} = (i+j-2)\mathcal{P}_{i+j-2}\partial_x + (j-1)\mathcal{P}_{i+j-2,x} - \\
&\qquad - \sum_{k=1}^{i-1} u_{i-k}[(k+j-2)\mathcal{P}_{k+j-2}\partial_x + (j-1)\mathcal{P}_{k+j-2,x}], \quad i \neq 1, \quad j \neq 1,
\end{aligned} \tag{5}$$

the $\mathcal{P}_s$ are the power sums [10]

$$\mathcal{P}_s := \frac{1}{s}(p_1^s + \cdots + p_N^s) = \sum_{1 \le i \le s}^{(s)} \frac{1}{i}(u_1 + \cdots + u_N)^i,$$

and the superscript (s) in the summation symbol indicates that only the terms of weight s are retained.

The next problem is to determine expressions for $\mathbf{a}$ (in (4)) depending on k and $\mathbf{u}$ such that flow (4) is consistent with the polynomial dependence of $\mathbf{u}$ on the variable k. In other words, if $d_n := \mathrm{degree}(u_n)$ are the degrees of the coefficients u_n as polynomials in k, then

$$\mathrm{degree}(J_0\mathbf{a})_n \le d_n, \quad n = 1, \dots, N,$$

must be satisfied. The strategy for finding consistent deformations [4], [5] is to use Lenard-type relations

$$J_0\mathbf{r} = 0, \qquad \mathbf{r} := (r_1, \dots, r_N)^{\mathrm{T}}, \quad r_j \in \mathbb{C}((k)), \tag{6}$$

and take $\mathbf{a} := \mathbf{r}_+$, where $(\,\cdot\,)_+$ and $(\,\cdot\,)_-$ indicate the respective parts of nonnegative and negative powers in k. It is now clear from the identity $J_0\mathbf{a} = J_0\mathbf{r}_+ = -J_0\mathbf{r}_-$ that a sufficient condition for the consistency of (4) is

$$\max_{m=1,\dots,N}\{\mathrm{degree}(J_0)_{nm}\} \le d_n + 1, \quad n = 1, \dots, N. \tag{7}$$

Imposing this condition, we obtain a sufficient condition for consistency that depends only on curve (1) and is independent of the particular solution of the Lenard relation used. All the solutions of (7) were given in [4], [6] for the cases $N = 2$ and $N = 3$ and in [7] for $N = 4$. It was thus found that the compatible degrees are

$$\begin{aligned}
N = 2: &\qquad d_1 \le d_2 + 1,\\
N = 3: &\qquad (0,0,1),\quad (0,1,0),\quad (0,1,1),\quad (0,1,2),\quad (1,0,0),\quad (1,0,1),\\
&\qquad (1,1,0),\quad (1,1,1),\quad (1,1,2),\quad (1,2,1),\quad (1,2,2),\quad (1,2,3),\\
N = 4: &\qquad (0,0,0,1),\quad (0,0,1,0),\quad (0,0,1,1),\quad (0,1,0,0),\\
&\qquad (0,1,0,1),\quad (0,1,1,0),\quad (0,1,1,1),\quad (0,1,1,2).
\end{aligned} \tag{8}$$

Also, a general result for $N \ge 5$ was proved in [7]; it is formulated as follows.

Theorem 2. *For each $N \in \mathbb{N}$ ($N \ge 5$), the degrees $(d_1, \dots, d_N)$ satisfy compatibility condition (7) if and only if they are*

$$d_j = 0, \quad j = 1, 2, \dots, N-3, \qquad d_{N-2}, d_{N-1}, d_N \le 1. \tag{9}$$

The next step for classifying the deformations is to determine solutions $\mathbf{r}$ of Lenard relation (6). It was proved in [4], [5] that the solution of the equation $J_0\mathbf{r} = 0$ is given by

$$\mathbf{r} = T\nabla_{\mathbf{u}}R, \qquad R = \sum_{j=1}^{N} g_j(z)p_j, \qquad \nabla_{\mathbf{u}}R = \left(\frac{\partial R}{\partial u_1}, \dots, \frac{\partial R}{\partial u_N}\right)^{\mathrm{T}}, \tag{10}$$

where

$$T := \begin{pmatrix} 1 & -u_1 & \cdots & -u_{N-1}\\ 0 & 1 & \cdots & -u_{N-2}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and $g_j \in \mathbb{C}((z))$. Hence, the remaining problem is determining the appropriate choices of the functions g_j

such that $R \in \mathbb{C}((k))$ (and consequently $\mathbf{r} \in \mathbb{C}((k))$). This problem was considered in [6] in terms of the element σ_0 of the Galois group of the curve,

$$\sigma_0(p_j)(z) := p_j(\epsilon_0 z), \qquad \epsilon_0 := e^{2\pi i/l_0}. \tag{11}$$

It is therefore clear that requiring $R \in \mathbb{C}((k))$ is equivalent to requiring that R be invariant under the action of σ_0, i.e.,

$$R(\epsilon_0 z, \sigma_0 \mathbf{p}) = R(z, \mathbf{p}). \tag{12}$$

In this paper, we focus on constructing functions R of form (10) satisfying invariance condition (12). Using these functions, we can give some examples of integrable systems of deformations. To determine the invariant functions R specified by (10)–(12), we must classify all the compatible degrees (8) and (9) of the potentials according to σ_0 and l_0.

This paper is organized as follows. In Sec. 2, we compute the Newton exponent l_0 and the element σ_0 in (11) of the Galois group of the curve in each compatible case in (8) and (9). In Sec. 3, we consider the problem of constructing the function R of form (10) satisfying (12) in each compatible case. Finally, we illustrate all these results in Sec. 4 with some examples of integrable deformations of algebraic curves and also compute some conservation laws associated with each integrable deformation.

2. Newton exponents and Galois group elements of the curve

Our next goal is to classify all the compatible cases in terms of the corresponding Newton exponent and the element σ_0 of the Galois group of the curve (see (11)). We first note that in the cases $N = 2, 3, 4$, the branches can be written in terms of the potentials. Consequently, expanding the branches in powers of z and using definition (11), we immediately obtain the results in Tables 1–3.

Table 1

σ_0	l_0	(d_1, d_2)
$\begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}$	2	$d_2 > 2d_1$, d_2 odd
$\begin{pmatrix} p_1 & p_2 \\ p_1 & p_2 \end{pmatrix}$	1	$d_2 < 2d_1$ or d_2 even

Values of σ_0 and l_0 for $N = 2$.

Table 2

σ_0	l_0	(d_1, d_2, d_3)
$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_1 \end{pmatrix}$	3	$(0,0,1)$ $(0,1,2)$
$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_1 & p_3 \end{pmatrix}$	2	$(0,1,0)$ $(0,1,1)$ $(1,0,0)$ $(1,1,2)$
$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$	1	$(1,0,1)$ $(1,1,0)$ $(1,1,1)$ $(1,2,1)$ $(1,2,2)$ $(1,2,3)$

Values of σ_0 and l_0 for $N = 3$.

Table 3

σ_0	l_0	(d_1, d_2, d_3, d_4)
$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_3 & p_4 & p_1 \end{pmatrix}$	4	$(0,0,0,1)$
$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_3 & p_1 & p_4 \end{pmatrix}$	3	$(0,0,1,0)$ $(0,0,1,1)$
$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & p_3 & p_4 \end{pmatrix}$	2	$(0,1,1,0)$ $(0,1,1,1)$ $(0,1,0,1)$
$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & p_4 & p_3 \end{pmatrix}$	2	$(0,1,0,0)$ $(0,1,1,2)$

Values of σ_0 and l_0 for $N = 4$.

Because of the Abel theorem, the branches in the general case $N \geq 5$ cannot be written in terms of the potentials. Nevertheless, all the algebraic curves corresponding to (9) are rational (k is a rational function of p). The classification can therefore be obtained by studying the asymptotic behavior of the N branches p_j, $j = 1, 2, \ldots, N$, as $k \to \infty$. To do that, we write the potentials as

$$u_n = \sum_{j=0}^{d_n} u_{nj} k^j.$$

We then have the following classification:

- Let $(d_1, \ldots, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = (0, \ldots, 0, 0, 0, 1)$. Then (1) can be written as

$$k = \frac{1}{u_{N1}} \left(p^N - \sum_{l=1}^{N} u_{l0} p^{N-l} \right).$$

  Hence,

$$p_j^N \sim u_{N1} k \quad \text{as } k \to \infty, \quad j = 1, 2, \ldots, N.$$

  Consequently, $p_j \in \mathbb{C}((k^{1/N}))$, $j = 1, 2, \ldots, N$, and

$$l_0 = N, \qquad \sigma_0 = \begin{pmatrix} p_1 & p_2 & \cdots & p_{N-1} & p_N \\ p_2 & p_3 & \cdots & p_N & p_1 \end{pmatrix}.$$

- Let $(d_1, \ldots, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = (0, \ldots, 0, 0, 1, 0)$. Then (1) can be written as

$$k = \frac{1}{u_{N-1\,1}} \left(p^{N-1} - \sum_{l=1}^{N} u_{l0} p^{N-l-1} - \frac{u_{N0}}{p} \right).$$

  Hence,

$$p_j^{N-1} \sim u_{N-1\,1} k \quad \text{as } k \to \infty, \quad j = 1, 2, \ldots, N-1,$$

$$p_N \sim -\frac{u_{N0}}{u_{N-1\,1}} \frac{1}{k} \quad \text{as } k \to \infty.$$

  Consequently,

$$l_0 = N - 1, \qquad \sigma_0 = \begin{pmatrix} p_1 & p_2 & \cdots & p_{N-1} & p_N \\ p_2 & p_3 & \cdots & p_1 & p_N \end{pmatrix}.$$

• Let $(d_1,\dots,d_{N-3},d_{N-2},d_{N-1},d_N)=(0,\dots,0,1,0,0)$. Then (1) can be written as

$$k=\frac{1}{u_{N-2\,1}}\left(p^{N-2}-\sum_{l=1}^{N-2}u_{l0}p^{N-l-2}+\frac{u_{N-1\,0}}{p}+\frac{u_{N0}}{p^2}\right).$$

Hence,

$$\begin{aligned}
&p_j^{N-2}\sim u_{N-2\,1}k \quad \text{as } k\to\infty,\quad j=1,2,\dots,N-2,\\
&p_j^2\sim \frac{u_{N0}}{u_{N-2\,1}}\frac{1}{k} \qquad \text{as } k\to\infty,\quad j=N-1,N.
\end{aligned}$$

Consequently, the Newton exponent is

$$l_0=\begin{cases}N-2 & \text{if } N \text{ is even},\\ 2(N-2) & \text{if } N \text{ is odd},\end{cases}$$

and the corresponding Galois group element is given by

$$\sigma_0=\begin{pmatrix}p_1 & p_2 & \cdots & p_{N-2} & p_{N-1} & p_N\\ p_2 & p_3 & \cdots & p_1 & p_N & p_{N-1}\end{pmatrix}.$$

Proceeding in the same way with the remaining cases, we obtain the results in Table 4.

Table 4

σ_0	l_0	$(d_1,\dots,d_N)$
$\begin{pmatrix}p_1 & p_2 & \dots & p_{N-1} & p_N\\ p_2 & p_3 & \dots & p_N & p_1\end{pmatrix}$	N	$(0,\dots,0,0,0,1)$
$\begin{pmatrix}p_1 & p_2 & \dots & p_{N-1} & p_N\\ p_2 & p_3 & \dots & p_1 & p_N\end{pmatrix}$	$N-1$	$(0,\dots,0,0,1,0)$ $(0,\dots,0,0,1,1)$
$\begin{pmatrix}p_1 & \dots & p_{N-2} & p_{N-1} & p_N\\ p_2 & \dots & p_1 & p_{N-1} & p_N\end{pmatrix}$	$N-2$	$(0,\dots,0,1,1,0)$ $(0,\dots,0,1,1,1)$ $(0,\dots,0,1,0,1)$
$\begin{pmatrix}p_1 & \dots & p_{N-2} & p_{N-1} & p_N\\ p_2 & \dots & p_1 & p_N & p_{N-1}\end{pmatrix}$	$N-2$, if N even $2(N-2)$, if N odd	$(0,\dots,0,1,0,0)$

Classification of (9) according to σ_0 and l_0.

3. The invariant function R

To obtain the hierarchy of integrable deformations (4), we must determine the function R of form (10) satisfying invariance condition (12). In view of (12), the different cases are classified according to the corresponding element σ_0 of the Galois group of the curve.

• Let

$$\sigma_0=\begin{pmatrix}p_1 & p_2 & \dots & p_{N-1} & p_N\\ p_2 & p_3 & \dots & p_N & p_1\end{pmatrix}.$$

According to Tables 1–4 in the preceding section, we have $l_0 = N$ ($\epsilon_0 = \epsilon = e^{2\pi i/N}$). To determine the invariant function R, we seek functions

$$R_k = \sum_{j=1}^{N} \alpha_j p_j \quad \text{such that } \sigma_0(R_k) = \epsilon_0^{-k} R_k, \quad k = 0, 1, \ldots, N-1.$$

It is easy to verify that

$$\sigma_0(R_k) = \alpha_N p_1 + \sum_{j=2}^{N} \alpha_{j-1} p_j.$$

Hence, the condition $\sigma_0(R_k) = \epsilon_0^{-k} R_k$ implies that

$$\alpha_{j-1} = \epsilon_0^{-k} \alpha_j, \quad j = 2, \ldots N-1, N, \qquad \alpha_N = \epsilon_0^{-k} \alpha_1.$$

This system admits the nontrivial solutions $\alpha_j = \epsilon_0^{j\,k} \alpha_N$. The functions R of form (10) satisfying (12) can therefore be written as

$$R = \sum_{k=0}^{N-1} z^k f_k(z^N) \sum_{j=1}^{N} \epsilon_0^{j\,k} p_j, \tag{13}$$

where $f_k \in \mathbb{C}((z^N))$, $k = 0, 1, \ldots, N-1$.

• Let

$$\sigma_0 = \begin{pmatrix} p_1 & \cdots & p_{N-2} & p_{N-1} & p_N \\ p_2 & \cdots & p_{N-1} & p_1 & p_N \end{pmatrix}.$$

Then the corresponding Newton exponent is $l_0 = N-1$ ($\epsilon_0 = e^{2\pi i/(N-1)}$). In this case, we have $\sigma_0(p_N) = p_N$ or, equivalently, $p_N \in \mathbb{C}((k))$. Proceeding as in the preceding case, we seek functions of the form

$$R_k = \sum_{j=1}^{N-1} \alpha_j p_j \quad \text{such that } \sigma_0(R_k) = \epsilon_0^{-k} R_k, \quad k = 0, 1, \ldots, N-2.$$

Because the action of σ_0 on the function R_k is given by

$$\sigma_0(R_k) = \alpha_{N-1} p_1 + \sum_{j=2}^{N-1} \alpha_{j-1} p_j,$$

the condition $\sigma_0(R_k) = \epsilon_0^{-k} R_k$ leads to

$$\alpha_{j-1} = \epsilon_0^{-k} \alpha_j, \quad j = 2, 3, \ldots, N-1, \qquad \alpha_{N-1} = \epsilon_0^{-k} \alpha_1.$$

Hence, $\alpha_j = \epsilon_0^{j\,k} \alpha_{N-1}$, and

$$R = \sum_{k=0}^{N-2} z^k f_k(z^{N-1}) \sum_{j=1}^{N-1} \epsilon_0^{j\,k} p_j + f_{N-1}(z^{N-1}) p_N. \tag{14}$$

Example 3.1. For $N = 3$,

$$R = f_0(z^2)(p_1 + p_2) + z f_1(z^2)(p_1 - p_2) + f_2(z^2) p_3.$$

• Let

$$\sigma_0 = \begin{pmatrix} p_1 & \dots & p_{N-2} & p_{N-1} & p_N \\ p_2 & \dots & p_1 & p_{N-1} & p_N \end{pmatrix}.$$

In this case, $l_0 = N-2$ ($\epsilon_0 = e^{2\pi i/(N-2)}$). We note that $p_{N-1}, p_N \in \mathbb{C}((k))$. We seek functions of the form

$$R_k = \sum_{j=1}^{N-2} \alpha_j p_j \quad \text{such that } \sigma_0(R_k) = \epsilon_0^{-k} R_k, \quad k = 0, 1, \dots, N-3.$$

We find that

$$\alpha_{j-1} = \epsilon_0^{-k} \alpha_j, \quad j = 2, 3, \dots, N-2, \qquad \alpha_{N-2} = \epsilon_0^{-k} \alpha_1.$$

Hence, $\alpha_j = \epsilon_0^{j\,k} \alpha_{N-2}$, and

$$R = \sum_{k=0}^{N-3} z^k f_k(z^{N-2}) \sum_{j=1}^{N-2} \epsilon_0^{j\,k} p_j + f_{N-2}(z^{N-2}) p_{N-1} + f_{N-1}(z^{N-2}) p_N. \tag{15}$$

Example 3.2. For $N = 4$,

$$R = f_0(z^2)(p_1 + p_2) + z f_1(z^2)(p_1 - p_2) + f_2(z^2) p_3 + f_3(z^2) p_4.$$

• Let

$$\sigma_0 = \begin{pmatrix} p_1 & \dots & p_{N-2} & p_{N-1} & p_N \\ p_2 & \dots & p_1 & p_N & p_{N-1} \end{pmatrix}.$$

As discussed in Sec. 3, the Newton exponent for this σ_0 depends on whether N is even or odd.

Let N be even. Then $l_0 = N-2$ ($\epsilon_0 = e^{2\pi i/(N-2)}$). It is easy to see that $p_{N-1} + p_N \in \mathbb{C}((k))$ and $\sigma_0(p_{N-1} - p_N) = -(p_{N-1} - p_N)$. On the other hand, because σ_0 acts on p_j, $j = 1, 2, \dots, N-2$, as in the preceding case and ϵ_0 coincides with the preceding value, we again have

$$R_k = \sum_{j=1}^{N-2} \epsilon_0^{j\,k} p_j, \quad k = 0, 1, \dots, N-3,$$

such that $\sigma_0(R_k) = \epsilon_0^{-k} R_k$. Hence, R is now given by

$$\begin{aligned} R = & \sum_{k=0}^{N-3} z^k f_k(z^{N-2}) \sum_{j=1}^{N-2} \epsilon_0^{j\,k} p_j + \\ & + z^{(N-2)/2} f_{N-2}(z^{N-2})(p_{N-1} - p_N) + f_{N-1}(z^{N-2})(p_{N-1} + p_N). \end{aligned} \tag{16}$$

Example 3.3. For $N = 4$,

$$R = f_0(z^2)(p_1 + p_2) + z f_1(z^2)(p_1 - p_2) + z f_2(z^2)(p_3 - p_4) + f_3(z^2)(p_3 + p_4).$$

Let N be odd. Then $l_0 = 2(N-2)$ ($\epsilon_0 = e^{\pi i/(N-2)}$). In this case, we again have $p_{N-1} + p_N \in \mathbb{C}((k))$ and $\sigma_0(p_{N-1} - p_N) = -(p_{N-1} - p_N)$. Moreover, if we seek functions of the form

$$R_k = \sum_{j=1}^{N-2} \alpha_j p_j \quad \text{such that } \sigma_0(R_k) = \epsilon_0^{-2k} R_k, \quad k = 0, \dots, N-3,$$

proceeding as in the preceding cases, then we find that $\alpha_j = \epsilon_0^{2j\,k} \alpha_{N-2}$ Hence,

$$R = \sum_{k=0}^{N-3} z^{2k} f_k(z^{2(N-2)}) \sum_{j=1}^{N-2} \epsilon_0^{2j\,k} p_j +$$

$$+ z^{N-2} f_{N-2}(z^{2(N-2)})(p_{N-1} - p_N) + f_{N-1}(z^{2(N-2)})(p_{N-1} + p_N). \tag{17}$$

Example 3.4. For $N = 5$,

$$R = f_0(z^6)(p_1 + p_2 + p_3) + z^2 f_1(z^6)(e^{2\pi i/3} p_1 + e^{4\pi i/3} p_2 + p_3) +$$

$$+ z^4 f_2(z^6)(e^{4\pi i/3} p_1 + e^{2\pi i/3} p_2 + p_3) + z^3 f_3(z^6)(p_4 - p_5) + f_4(z^6)(p_4 + p_5).$$

The integrable deformations are determined by the expressions for R in (13)–(17) depending on σ_0 and the Newton exponent l_0.

4. Some examples and associated conservation laws

In this section, we include some examples of integrable deformations of algebraic curves corresponding to the cases $N = 3$ and $N = 4$. First, we note that expanding the components of the vectors $\mathbf{p}$ and $\mathbf{Q}$ in series in z in our first equation for deformations (2) leads to an infinite set of conservation laws. This is the sense in which we say that the deformations are integrable. In our examples, we determine the first members of these systems of conservation laws.

Example 4.1. Let $N = 3$ and $(d_1, d_2, d_3) = (0, 0, 1)$. This choice of the degrees corresponds to the irreducible case and consequently to the Newton exponent $l_0 = 3$. From (4)–(6), we find the trivial equations $u_{10,t} = u_{31,t} = 0$. We hence take

$$u_1 = 1, \qquad u_2 = u_{20}, \qquad u_3 = k + u_{30},$$

and choosing

$$R = \frac{27}{4}(1 - i\sqrt{3}) z^5 \mathcal{L}_2,$$

we find that the deformation is given by the system of conservation laws

$$\begin{aligned} u_{20,t} &= \frac{5}{18} \partial_x (7u_{20} + 18u_{20}^2 + 9u_{20}^3 + 54u_{20}u_{30} + 12u_{30} + 81u_{30}^2), \\ u_{30,t} &= \frac{5}{18} \partial_x (2u_{20}^2 + 27u_{30}u_{20}^2 - u_{30} - 27u_{30}^2 + 6u_{20}^3). \end{aligned} \tag{18}$$

This system can also be obtained from (2). Indeed, the coefficients of z^{-1} and z^{-2} in (2) lead to (18). On the other hand, the coefficient of z^{-3} gives trivial equations, while the coefficient of z^{-4} leads to the conservation law

$$\partial_t (5u_{20} + 9u_{20}^2 + 9u_{30} + 27u_{20}u_{30}) =$$

$$= \partial_x \Big(\frac{175}{18} u_{20} + \frac{95}{2} u_{20}^2 + \frac{195}{2} u_{20}^3 + \frac{135}{2} u_{20}^4 + \frac{85}{6} u_{30} + \frac{255}{2} u_{20}u_{30} + \frac{675}{2} u_{20}^2 u_{30} +$$

$$+ \frac{675}{2} u_{20}^2 u_{30} + \frac{405}{2} u_{20}^3 u_{30} + 90u_{30}^2 + 405u_{20}u_{30}^2 + 405u_{30}^3 \Big).$$

The coefficients of z^{-j}, $j > 4$, lead to more involved conservation laws.

Example 4.2. Let $N = 3$ and $(d_1, d_2, d_3) = (1, 0, 0)$. Then the Newton exponent is $l_0 = 2$, and (4)–(6) imply that $(u_{30}/u_{11})_t = 0$. Setting

$$u_1 = u_{11}k + u_{10}, \qquad u_2 = u_{20}, \qquad u_3 = u_{11}$$

and choosing R as

$$R = z^4(p_1 + p_2),$$

we obtain

$$\begin{gathered} u_{11,t} = \partial_x\left(-\frac{2u_{10}}{u_{11}}\right), \qquad u_{10,t} = \partial_x\left(-\frac{u_{10}^2 + 2u_{20}}{u_{11}^2}\right), \\ u_{20,t} = \frac{1}{u_{11}^3}[2u_{10}u_{20}u_{11,x} + u_{11}(-2u_{20}u_{10,x} + 4u_{11,x})], \end{gathered} \tag{19}$$

where only the two first equations are conservation laws. New conservation laws can be obtained from (2). For example, from the coefficients of z^{-2}, z^{-3}, and z^{-4}, we obtain

$$\begin{gathered} \partial_t\left(\frac{u_{20}}{u_{11}}\right) = \partial_x\left(-\frac{2}{u_{11}^2}\right), \qquad \partial_t\left(\frac{4u_{10}}{u_{11}} + \frac{u_{20}^2}{u_{11}^2}\right) = \partial_x\left(-\frac{8u_{20}}{u_{11}^3}\right), \\ \partial_t\left(\frac{1}{u_{11}} - \frac{u_{10}u_{20}}{u_{11}^2}\right) = \partial_x\left(\frac{2u_{10}}{u_{11}^3} + \frac{u_{20}^2}{u_{11}^4}\right). \end{gathered}$$

Example 4.3. Let $N = 4$ and $(d_1, d_2, d_3, d_4) = (0, 1, 0, 0)$. For this choice of the degrees, the Newton exponent is $l_0 = 2$, and $(u_{40}/u_{21})_t = 0$. We then set

$$u_1 = u_{10}, \qquad u_2 = u_{21}k + u_{20}, \qquad u_3 = u_{30}, \qquad u_4 = u_{21}.$$

Labeling the branches of $\mathcal{C}$ such that $p_{1,2} \sim z + O(1)$ and $p_{3,4} \sim z^{-1} + O(z^{-2})$ and taking

$$R = z^4(p_3 + p_4),$$

we obtain

$$\begin{aligned} u_{10,t} &= \partial_x\left(\frac{2u_{10}^2u_{20} + 2u_{20}^2 - 4u_{21} + 2u_{10}u_{30}}{u_{21}^2}\right), \\ u_{21,t} &= 4u_{21}^{1/2}\partial_x\left(\frac{2u_{10}u_{20} - 2u_{30}}{u_{21}^{3/2}}\right), \\ u_{20,t} &= u_{21}^{-2}[(4u_{20}^2 - 2u_{10}u_{30} + 4u_{21})u_{10,x} + (4u_{10}u_{20} + 2u_{30})u_{20,x} - 2u_{10}^2u_{30,x}] + \\ &\quad + u_{21}^{-3}(6u_{10}u_{21} - 6u_{10}u_{20}^2 + 4u_{10}^2u_{30} - 2u_{20}u_{30} - 14u_{10}u_{21})u_{21,x}, \\ u_{30,t} &= u_{21}^{-2}[(4u_{20}u_{30} - 4u_{10}u_{21})u_{10,x} + (4u_{10}u_{30} + 4u_{21})u_{20,x} - 4u_{30}u_{30,x}] + \\ &\quad + u_{21}^{-3}[(4u_{20} - 2u_{10}^2 - 12u_{20})u_{21} - 6u_{10}u_{20}u_{30} + 6u_{30}^2 + 6u_{10}^2u_{21}]u_{21,x}. \end{aligned} \tag{20}$$

Here, the first equation is given in conserved form, and the second can be transformed into a conservation law for $u_{21}^{1/2}$. New conservation laws associated with (20) can be derived from (2). From the coefficients of z^{-1}, we obtain

$$\partial_t\left(\frac{4u_{10}^2 + 4u_{20}}{u_{21}^{1/2}}\right) = \partial_x\left(\frac{2u_{10}^3u_{20} + 8u_{10}u_{20}^2 + 16u_{10}u_{21} - 2u_{10}^2u_{30} + 8u_{20}u_{30}}{u_{21}^{5/2}}\right),$$

and from the coefficients of z^{-j}, $j = 2, 3, 4$, of the last two components of (2), we obtain the corresponding conservation laws

$$\partial_t\left(\frac{u_{30}}{u_{21}}\right) = \partial_x\left(\frac{4u_{20} - 2u_{10}^2}{u_{21}^2}\right),$$

$$\partial_t\left(\frac{4u_{20}u_{21} + u_{30}^2}{u_{21}^2}\right) = \partial_x\left(\frac{16u_{10}u_{21} - 8u_{10}^2u_{30} + 16u_{20}u_{30}}{u_{21}^3}\right),$$

$$\partial_t\left(\frac{u_{10}u_{21} + u_{20}u_{30}}{u_{21}^2}\right) = \partial_x\left(-\frac{2}{u_{21}^2} + \frac{4u_{20}^2 - 2u_{10}^2u_{20} + 6u_{10}u_{30}}{u_{21}^3} + \frac{(2u_{20} - u_{10}^2)u_{30}^2}{u_{21}^4}\right).$$

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