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Note

New results on the Zarankiewicz problem

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Abstract

Let (X, Y) denote a bipartite graph with classes X and Y such that $|X| = m$ and $|Y| = n$. A complete bipartite subgraph with s vertices in X and t vertices in Y is denoted by $K_{(s,t)}$. The Zarankiewicz problem consists in finding the maximum number of edges, denoted by $z(m, n; s, t)$, of a bipartite graph (X, Y) with $|X| = m$ and $|Y| = n$ without a complete bipartite $K_{(s,t)}$ as a subgraph. First, we prove that $z(m, n; s, t) = mn - (m + n - s - t + 1)$ if $\max\{m, n\} \leqslant s + t - 1$. Then we characterize the family $Z(m, n; s, t)$ of extremal graphs for the values of parameters described above. Finally, we study the $s = t$ case. We give the exact value of $z(m, n; t, t)$ if $2t \leqslant n \leqslant 3t - 1$ and we characterize the extremal graphs if either $n = 2t$ or both $2t < n \leqslant 3t - 1$ and $m \leqslant \lfloor (3t - 1)/2 \rfloor$.

Keywords: Zarankiewicz problem; Extremal graph theory; Forbidden subgraphs

1. Introduction

Throughout this paper only undirected simple graphs without loops or multiple edges are considered. Unless stated otherwise, we follow the book by Bollobás [1] for terminology and definitions.

Let (X, Y) denote a bipartite graph with classes X and Y such that $|X| = m$ and $|Y| = n$. A complete bipartite subgraph with s vertices in X and t vertices in Y is denoted by $K_{(s,t)}$. Given any integers m, n, s, t, what is the maximum size of a bipartite graph (X, Y) if it does not contain a $K_{(s,t)}$? Denote this maximum by $z(m, n; s, t)$ and let $Z(m, n; s, t)$ stand for the family of extremal bipartite graphs (X, Y) free of $K_{(s,t)}$ whose size is $z(m, n; s, t)$; when $m = n$ and $s = t$, simply put $z(n; t) = z(n, n; t, t)$ and $Z(n; t) = Z(n, n; t, t)$. To avoid the trivial cases we shall suppose that $2 \leqslant s \leqslant m$, $2 \leqslant t \leqslant n$. In 1951 Zarankiewicz [9] posed the problem of determining $z(n; 3)$ for $n = 4, 5, 6$ and the general problem has also become known as *the problem of Zarankiewicz*. It is worth noting that a related problem to this is to find the maximum number of edges of a bipartite graph (X, Y) without $K_{s,t}$ as a subgraph (i.e., (X, Y) is free of both $K_{(s,t)}$ and $K_{(t,s)}$). This number is denoted by $\mathrm{ex}(m, n; K_{s,t})$ and we use $\mathrm{EX}(m, n; K_{s,t})$ to denote the corresponding family of extremal graphs with $\mathrm{ex}(m, n; K_{s,t})$ edges. Clearly,

$$\mathrm{ex}(m, n; K_{s,t}) \leqslant z(m, n; s, t).$$

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A survey of work on Zarankiewicz problem appears in Section VI.2 of [1]. Some of the more recent work includes the papers [3–8]. In [3,5,6], bounds for the function $z(m, n; s, t)$ are given in an asymptotical way, i.e., when s, t are fixed and m, n are much larger than s, t. In this work we are concerned with the exact values of the function $z(m, n; s, t)$. In this regard Culik [2] has shown that if s, t, m are fixed then

$$z(m, n; s, t) = (s-1)n + (t-1)\binom{m}{s} \quad \text{for all } n > (t-1)\binom{m}{s}. \tag{1}$$

It is known [1] that $z(n; 2) \leqslant (n + n\sqrt{4n-3})/2$ and equality holds when $n = q^2 + q + 1$ for a prime power q. Goddard et al. [4] found the exact values of $z(n; 2)$ for $n \leqslant 20$ and showed that in some cases the family $Z(n; 2)$ is formed by only one extremal graph.

The problem of Zarankiewicz can be formulated using matrix terminology. More precisely, $z(m, n; s, t)$ might be defined as the maximum number of ones in an $m \times n$ $\{0, 1\}$-matrix free of $s \times t$ *submatrix* of ones (where a submatrix is the intersection of s rows and t columns). Griggs et al. [7,8] studied the so-called "half–half" case, or in other words the function $z(2s, 2t; s, t)$. Thus, supposing $s \leqslant t$, the following bounds are stated in [8]:

$$4st - (2t + 2s - \gcd(s, t) + 1) \leqslant z(2s, 2t; s, t) \leqslant 4st - (2t + s + 1),$$

where $\gcd(s, t)$ being the greatest common divisor of s and t. Furthermore, it is proved that if $t = ks + r$, $0 \leqslant r < s$, such that either $r = 0$ or $r > 0$ and $s \leqslant k + r$ (which is always true for $t > (s-1)^2$) then

$$z(2s, 2t; s, t) = 4st - (2t + s + 1). \tag{2}$$

Later [7] deals with the case $t = ks + 1$, evaluating $z(2s, 2t; s, t)$ for large enough s and t, and providing the exact values for $t \leqslant 20$ and $s \leqslant 7$.

In this work we determine the exact value of $z(m, n; s, t)$ and we characterize the family $Z(m, n; s, t)$ of extremal graphs if $\max\{m, n\} \leqslant s + t - 1$. Besides, we study the $s = t$ case providing the exact value of the function $z(m, n; t, t)$ if $2t \leqslant n \leqslant 3t - 1$. Finally, we characterize the family $Z(m, n; t, t)$ of extremal graphs if either $n = 2t$ or both $2t < n \leqslant 3t - 1$ and $m \leqslant \lfloor (3t-1)/2 \rfloor$.

Let us define an *X-matching* in a complete bipartite graph $K_{(m,n)} = (X, Y)$ as the edges of a set of pairwise disjoint $K_{(j,1)}$ with $j \geqslant 1$; that is, such that all the vertices of degree greater than 1 (if any) belong to the Y-class of $K_{(m,n)}$. A *Y-matching* in $K_{(m,n)}$ is defined analogously. Observe that a matching in $K_{(m,n)}$ is always both an X-matching and a Y-matching.

If $s = m$ by (1) it follows that for every $n \geqslant t$, $z(m, n; m, t) = mn - (n - t + 1)$. Analogously if $t = n$ by (1) it follows that for every $m \geqslant s$, $z(m, n; s, n) = mn - (m - s + 1)$. Our first result concerns with the extremal family for these cases.

Theorem 1.1. *Let m, n, t be integers with $m > 2$, $2 \leqslant t < n$. Let us denote by X the m-vertices class and by Y the n-vertices class of $K_{(m,n)}$. Then*

$$Z(m, n; m, t) = \{K_{(m,n)} - M\},$$

where M is any Y-matching in $K_{(m,n)}$ with cardinality $n - t + 1$.

It is worth noting that for all values of the parameters m, n, s, t we have $z(m, n; s, t) = z(n, m; t, s)$. Moreover, the extremal graphs of the family $Z(n, m; t, s)$ are obtained by swapping the vertices classes of each one of the graphs in $Z(m, n; s, t)$. Therefore, Theorem 1.1 also provides the extremal family of graphs $Z(m, n; s, n)$ in terms of the X-matchings of cardinality $m - s + 1$ of the complete bipartite graph $K_{(m,n)}$.

Next, we present a theorem which extends (1) and allows us to obtain the exact value of the Zarankiewicz number $z(m, n; s, t)$ whenever the parameters m, n, s, t are related by $\max\{m, n\} \leqslant s + t - 1$.

Theorem 1.2. *Let m, n, s, t be integers with $2 \leqslant s < m$, $2 \leqslant t < n$ and such that $\max\{m, n\} \leqslant s + t - 1$. Then*

$$\begin{cases} z(m, n; s, t) = \mathrm{ex}(m, n; K_{s,t}) = mn - (m + n - s - t + 1), \\ Z(m, n; s, t) = \mathrm{EX}(m, n; K_{s,t}) = \{K_{(m,n)} - M\}, \end{cases}$$

where M is any matching of cardinality $m + n - s - t + 1$.

Note that both Theorem 1.2 and (1) give the same value when $t=2$ and $n=m=s+1$. Next, we present a theorem which can be compared with the "half–half" case (2) in the sense that we find the exact value for the function $z(m, 2t; t, t)$ for every $m \leqslant 2t$.

Theorem 1.3. *Let m, t be integers such that $2 \leqslant t \leqslant m \leqslant 2t$. Then*

$$z(m, 2t; t, t) = m \cdot 2t - (2m - t + 1).$$

Moreover,

$$Z(m, 2t; t, t) = \{K_{(m,2t)} - (M_1 \cup M_2)\},$$

where M_1 is a matching on $t - 1 - 2r$ edges with $r \in \{0, \dots, \lfloor (t-1)/2 \rfloor\}$, and M_2 is a subset of edges of $K_{(m,2t)}$ for which one of the following assertions hold:

(i) *$m = 2t$, $r = 0$ and M_2 induces a cycle of length $2(t + 1)$;*
(ii) *$m < 2t$, $r = 0$ and M_2 induces a forest with $m - (t - 1)$ vertices of degree 2 in the m-class;*
(iii) *$m < 2t$, $r \neq 0$ and M_2 consists of the edges of $m - (t - 1) + r$ disjoint copies of $K_{(1,2)}$,*

and such that the induced subgraphs by M_1 and M_2 are disjoint.

Observe that when $m=2t$ item (i) of the theorem gives the same result as (2). Hence Theorem 1.3(i) is an improvement of the "half–half" case when $s = t$.

Finally, we partially extend the latter result for greater values of the parameter n.

Theorem 1.4. *Let m, n, t be integers such that $2 \leqslant t \leqslant m \leqslant n$ and $2t < n \leqslant 3t - 1$. Then*

$$z(m, n; t, t) = mn - (2m + n - 3t + 1).$$

Moreover, if $m \leqslant \lfloor (3t - 1)/2 \rfloor$ then

$$Z(m, n; t, t) = \{K_{(m,2t)} - (M_1 \cup M_2)\},$$

where M_1 is a matching on $n - t + 1 - 2r$ edges with $r \in \{2, \dots, \lfloor (n - t + 1)/2 \rfloor\}$, and M_2 consists of the edges of $m - t + r$ disjoint copies of $K_{(1,2)}$ such that the induced subgraphs by M_1 and M_2 are disjoint.

We must bear in mind the fact that if $s = t$ then both extremal problems, $z(m, n; t, t)$ and $\mathrm{ex}(m, n; K_{t,t})$, become equivalent. Therefore, Theorems 1.3 and 1.4 not only give us the solution for the Zarankiewicz problem but also for the problem of bipartite graphs free of $K_{t,t}$.

2. Proofs

The degree of a vertex w in a graph G is denoted by $d_G(w) = d(w)$, $N_G(w)$ is the set of vertices that are adjacent to w in G, and $N_G(T) = \bigcup_{w \in T} N_G(w)$ denotes the neighborhood of a subset of vertices T. We will also use $e(G)$ to denote the number of edges of G, and $G[V']$ stands for the induced subgraph in G by the set of vertices $V' \subseteq V(G)$.

Proof of Theorem 1.1. Let $G = (X, Y)$ be a bipartite graph with m-class X and n-class Y. We claim that G does not contain a bipartite $K_{(m,t)}$ as a subgraph if and only if there exists a set $U \subset Y$ of $n - t + 1$ vertices which are not adjacent to some vertex of X. Indeed, every t-subset $T \subset Y$ contains at least one vertex of U, hence $G[X \cup T] \neq K_{(m,t)}$; furthermore, if no such U exists, at most $n - t$ vertices of Y are not adjacent to some vertex of X, hence we can find a t-subset of Y whose vertices are adjacent to all the vertices of X, so $K_{(m,t)} \subset G$. As a consequence, $z(m, n; m, t) = e(K_{(m,n)}) - (n - t + 1) = mn - (n - t + 1)$, and every extremal graph in $Z(m, n; m, t)$ must be obtained after deleting $n - t + 1$ edges (incident with $n - t + 1$ different vertices of Y) from a complete bipartite $K_{(m,n)}$, so the result follows. □

We focus our attention on bipartite graphs (X, Y) with no subgraph $K_{(s,t)}$ such that both $s < m$ and $t < n$, as the cases $s = m$ and $t = n$ have been treated in Theorem 1.1. To do that, next we prove the following lemma.

Lemma 2.1. *Let m, n, s, t be integers such that $2 \leqslant s < m$ and $2 \leqslant t < n$. Let $G = (X, Y)$ be a bipartite graph without $K_{(s,t)}$ such that $|X| = m$ and $|Y| = n$. Then the following assertions hold*:

(i) $e(G) \leqslant mn - (m + n - s - t + 1)$.
(ii) *If $e(G) = mn - (m + n - s - t + 1)$ then* $\max\{m, n\} \leqslant s + t - 1$ *and the m-class X of G consists of vertices of degree at least $n - 1$ and the n-class Y of G consists of vertices of degree at least $m - 1$.*

Proof. Let us order the vertices of X in such a way that $d(x_i) \geqslant d(x_{i+1})$, for $i = 1, \ldots, m - 1$. Let us also denote by $S = \{x_1, \ldots, x_s\}$. Then $d(x_s) \leqslant n - 1$ because otherwise $d(x_i) = n$, for all $i = 1, \ldots, s$ and therefore, $G[S \cup Y] = K_{(s,n)} \supseteq K_{(s,t)}$, against our assumptions. Thus, we have $d(x_s) \leqslant n - 1$, which implies that $d(x_i) \leqslant n - 1$, for all $i = s, \ldots, m$. Since G does not contain $K_{(s,t)}$ as a subgraph, by (1) it follows that $e(G[S \cup Y]) \leqslant sn - (n - t + 1)$. So,

$$\begin{aligned} e(G) &= e(G[S \cup Y]) + e(G[(X \backslash S) \cup Y]) \\ &\leqslant sn - (n - t + 1) + \sum_{i=s+1}^{m} d(x_i) \\ &\leqslant sn - (n - t + 1) + (m - s)d(x_{s+1}) \\ &\leqslant sn - (n - t + 1) + (m - s)(n - 1) \\ &= mn - (m + n - s - t + 1), \end{aligned} \tag{3}$$

and assertion (i) of the lemma is true.

Now suppose that $e(G)$ attains its maximum value. Then all the above inequalities of (3) become equalities. So, we deduce that $d(x_i) = n - 1$ for all $i = s, \ldots, m$ and $n - 1 \leqslant d(x_i) \leqslant n$ for all $i = 1, \ldots, s - 1$. Moreover, $G[S \cup Y]$ is the graph resulting from the complete bipartite $K_{(s,n)}$ after deleting exactly $n - t + 1$ edges, no two of them being incident with the same vertex of X because of the degrees of $x_1, \ldots, x_m$. Therefore $n - t + 1 \leqslant s$. The argument for the class Y is similar. This proves item (ii). □

Recall that the cardinality of a maximum matching in a complete bipartite $K_{(m,n)}$ is $\min\{m, n\}$. Next, we present a result concerning the structure of the remaining graph after deleting a matching from $K_{(m,n)}$.

Lemma 2.2. *Let m, n, s, t be integers such that $2 \leqslant s \leqslant m$, $2 \leqslant t \leqslant n$ and* $\max\{m, n\} \leqslant s + t - 1$. *Let M be a set of edges of $K_{(m,n)}$ of cardinality $m + n - s - t + 1$. If M is a matching then the bipartite graph $K_{(m,n)} - M$ has neither $K_{(s,t)}$ nor $K_{(t,s)}$ as subgraphs.*

Proof. Let $K_{(m,n)} = (X, Y)$ and let S be any subset of X of cardinality s and T any subset of Y of cardinality t. Since M is a matching then the number of edges of M which are incident with some vertex in $(X \backslash S) \cup (Y \backslash T)$ is at most $|X \backslash S| + |Y \backslash T| = (m - s) + (n - t) = |M| - 1$. It follows that there exists some edge $ab \in M$ with $a \in S$ and $b \in T$. Hence $K_{(s,t)}$ cannot be a subgraph of $K_{(m,n)} - M$. Similarly, it is proved that $K_{(m,n)} - M$ does not contain $K_{(t,s)}$ as a subgraph. □

The previous result provides a lower bound for the extremal function $\mathrm{ex}(m, n; K_{s,t})$, which allows us to prove Theorem 1.2.

Proof of Theorem 1.2. The existence in $K_{(m,n)}$ of some matching M with cardinality $m+n-s-t+1$ follows from the condition $\max\{m, n\} \leqslant s + t - 1$. Hence, for any such matching M, from Lemma 2.2 it follows that $K_{(m,n)} - M$ does not contain neither $K_{(s,t)}$ nor $K_{(t,s)}$ as subgraphs, so $\mathrm{ex}(m, n; K_{s,t}) \geqslant mn - (m+n-s-t+1)$. Furthermore, by Lemma 2.1, $z(m, n; s, t) \leqslant mn-(m+n-s-t+1)$. Therefore we conclude that $\mathrm{ex}(m, n; K_{s,t}) = z(m, n; s, t) = mn-(m+n-s-t+1)$, and also that $K_{(m,n)} - M \in \mathrm{EX}(m, n; K_{s,t}) \subset Z(m, n; s, t)$ for every matching M in $K_{(m,n)}$ with $|M| = m+n-s-t+1$.

Next, we show that there are no other extremal graphs in $Z(m, n; s, t)$. Let G be a graph of the family $Z(m, n; s, t)$ with classes X and Y such that $|X| = m$ and $|Y| = n$. Clearly $e(G) = mn - (m + n - s - t + 1)$. Lemma 2.1 allows us to guarantee that $\min\{d_G(x) : x \in X\} = n - 1$ and $\min\{d_G(y) : y \in Y\} = m - 1$, hence there must exist a matching M in $K_{(m,n)}$ of cardinality $m + n - s - t + 1$ such that $G = K_{(m,n)} - M$. So the proof is complete. □

Next, we concentrate upon the $s = t$ case for which we are going to prove similar results to the ones obtained for the general case.

Lemma 2.3. *Let m, n, t be integers such that $2 \leqslant t \leqslant m$ and $2t \leqslant n \leqslant 3t - 1$. Let $G = (X, Y)$ be a bipartite graph without $K_{(t,t)}$ such that $|X| = m$ and $|Y| = n$. Then the following assertions hold*:

(i) $e(G) \leqslant mn - (2m + n - 3t + 1)$.
(ii) *If $e(G) = mn - (2m + n - 3t + 1)$ then the m-class X of G consists of vertices of degree at least $n - 2$.*

Proof. Let us order the vertices of X as in Lemma 2.1. Let us also denote by $T = \{x_1, \ldots, x_t\}$. First, since G does not contain $K_{(t,t)}$ as a subgraph, by (1) it follows that $e(G[T \cup Y]) \leqslant tn - (n - t + 1)$. Then $d(x_t) \leqslant n - 2$ because otherwise $d(x_i) \geqslant n - 1$, for all $i = 1, \ldots, t$ and therefore, $e(G[T \cup Y]) \geqslant tn - t > tn - (n - t + 1) \geqslant e(G[T \cup Y])$, against our assumptions. Thus, we can assume that $d(x_t) \leqslant n - 2$, which implies that $d(x_i) \leqslant n - 2$, for all $i = t, \ldots, m$. So,

$$\begin{aligned} e(G) &= e(G[T \cup Y]) + e(G[(X \backslash T) \cup Y]) \\ &\leqslant tn - (n - t + 1) + (m - t)(n - 2) \\ &= mn - (2m + n - 3t + 1), \end{aligned} \tag{4}$$

and assertion (i) of the lemma is proved.

Next, if $e(G) = mn - (2m + n - 3t + 1)$ then all the above inequalities of (4) become equalities. Therefore, $d(x_i) = n - 2$ for all $i = t, \ldots, m$ and $n - 2 \leqslant d(x_i) \leqslant n$ for all $i = 1, \ldots, t - 1$, which prove item (ii). Furthermore, notice that $e(G[T \cup Y]) = tn - (n - t + 1)$, and the edge set of $K_{(t,n)} - G[T \cup Y]$ is a Y-matching because $K_{(t,t)} \nsubseteq G$. □

Let us denote by G^c the bipartite complement of a bipartite graph $G = (X, Y)$, that is $G^c = K_{(m,n)} - E(G)$, where $E(G)$ denotes the edge set of G.

Lemma 2.4. *Let m, n, t be integers such that $2 \leqslant t \leqslant m \leqslant n$. The bipartite graph $K_{(m,n)} - (M_1 \cup M_2)$, M_1 and M_2 inducing disjoint subgraphs of $K_{(m,n)}$, does not contain $K_{(t,t)}$ as a subgraph if one of the following conditions hold*:

(i) *M_1 is a matching of cardinality $t - 1$ and M_2 induces a path of length $2(m - t + 1)$ with end vertices in Y, provided that $m < n = 2t$.*
(ii) *M_1 is a matching of cardinality $t - 1$ and M_2 induces a cycle of length $2(t + 1)$, provided that $m = n = 2t$.*
(iii) *M_1 is a matching on $n - t + 1 - 2r$ edges with $r \in \{2, \ldots, \lfloor (n - t + 1)/2 \rfloor\}$, and M_2 induces $m - t + r$ disjoint copies of $K_{(1,2)}$, provided that $m \leqslant \lfloor (3t - 1)/2 \rfloor$ and $2t < n \leqslant 3t - 1$.*

Proof. First, suppose that $m < n = 2t$ and let us consider the graph G described in item (i). The existence of such sets of edges M_1 and M_2 follows from $(t - 1) + (m - t + 2) = m + 1 \leqslant n$. Let us denote by T any subset of t vertices of the m-class X. The set T is formed by j vertices incident with j edges of M_1 and $t - j$ vertices incident with $2(t - j)$ edges of M_2, where $j \in \{0, \ldots, t\}$. These j vertices are adjacent in G^c to j vertices in Y because the set of edges of M_1 is a matching. The remaining $t - j$ vertices are adjacent to at least $t - j + 1$ vertices in Y because M_2 induces a path. Then the number of vertices of the n-class Y that are adjacent in G^c to some vertex in T is at least $j + t - j + 1 = t + 1$. That is to say, because $|Y| = 2t$, the number of vertices of Y that are adjacent in G to all vertices in T is at most $2t - (t + 1) = t - 1$ and therefore $K_{(t,t)} \nsubseteq G$. This proves item (i). The proof for items (ii) and (iii) is analogous. □

Note that the size of the graphs described in the above lemma is $mn - (2m + n - 3t + 1)$.

Proof of Theorem 1.3. By item (i) of Lemma 2.3 we have $z(m, 2t; t, t) \leqslant mn - (2m - t + 1)$. The other inequality comes from items (i) and (ii) of Lemma 2.4. This proves the exact value of the extremal function $z(m, 2t; t, t) = mn - (2m - t + 1)$. Let $G = (X, Y)$ be a graph of the family $Z(m, 2t; t, t)$. From now on we consider the bipartite complement G^c of the graph G. Let us assume that the vertices of G are ordered as in Lemma 2.1, so $d(x_1) \leqslant \cdots \leqslant d(x_m)$ in G^c. By the proof of Lemma 2.3 we know that $d(x_t) = \cdots = d(x_m) = 2$ and $e(G^c[T \cup Y]) = t + 1$, where T denotes the set of vertices $\{x_1, \ldots, x_t\}$. First, suppose that $d(x_{t-1}) = 1$. Then $d(x_i) = 1$ for all $i = 1, \ldots, t-1$ because $e(G^c[T \cup Y]) = t+1$, $d(x_t) = 2$ and $d(x_i) \leqslant d(x_{i+1})$ in G^c. Moreover, for all $j = t, \ldots, m$ and all $i = 1, \ldots, t-1$ we have $N_{G^c}(x_j) \cap N_{G^c}(x_i) = \emptyset$. Otherwise, we can consider the set $T' = \{x_1, \ldots, x_{t-1}, x_j\}$ which satisfies $|N_{G^c}(T')| = t$, following $K_{(t,t)} \subseteq G$ because $|Y| = 2t$, against our assumptions. Then the edge set $M_1 = E(G^c[\{x_1, \ldots, x_{t-1}\} \cup Y])$ is a matching on $t - 1$ edges. Furthermore, if the edge set $M_2 = E(G^c[\{x_t, \ldots, x_m\} \cup Y])$ induces a cycle of length $2l$, with $2 \leqslant l \leqslant t$, then we can consider the t-subset T' of X formed by the l vertices of X on the cycle and $t - l$ vertices incident with the edges of M_1. Clearly, $|N_{G^c}(T')| = t$ which again leads us to an absurdity. Therefore the edge set of G^c is $M_1 \cup M_2$, where M_1 and M_2 induce disjoint subgraphs of $K_{(m,2t)}$. More precisely, M_1 is a matching of cardinality $t - 1$ and either M_2 induces a cycle of length $2(t + 1)$ (which implies $m = 2t$) or M_2 induces an acyclic subgraph. Note that in this latter case M_2 induces a forest such that all the vertices in the X class have degree 2. Second, suppose $d(x_{t-1}) = 2$. Then it is easy to check, reasoning as above, that M_1 is a Y-matching on $t - 1$ edges formed by the edges of disjoint copies of $K_{(1,1)}$ and $K_{(1,2)}$. Since $d(x_t) = d(x_{t-1}) = 2$ and $e(G^c[T \cup Y]) = t + 1$ hence $d(x_1) = 0$. If there exists $\tilde{x}_1, \tilde{x}_2 \in \{x_t, \ldots, x_m\}$ such that $N_{G^c}(\tilde{x}_1) \cap N_{G^c}(\tilde{x}_2) \neq \emptyset$ then we can consider the subset $T' = \{x_1, x_2, \ldots, x_{t-2}, \tilde{x}_1, \tilde{x}_2\}$ of X, and clearly $|N_{G^c}(T')| \leqslant t$ which is impossible. Therefore M_2 consists of the edges of $m - (t - 1)$ disjoint copies of $K_{(1,2)}$. This proves the theorem. □

Proof of Theorem 1.4. By item (i) of Lemma 2.3 we have $z(m, n; t, t) \leqslant mn - (2m + n - 3t + 1)$. The other inequality comes from item (iii) of Lemma 2.4. Then $z(m, n; t, t) = mn - (2m + n - 3t + 1)$. Let G be an extremal graph of the family $Z(m, n; t, t)$ and let us follow the same notation and assumptions as in the proof of Theorem 1.3. Notice that $e(G^c) = 2m + n - 3t + 1 \leqslant n$ because $m \leqslant \lfloor (3t - 1)/2 \rfloor$, hence $E(G^c)$ may be a Y-matching. By the proof of Lemma 2.3 we know that $d(x_t) = \cdots = d(x_m) = 2$ in G^c, and $E(G^c[T \cup Y])$ is a Y-matching with $n - t + 1$ edges. Since $d(x_i) \leqslant d(x_{i+1})$ in G^c, then $G^c[T \cup Y]$ consists of $n - t + 1 - 2r$ disjoint copies of $K_{(1,1)}$ and r disjoint copies of $K_{(1,2)}$. Since $n \geqslant 2t + 1$ hence $e(G^c[T \cup Y]) = n - t + 1 \geqslant t + 2$, that is to say, there exists at least two vertices in T of degree 2 and therefore $2 \leqslant r \leqslant \lfloor (n - t + 1)/2 \rfloor$. Reasoning as in the proof of Theorem 1.3 we can check that the edge set of the subgraph $G^c[(X \setminus T) \cup Y]$ is a Y-matching with $m - t$ vertices of degree 2 in X. Consequently, $G = K_{(m,n)} - (M_1 \cup M_2)$ where $M_1 = E(G^c[\{x_1, \ldots, x_{t-r}\} \cup Y])$ is a matching on $n - t + 1 - 2r$ edges, and $M_2 = E(G^c[\{x_{t-r+1}, \ldots, x_m\} \cup Y])$ consists of the edges of $m - t + r$ disjoint copies of $K_{(1,2)}$. □

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