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Numerical solving of equations in the work of José Mariano Vallejo

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Abstract The progress of Mathematics during the nineteenth century was characterised both by an enormous acquisition of new knowledge and by the attempts to introduce rigour in reasoning patterns and mathematical writing. Cauchy's presentation of Mathematical Analysis was not immediately accepted, and many writers, though aware of that new style, did not use it in their own mathematical production. This paper is devoted to an episode of this sort that took place in Spain during the first half of the century: It deals with the presentation of a method for numerically solving algebraic equations by José Mariano Vallejo, a late Spanish follower of the Enlightenment ideas, politician, writer, and mathematician who published it in the fourth (1840) edition of his book *Compendio de Matemáticas Puras y Mistas*, claiming to have discovered it on his own. Vallejo's main achievement was to write down the whole procedure in a very careful way taking into account the different types of roots, although he paid little attention to questions such as convergence checks and the fulfilment of the hypotheses of Rolle's Theorem. For sure this lack of mathematical care prevented Vallejo to occupy a place among the forerunners of Computational Algebra.

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1 A biographical sketch

Inspired by Enlightenment ideas, José Mariano Vallejo Ortega (Vallejo from now on) was a prominent character of the Spanish social scene in the first half of the nineteenth century. He was a civil servant, a politician, a prolific writer on a wide range of topics, from Agriculture to Grammar and Railways, and a mathematician. Vallejo was born in 1779 at Albuñuelas, a small village in the province of Granada, Andalusia. It is known [9, p. 105] [10, p. 382] that he spent some time as a college student in Granada, and in 1801, [10, p. 383] he was an appointed Lecturer of Mathematics at the *Real Academia de San Fernando*[1] in Madrid. In 1802, although he had not yet finished his studies, Vallejo won a public contest for the Chair of *Matemáticas, Ataque, Fortificación y Defensa de Plazas* at the *Real Seminario de Nobles in Madrid*. According to [11, pp. 32-34], he was a brilliant, promising student, but the complex political situation in Spain kept him from becoming a first rate mathematician.

Vallejo served as a representative of Granada at the *Cortes de Cádiz* in 1812, and he kept a good relationship with politics during the first half of Fernando VII's regime, from 1814 to 1820. He was socially very active and held positions such as Librarian of the *Sociedad Económica de Madrid*, Chief Accountant in the *Colegio de Sordomudos*, Director of the *Gabinete Geográfico*, and some more. He was an active liberal and during the *trienio liberal* he served as a member of the *Dirección General de Estudios* at the *Ministerio del Interior*. The subsequent ten years or *década ominosa* (1823–1833), when the Spanish political situation reverted to royal absolutism, he had to flee from the royal persecution and went into exile. Vallejo spent those years travelling through France, Belgium, Britain, and the Netherlands, and he took profit of his exile by visiting most of the leading European scientists of his time. We read in the foreword to his *Tratado* his acknowledgement and gratitude towards a number of cultivated people (all quotations from Vallejo have been translated into English by the authors):

> In finishing this foreword, I cannot but express my gratitude to several wise men, who have favoured me with their attention during my travels around Europe. They are, in alphabetical order in Paris: MM. Ampère, Arago, Beudant, Biot, Brochant, Brogniart, Cauchy, Chaptal, Dégérando, Elie de Beaumont, Fourier, Francoeur, Gay-Lussac, Girard, Hachette, Jornard, Lacroix, Laplace, Lasteyrie, Legendre, Navier, Poisson, Prony, Puissant, and Tenard (…) in Brussels: Mr. Quételet and Mr. Lejoinne [24, Vol. II, p. XV].[2]

The European scientific community accepted Vallejo as a relevant member, as shown by Valson's biography of Cauchy, where Vallejo appears among the attendants to Cauchy's lectures which included many other well known scientists:

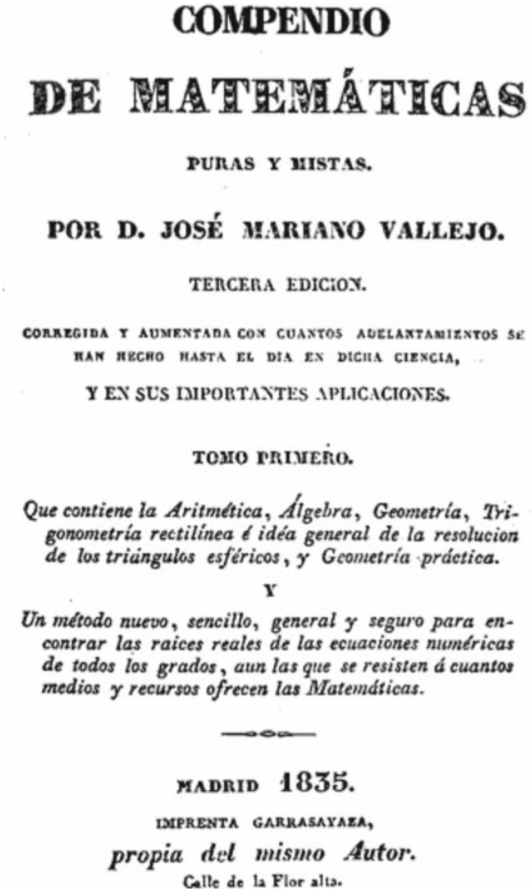
[1] For this and other untranslated Spanish terms, see the glossary at the end of the article.

[2] Al concluir este prólogo, no puedo menos de manifestar mi gratitud a varios sabios, a quienes he debido atenciones, durante mis viajes por Europa, y que me han favorecido con su ilustración. Estos, colocados por orden alfabético, son:…

José Mariano Vallejo Ortega (1779–1846)

> A côté de simples étudiants, on y voyait les hommes les plus illustres dans les sciences mathématiques, tels que MM. Ampère, Sturm, Coriolis, Lamé et la plupart des géomètres de ce temps. Il lui venait même des auditeurs de l'étranger, et ce devait être, pour le profeseur, l'occasion d'un sentiment d'orgueil bien légitime d'avoir à porter la parole devant des hommes tels que MM. Lejeune-Dirichlet, de Berlin, Vallejo, de Madrid (our emphasis), Ostrogradsky et Bouniakousky, de l'Académie de Saint-Pétersbourg, devant des savants qui s'étaient déjà illustres par des travaux personnels de premier ordre et que la réputation de Cauchy avait ré unis de si loin autour de sa chaire [30, Vol. I, p. 66].

After the death of king Fernando VII, Vallejo returned to Spain in 1833 and resumed his previous intense political activity. He came back to the Government sphere and served as *Director General de Estudios* in 1835. The next year he was re-elected as a representative of Granada and involved himself in a plethora of projects, among them the introduction and extension of the Metric System, the division of Spain into


COMPENDIO
DE MATEMÁTICAS
PURAS Y MISTAS.
POR D. JOSÉ MARIANO VALLEJO.
TERCERA EDICION.
CORREGIDA Y AUMENTADA CON CUANTOS ADELANTAMIENTOS SE HAN HECHO HASTA EL DIA EN DICHA CIENCIA,
Y EN SUS IMPORTANTES APLICACIONES.
TOMO PRIMERO.
Que contiene la Aritmética, Álgebra, Geometría, Trigonometría rectilínea é idéa general de la resolucion de los triángulos esféricos, y Geometría práctica.
Y
Un método nuevo, sencillo, general y seguro para encontrar las raices reales de las ecuaciones numéricas de todos los grados, aun las que se resisten á cuantos medios y recursos ofrecen las Matemáticas.
MADRID 1835.
IMPRENTA GARRASAYAZA,
propia del mismo Autor.
Calle de la Flor alta.

Compendio, 1835. The first edition to include Vallejo's Method.

provinces, Railways, and the renovation of teaching methods. From 1843 until his death in 1846 he was a Senator.

As a mathematician, Vallejo wrote several books and memoirs. His first one, from 1804, was a teaching aid for children, a topic he returned to in his 1826 and 1840 books. This paper is focused on the study of some ideas about numerical computations appearing in his *Tratado* [24] and *Compendio* [22]. Both texts had many printings and editions along the years, and as late as 1856 the estate of Vallejo published a sort of ultimate edition.

His two mathematical memoirs deal with the curvature of lines (1807) [20] and some addenda (1806) [19] to the classical Spanish *Instituciones de Geometría práctica* (1795) by Benito Bails (1730–1797) [2].

The pedagogical bias in Vallejo's mathematical writings is most interesting and deserves comment. Above all, he really believed in the need of promoting mathematical literacy and awareness, much in the line started in the middle of the eighteenth century, when several European governments asked known scientists and philosophers to write texts for school usage. For instance, the well known *Logic* by E. Condillac (1714–1780) was written for its use in Polish schools as shown in the Introduction to [6]. Many examples can be found in the books by Vallejo supporting this view. The writing is extremely rhetoric, keeping formulae to a bare minimum and with various mnemonic rules. In fact, his teaching techniques encouraged learning by rote. Here is the list of the works by Vallejo (full bibliographical details are provided in the bibliography):

- *Aritmética para niños*. Madrid, 1804.
- *Adiciones a la Geometría de don Benito Bails*. Madrid, 1806.
- *Memoria sobre la curvatura de líneas*. Madrid, 1807.
- *Tratado Elemental de Matemáticas*, (3 Vol.). Palma de Mallorca, 1812 (Vol.I and II) and Valencia, 1817 (Vol. III).
- *Compendio de Matemáticas puras y mistas*. Valencia 1819.
- *Ideas primarias que deben darse a los niños …acerca de los números*. Paris, 1826.
- *Definiciones y extracto de las principales reglas y operaciones de la Aritmética*. Madrid, 1840.
- *Explicación del sistema decimal*. Madrid, 1840.
- *Tratado completo de Matemáticas y Álgebra*. Paris, 1856. (Posthumously published by his estate).

2 Numerical solution of equations in Vallejo's *Compendio*

The method Vallejo claims to be his discovery for numerical solving of (polynomial) equations of any degree is presented [25, Vol. I, pp. 191–290] in the fourth (1840) edition of his *Compendio*, where the reader is informed on the front page that the book contains

> A new, simple, general and reliable method to find real roots of numerical equations of any degree, even those that resist all means and resources offered by Mathematics, *even those provided by the Infinitesimal Calculus* [25].[3]

The history of how this topic came to be included in this edition is the following: Neither the first nor the second edition (1826) of *Compendio* dealt with the subject of numerical solution of equations, but the 1841–1844 edition of *Tratado* reads:

> In my *Tratado sobre el movimiento y aplicaciones de las aguas* (A treatise on the movement and applications of waters), book 3, I solved through a simplified version of this method some questions that were impossible to solve by any other method. Later on, when I was preparing the third edition of *Compendio* I discovered that the method could be generalised and simplified in a most admirable way. I was involved in this research by 1835, and the printing of the first book of *Compendio* arrived at the point where this doctrine should be inserted, so I

[3] Un método Nuevo, sencillo, general y seguro para encontrar las raíces reales de las ecuaciones numéricas de todos los grados, aun las que se resisten á cuantos medios y recursos ofrecen las Matemáticas, *inclusos los que suministra el Cálculo Infinitesimal*.[Our emphasis.]

had to stop solving a seventh degree equation and delayed its conclusions to an appendix in the second Volume of *Compendio*. As stated in the notes on pages 218 and 263, I did this work while staying in bed [28, Vol. I, p. 368].[4]

Then follows the announcement [28, Vol. I, p. 369] that this new technique will be included in the next printing of *Compendio.* This printing, or rather a new edition (fourth) of *Compendio* appeared in 1840 [25], and will be the standard reference in this article.

In pages 150–167 of *Compendio* Vallejo makes a succinct though interesting historical description of the art of numerically solving equations. He is well aware of the work by several previous authors: Viète, Harriot, Oughtred, Pell, Newton, Halley, Lagni, Raphson, Jakob, Johann and Daniel Bernoulli, Euler, Lagrange, and Budan de Bois-Laurent. The purpose of this introductory report is to highlight the difficulties he had found in the practical application of these methods. In his survey Vallejo comments on several possible causes for the malfunction of the methods he knew:

(a) Some of them, like Newton's, depend heavily on the knowledge of a good initial estimate. Vallejo writes:
This method of Newton needs the estimation of the root with an error less than one tenth of the difference (Vallejo refers to the example given by Newton in *Analysis*, where he states: "et sit 2 numerus qui minus quam decima sui parte differt a Radice quæsita" [13, p. 8]) between the initial estimate and the true value. In the case of a complicated equation this estimation may be more cumbersome than directly solving the equation by my method [25, Vol. I, p. 153].[5]
On the contribution by Raphson:
As a rule, for the methods by Newton and Raphson to produce their effect, the difference between the initial estimate and the true value must be less than one tenth of the true value. Should this not happen, the approximation would be much slower and a greater number of calculations would be needed. Occasionally the method would "fall in defect" [25, Vol. I, p. 154].[6]

[4] En mi *Tratado sobre el movimiento y aplicaciones de las aguas* (§§130, 131, 132, 133, 140, 141, 142 y 276 del libro 3 ^{o}) resolví por este método, algo simplificado, varias cuestiones que no se podían resolver por ningún otro procedimiento; y al preparar el original para la tercera edición de mi *Compendio de Matemáticas*, llegué a descubrir, que este procedimiento se podía simplificar y generalizar del modo más admirable y portentoso; y continuaba estas investigaciones en 1835, cuando ya la impresión del primer tomo llegó hasta el parage en que se insertaba esta doctrina; y tuve que suspender la resolución de una ecuación del 7 ^{o} grado, reservándome insertar su conclusión y demás investigaciones en el apéndice al fin del 2 ^{o} tomo de aquella edición del Compendio. Y consta por la notas de las páginas 218 y 263, que este trabajo lo hice yo estando en cama.

[5] Este método de Newton estriba en que por tanteos se conozca ya el verdadero valor de la raíz con menos de una décima parte de diferencia; lo cual en ecuaciones complicadas exige más molestia y trabajo que la resolución completa de la ecuación por mi método.

[6] En general, para que el método de Newton y el de Raphson produzcan su efecto, conviene que se conozca un valor a de la raíz, que difiera del verdadero valor de x en menos de la décima parte de dicho valor; si esto no se verifica la aproximación será más lenta, y la operación exigirá un gran número de sustituciones, pudiendo llegar el caso de caer en defecto.

Here "fall in defect" means "be not convergent". And on the method by Lagrange:

This method is based, as Newton's, on the hypothesis that the integer part of a root is known [25, Vol. I, p. 155].[7]

(b) Other methods are complicated, tedious and leading to hard computations. Vallejo comments:

Lagrange's method often leads to long, troublesome, and painful calculations. Therefore, in practice Newton's method is preferred, as well as some other procedures by Kramp, Cagnoli.[8], etc. [25, Vol. I, p. 155][9]

The method of Daniel Bernoulli is also criticised. Vallejo referred to slow convergence as weak approximation:

The method by Bernoulli is the computation of a recursive series such that the quotient of consecutive terms yields better approximations to the root as the index grows. This method is not suitable for every equation, for quite often the number of terms needed is very high and only a weak approximation is obtained.[10]

(c) Difficulties in estimating and bounding the error of the computed approximation:

It is rather difficult to know the degree of approximation obtained with the preceding methods, i.e. which is the right number of decimal figures where one should stop in order that the subsequent computations do not become too laborious [25, Vol. I, p. 154].[11]

Moreover, Vallejo insists on the fact that all these methods need Differential Calculus or *Matemática sublime*:

> In addition to incompleteness, all these methods do need higher Mathematics [25, Vol. I, p. 155].[12]

Finally, the author states his teaching interests and writes:

[7] Supone, como el de Newton, que ya por tanteos se conozca que una de las raíces se halla entre a y $a + 1$; o de la cual a es la parte entera.

[8] He refers to the astronomers Christian Kramp (1770–1826) and Antonio Cagnoli (1743–1816).

[9] El método de Lagrange conduce muchas veces a cálculos tan largos, molestos, penosos y desagradables, que en la práctica se prefiere aún el de Newton y algunos otros procedimientos usados por Kramp, Cagnoli, etc.

[10] El método de Bernoulli consiste en hallar una serie recurrente, tal que uno de sus términos, dividido por el precedente, dé un valor más y más aproximado a una raíz de la ecuación, según los términos empleados sean mayores.

No todas las ecuaciones son de naturaleza de podérselas aplicar este método con ventaja; y frecuentemente hay precisión de calcular un muy gran número de términos de la serie, para obtener una dé bil aproximación.

[11] Es bastante difícil poder conocer exactamente el grado de aproximación que se obtiene por los métodos precedentes, a saber: en cada operación, cuáles son las cifras decimales, en que se debe uno detener para no hacer inúltilmente los cálculos sucesivos demasiado laboriosos.

[12] Todos estos métodos, además de lo defectuoso e incompletos que son, necesitan para poderse emplear, conocimientos de los más sublimes de las Matemáticas.

I think I could say that my method can be applied by pupils at schools [25, Vol. I, p. 156].[13]

It is most interesting to notice the absence of references in the quotations by Vallejo to two authors who made first-order contributions to the field. These are Paolo Ruffini's (1765–1822) *Sopra la Determinazione delle Radici nelle Equazione numeriche di qualunque Grado* [15] and William Horner's (1786–1837) *A new method of solving numerical equations of all orders by continuous approximation* [12]. There is not a clear explanation of this fact. It is quite possible that Vallejo was not aware of them, or maybe the very strong influence of French culture and Mathematics in Spain—and most notably Cauchy's (cf. [31,32])—was determinant in his ignorance. This should be no surprise because, as Cajori points out in [3], the works of Horner and Ruffini were relegated in French treatises to a few footnotes until the beginning of the twentieth century. On the other hand, Budan's method based on polynomial transformations was indeed known to Vallejo (cf. [25, Vol. I, p. 156]), even though this method did not count among the favourites of Cauchy, whose opinion on it is described in his *Mémoire sur la résolution numérique des équations...* with the words "*...sont appuyés sur des théories étrangères aux éléments d'Algèbre*" [5, 2ª ser., T. 9, p. 88].

3 Numerical solution of equations according to Vallejo

The work of Vallejo on this topic is presented in pages 191–290 of the 1840 edition of *Compendio*. Pages 191–200 contain a study of *Regula Falsi*, 200–209 deal with errors, and his own method is explained in twenty pages from 209 to 229. The remaining pages contain a number of fully worked out exercises illustrating the method at work in several polynomial equations with degrees ranging from two to six. Further on the reader finds a very long footnote on logarithms, extending from page 312 to page 335, where the application of the method to seventeen examples of transcendental equations is considered. More applications can be found in an appendix to Volume II of *Compendio*, where he presents the work of students at *Escuelas Normales* (institutions for the training of future schoolteachers) on 29 more equations, even one of the $80th$ degree.

3.1 The toolbox

As an introduction to his method, Vallejo presents in *Compendio* the fundamental results on which it is based. Six theorems are studied, and he claims they contain everything that is needed for the method to work.

The first theorem, for which a standard proof is provided, is the well known factor theorem stated this way:

[13] creo haber llegado hasta el punto de poderse aplicar mi método por los discípulos de las escuelas.

> If a is root of the equation $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \cdots + Kx + L = 0$, then $x - a$ is a factor of the left member [25, Vol. 1, p. 157].[14]

Second comes the theorem now known as Bolzano's, rhetorically formulated in a complicated nineteenth century language. It goes like this:

> Let two different numbers be substituted for the unknown in the left member of a (polynomial) equation, such that they yield numerical values with different signs. Then it is verified that the equation must have at least one root comprised between the two chosen numbers [25, Vol. I, p. 158].[15]

Vallejo offers a "proof" which strongly contrasts with the rigorous style of Cauchy:

> Let the polynomial $p(x)$ be written by grouping together all positive and negative terms, $p(x) = P(x) - N(x)$. Now consider two numbers a and b such that $p(a) < 0$ and $p(b) > 0$. It follows that $P(a) < N(a)$ and $P(b) > N(b)$. Therefore some z must exist between a and b which satisfies $P(z) = N(z)$ or equivalently $p(z) = 0$.

This is a circular argument: To infer the existence of z from $P(a) < N(a)$ and $P(b) > N(b)$, Bolzano's theorem is needed, but it is the one Vallejo wants to prove. In his defense it must be noted that his argument is more a geometric intuition, quite possibly inspired in the *Cours d'Analyse* [4, p. 43], than a proof.

A third theorem establishes the fact that *every odd degree polynomial has a real root whose sign is opposite to that of its independent term* [25, T. I, p. 159].[16]

The proof offered by Vallejo combines Bolzano's theorem with the following result contained in the second Volume of *Compendio*:

> For any polynomial of degree n, a value z can be chosen for the unknown such that the term containing the highest degree term z^n is larger than the sum of all the remaining terms [25, Vol. II, pp. 50–52].[17]

Next comes the related theorem for polynomials of even degree, proved in an analogous way: *Every even degree polynomial whose independent term is negative has at least two real roots, a positive and a negative one* [25, T. I, p. 159].[18]

The two previous results are combined with the change of variable $x^2 = z$ in order to show that: *Every polynomial whose terms are all of even degree and having a neg-*

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[14] Si a es raíz de la ecuación $x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \cdots + Kx + L = 0$, el primer miembro de dicha ecuación será divisible por $x - a$.

[15] Si de sustituir dos números por la incógnita en una ecuación, el primer miembro da valores de signos contrarios, se verifica, que dicha ecuación ha de tener al menos una raíz real comprendida entre aquellos dos valores de la incógnita.

[16] Toda ecuación de grado impar, tiene al menos una raíz real de signo contrario al de su último término.

[17] En toda serie ordenada por potencias de una sola variable, se le puede dar a esta un valor tal que un término cualquiera sea mayor que la suma de todos los que le siguen.

[18] Toda ecuación de grado par, cuyo último término es negativo, tiene al menos dos raíces reales, una positiva y otra negativa.

ative independent term has at least two real roots of equal absolute value, a positive and a negative one [25, T. I, p. 160].[19]

The toolbox is completed by adding the last item, an easy consequence of the above theorems: *Every polynomial whose terms are all of even degree and with positive coefficients has no real roots, only imaginary ones* [25, T. I, p. 160].[20]

A remark on nineteenth century nomenclature is that our "absolute value" and "complex numbers" are indicated as "numerical value" and "imaginary quantities".

Now Vallejo proudly proclaims in his flourished style: *On the only basis of the preceding results I shall solve any sort of numerical equations, even the most difficult ones solved up to date by any other known method* [25, T. I, p. 161][21]...

As an application, he adds that the 64 roots of $u^{64} - 7u^{48} + 21u^{32} - 29u^{16} + 14 = 0$ are computed by his purely arithmetical method.

3.2 Vallejo and Rolle's Theorem

Vallejo is extremely confident on the validity of his "completely arithmetical" method. Nevertheless, he knows well that location of the roots can be assessed from an application of Rolle's theorem, and when it comes to looking for bounds of the roots, he does use the theorem in a rather rhetoric way without giving it any name. In *Compendio* [25, Vol. I, pp. 217–218], instead of simply stating that *the real roots of the derivative are separated by the real roots of the polynomial* or any similar formulation, he proceeds to explain how to write down the derivative of a polynomial and how to use its roots in order to compute intervals such that the original polynomial has different signs at the endpoints. His idea is to provide the user with a simple rule to be used in a purely algorithmic way as a mere complement to his arithmetical conception:

> To form the derivative of the equation, just multiply the coefficient of each term by the exponent and reduce this in one unit, eliminate the independent term and equate everything to zero. This new equation will be solved by my method in an easier way because its degree is one less than that of the original equation. The values obtained are employed as supposed numbers in the equation, and if their errors are of different signs [25, Vol. I, pp. 217–218].[22]

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[19] La ecuación en que todos los exponentes de la incógnita sean números pares, y su último término sea negativo, tendrá al menos dos raíces reales de igual valor numérico.

[20] La ecuación en que todos los exponentes sean números pares y todos los coeficientes sean números positivos, no tiene ninguna raíz real; y todas serán imaginarias.

[21] Fundándome únicamente en lo que precede resolveré por mi mé todo toda clase de ecuaciones numéricas; contrayéndome desde luego a las ecuaciones más difíciles, que cuantas se han resuelto hasta el día por todas los otros métodos conocidos...

[22] Fórmese la ecuación derivada de la ecuación primitiva; lo que se consigue multiplicando el exponente de la incógnita en cada término, por su coeficiente, disminuyendo cada exponente en una unidad, suprimiendo el término constante de la primitiva e igualando a cero todo este conjunto de términos. Se resolverá esta ecuación derivada encontrando sus raíces reales por el procedimiento anterior lo cual será mas fácil, pues la ecuación derivada resulta siempre un grado inferior a la primitiva. Los valores, que se obtengan para las raíces de la ecuación derivada, se sustituirán en la ecuación primitiva; si diesen resultados de signos con-

He also thinks that the lower the degree is, the easier to solve is the polynomial equation. In general, this is not true: A simple example is provided by the polynomial $(x-1)(x-2)(x-3)(x-4)(x-5)$. It is enough to expand it and to compute its derivative to check that Vallejo's assumption is a risky one.

3.3 The method

Strictly speaking, the method of Vallejo for the numerical solution of polynomial equations is a clear example of the general shape of a numerical method:

(a) A starting two-point (or double) Regula Falsi, including possible checks by Bolzano's theorem when needed.
(b) Iteration of (a).

The theory starts with the statement: *The doctrine of the false position, either simple or double, can only be applied to problems where the left hand side of the equation is proportional with the sought number* [25, Vol. I, pp. 197–198],[23] and Vallejo illustrates it by writing $x^n + Ax^{n-1} + Bx^{x-2} + \cdots + Kx + L = 0$ in the form $x\left(x^{n-1} + Ax^{n-2} + Bx^{n-3} + \cdots + K\right) = -L$ and observing that in this last expression the first factor is indeed proportional to x and though the second one *...is not strictly proportional to x , it does vary with it; therefore variations in the value of the left hand side are dominated by its being proportional to x, and any formulae we may apply to the left hand side, obtained on the hypothesis of this proportionality, will yield results that, although not exact ones, will certainly approximate the true ones. And this is the basis of the most essential part of my method* [25, Vol. I, pp. 197–198].[24]

It is difficult to understand what Vallejo had in mind, and little help is obtained when he presents a first estimate[25] to the actual root obtained by means of the "method of the secant" (see Fig. 1), starting from any two numbers a and b such that $|f(a)| < |f(b)|$:

Here $\bar{x} \approx c = a + \frac{f(a)(a-b)}{f(b)-f(a)}$, a formula easily derived from the similarity of triangles $B'DA'$ and $A'AC$. In fact, the method actually introduces a forward finite difference approximation to $f'(a)$ in the Newton–Raphson formula. In the language of Vallejo a and b are called *supposed numbers* (Spanish: *supuestos*), $f(a)$ and $f(b)$ are *errors*, $|f(a)|$ and $|f(b)|$ are *numerical errors*, where a and b are chosen in order that $|f(a)| < |f(b)|$, and $\frac{f(a)(a-b)}{f(b)-f(a)}$ is the *correction term*. As a rule, the author systematically begins with the supposed numbers $a = 1$, $b = 2$ and then checks that the

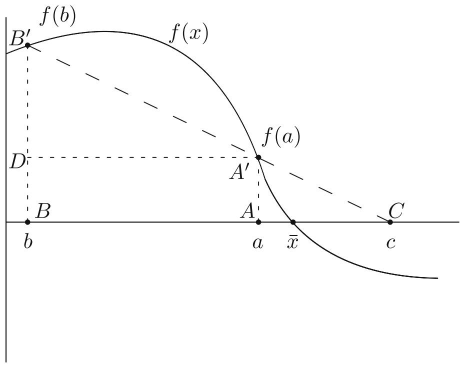
trarios a los que se hayan obtenido por lo otros supuestos, se tomarán para dos primeros números supuestos respecto a esta combinación, (...) y con ellos se continuará hasta encontrar las raíces reales que contengan.

[23] La doctrina de la falsa posición, simple o doble sólo se puede aplicar a cuestiones en que el primer miembro de la ecuación a que conduzcan sea proporcional con el número buscado.

[24] ... aunque no es esencialmente proporcional con x, varía con x; luego siempre la parte dominante de las variaciones del primer miembro es el ser proporcional a x; y cuantas fórmulas apliquemos a dicho primer miembro, que estén sacadas en el supuesto de la proporcionalidad con x , darán resultados que, aunque no serán exactos, se aproximarán a los verdaderos. Y en esto se funda la parte más esencial de mi nuevo método.

[25] The first step in the numerical computation of a root is the choice of an initial trial estimate, but it may happen that an unadequate one will spoil the calculations. Therefore this choice must be optimised through some *a priori* knowledge on the location of the roots of $p(x)$ before embarking on the solution of $p(x) = 0$.

Fig. 1



difference $f(b) - f(a)$ is large enough in order to guarantee that the first trial interval is not too long. It is known that the length of this interval will strongly influence the number of iterations.

Then, a detailed discussion of every possible case is presented. For instance, under the hypothesis $f(b)f(a) > 0$, a first case would be finding some c such that $f(a)f(c) > 0$ and $|f(b)| > |f(a)| > |f(c)| > 0$. When this happens, iteration of the procedure follows taking a and c as new supposed numbers, and continues in this way until either $f(c^*) = 0$ for some iteration, or $|f(c^*)|$ is small enough according to the needs of the calculation under way. The exposition is found in [25, Vol. I, pp. 212–213] and a number of worked examples are given in [25, Vol. II, pp. 438–451].

Vallejo does not explicitly describe any stopping criterion. Intuitively, he thinks that if the sequence of supposed numbers defines a monotone decreasing sequence of *numerical errors*, then this last one converges to some root of the equation, but no further convergence discussion is offered.

The second interesting case appears whenever $f(b) > f(a) > 0$ and $f(c) < 0$ or $f(b) < f(a) < 0$ and $f(c) > 0$. Now *there exists at least one root of the equation between*[26] a and c [25, Vol. I, p. 213], and Vallejo proceeds by applying again *Regula Falsi* with the new supposed numbers a and c. Furthermore, Vallejo explicitly cites convergence of *Regula Falsi*, to our knowledge for the first time in history [25, Vol. I, p. 214].

Further analyses are subtler. The third case is when the sequence of *numerical errors* $|f(b)|$, $|f(a)|$, $|f(c)|$ is not monotonically decreasing. In fact, the hypothesis $|f(c)| > \min\{|f(b)|, |f(a)|\}$ is considered. Now the method is to systematically explore the behaviour of the polynomial at the supposed numbers 10, 100, 1000,…, and if no change of sign is detected, at –10, –100, –1000,…, until a change of sign is found or it is clear from the nature of the equation that this will never happen [25, Vol. I, p. 214].[27] Surprisingly, Vallejo does not give any criterion for stopping this search. He certainly knew the *Cours d'Analyse* [4, pp. 470–479], where the estimate

[26] entonces hay al menos una raíz real comprendida entre.

[27] hasta que se logre que cambie de signo la equivocación o error, o se vea por la naturaleza de la ecuación que esto jamas se logrará.

$|\bar{x}| < 1 + \max_{0 \le i \le n} \left| \frac{a_i}{a_n} \right|$ for any root of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is given, a formula already found by Rolle [16, p. 254]. This method certainly works for odd degree polynomials, but for even degree polynomials it can skip pairs of real roots located between successive powers of 10. Vallejo writes on this in [25, Vol. I, p. 217], and he points out that the derivative—whose roots can also be computed by his method—must be calculated in order to apply the theorem of Rolle. In a long footnote in p. 219 [25, Vol. I] he describes the behaviour of $f(x)$ when $f'(x) = 0$. In short, his discussion can be summarised in three points:

If x is a root of $f'(x) = 0$, it can happen that:

(a) It is a root of $f(x) = 0$ as well.
(b) The sign of $f(x)$ is different from that of the values in previous supposed numbers. In this case, the theorem of Bolzano is applied in the interval defined by x and the nearest one of the supposed numbers.
(c) The sign of $f(x)$ is not different from that of the values in previous supposed numbers. Now we must look for other roots of the derivative until (a) or b) holds, or else, when no more roots of the derivative can be computed, we find that the original polynomial is of constant sign and therefore has no real roots.

With the above procedure, Vallejo is able either to find a root $\bar{x}$ of the equation, or to decide that it has no real roots. In the first case, further roots can be sought in the new equation $\frac{f(x)}{x-\bar{x}} = f_1(x) = 0$, to which the method is again applied and can be repeated until some rootless quotient $f_N(x)$ is found.

The author thinks that his procedure is robust in the sense that it is immune to the effect of erroneous computations. Since no convergence proofs are offered, it seems that he was led to this conclusion only on empirical evidence, because in every example he presents, the new supposed number is a rounded value of the previous one. Moreover, the method has no predilection for any particular root. To summarise, it always finds some root (or the lack of them), and as long as his only aim is the solving of equations, Vallejo is certainly satisfied with the final result.

4 The method from today's viewpoint

Numerical equation solving is above all a practical matter. In the first half of the nineteenth century, numerical computations were done by hand and there was little knowledge on the role of accumulative rounding errors. In some cases, especially when dealing with multiple roots, rounding errors can lead to wrong conclusions:

(a) For the first case studied by Vallejo, $|f(b)| > |f(a)| > |f(c)| > 0$, we know today that the iterates of the secant method may not converge toward a multiple root.
(b) In the second case $f(b) > f(a) > 0$ and $f(c) < 0$ (or $f(b) < f(a) < 0$ and $f(c) > 0$), it can happen that rounding errors lead to non accurate sign determinations, so the interval where Bolzano's Theorem is applied will not be the correct one, yielding a senseless computation.

(c) The deflating process by which the lower degree polynomial $f_1(x)$ is obtained after division of $f(x)$ by $x - \bar{x}$ can be unstable from the computational viewpoint. As a rule, small errors in the determination of $\bar{x}$ will give small errors in the coefficients of $f_1(x)$ and in the determination of its roots, but examples can be found in the literature [1,8] where this assumption is a false one: In [8, pp. 70–71] the following example is presented: Let $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)$, where the coefficient of x^6 is -28. If we take -28.002 as an approximate coefficient with eight significant digits, then the two roots 5 and 6 switch into a pair of complex roots $5.4586758 + 0.54012578i$, 5.4586758–0.54012578i . In modern language, this is a case of bifurcation of the roots as functions of the polynomial coefficients. Some interesting references on this "perfidious polynomial" are [7,33].

(d) It has been already remarked that Vallejo does not explicitly give stopping criteria for the iterative processes contained in his method. Instead, the user may decide when to stop calculations [25, Vol. I, p. 213] by determining some k such that $f(c) \leq 10^{-k}$. A simple example shows that false roots could be determined: If $f(x) = (x-p)^q + 10^{-r}$ and $r > k$, p could be taken as a root because $f(p) < 10^{-k}$. Indeed Vallejo should be forgiven for not having considered such special cases: His aim was to show how to train human computers and from his viewpoint these strange could be left to first-rate analysts or skilled mathematicians, if needed.

5 Conclusions

As a first result, it can be concluded that Vallejo contributed a formulation of *Regula Falsi* and the secant methods in almost present day terms, hoping that it could yield a Calculus-free approach to the problem of solving equations numerically. *Regula Falsi* as a method in its own right had somehow been abandoned during the two preceding Centuries in favour of methods derived from the new Differential Calculus, infinite series, and continued fractions. The secant method is simply a finite difference formulation of the Newton–Raphson procedure, where the derivative is approximated by a forward difference quotient.

Moreover, Vallejo combined his *Regula Falsi* with iteration, and was able to write down a rather straightforward procedure—or algorithm—that worked well in a large number of examples presented in [25]. Of course these examples do not contain any badly behaved polynomial, and only a few transcendental equations are dealt with. It is highly remarkable that his emphasis on the algorithmic nature of computations is ahead of his time by more than one hundred years. Vallejo insists on the main feature of his method: It is a global one, in the sense that it always arrives either at a root or to the lack thereof regardless of the two first supposed numbers employed, so there is no need of a previous estimation of a candidate root.

The second result is the observation that Vallejo is reluctant to accept the tools of Calculus into his method, and only uses Rolle's theorem when he finds that his arithmetical viewpoint is unable of further progress. Had he stated the method more completely by including Rolle's theorem as a routine check, he would certainly have been a real forerunner of Computational Algebra.

As a third result, it has been found that Vallejo was aware of the state of Mathematics in the first third of the nineteenth century and of the developments leading to the introduction of rigorous reasoning, whose main character was Cauchy. In spite of this, Vallejo did not formulate his theory and methods for solving equations in a rigorous way. Some proofs are simple geometric intuitions and long rhetoric arguments are offered in order to convince the reader of the validity of results. Rigour as a standard requirement in mathematical writing was starting its way, and the acceptation of this idea in Spanish Mathematics during the nineteenth century is the core of the doctoral dissertation [17] by one of the authors (CSA).

Spanish–English Glossary

Real Academia:	Royal Academy
Real Seminario de Nobles:	Royal Seminar or High School for Nobles
Cortes de Cádiz:	The first democratic Parliament elected in Spain. Its sessions took place in Cádiz the year 1812.
Sociedad Económica:	A type of private philanthropic society for the economic and cultural promotion of some city or area. They still survive in many parts of Spain.
Colegio de Sordomudos:	Special school for the deaf-mute.
Gabinete Geográfico:	Geographical Cabinet.
Trienio liberal:	Three year period (1820–1823) in the reign of Fernando VII during which all constitutional rights were respected.
Década ominosa:	Ominous decade. Ten year period following the trienio liberal where absolute monarchy was restored. It ended with the death of Fernando VII the year 1833.
Dirección General de Estudios:	General Directorate of Studies. Its headperson is the *Director General de Estudios*.
Tratado:	Treatise.
Compendio:	Compendium.

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