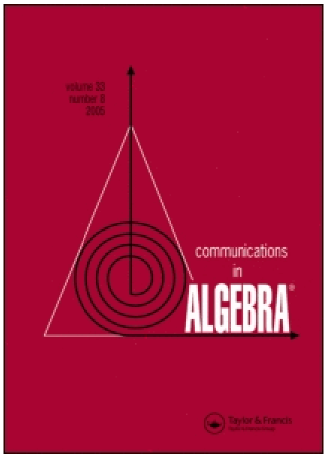

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ABSOLUTE VALUED STRONGLY POWER-ASSOCIATIVE TRIPLE SYSTEMS

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Absolute valued triple system are the natural ternary extension of absolute valued algebras. In this article we classify strongly power-associative absolute valued triple systems.

Key Words: Absolute values; Alternative algebras; Structure group of a Jordan and an alternative algebra; Triality; Triple systems.

AMS Subject Classification: 17A40; 17A80; 17C30; 17D05.

1. INTRODUCTION AND PRELIMINARIES

1.1.

Let $\mathbb{K}$ denote the field of real or complex numbers. An absolute valued algebra over $\mathbb{K}$ is a nonzero algebra A over $\mathbb{K}$, endowed with a norm $|\cdot|$ satisfying $|xy| = |x||y|$ for all $x, y \in A$. The most natural examples of absolute valued algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the algebra of Hamilton quaternions), and $\mathbb{O}$ (the algebra of Cayley numbers), with norms equal to their usual absolute values. In the early article of Albert (1947), it is proved that the only finite dimensional absolute valued algebra with a unit is $\mathbb{C}$ in the complex case and $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ in the real one, that any finite dimensional absolute valued algebra has dimension 1 in the complex case, and $1, 2, 4$, or 8 in the real one; and that the absolute values are the usual euclidean norms. Since Albert (1947), absolute valued algebras have been intensively studied by many authors (see for instance the excellent survey Rodríguez, 2004 and Albert,

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1949; Cuenca and Rodríguez, 1995; El-Amin et al., 1997; El-Mallah, 1998, 2001; Ramírez, 1999; Rochdi, 2005; Rodríguez, 1994; Urbanik and Wright, 1960).

Clearly, any finite dimensional absolute valued algebra is a division algebra, conversely, absolute valued division algebras are finite dimensional (Wright, 1953). It is easy to see that if two norms on a finite dimensional algebra convert it into an absolute valued algebra, then they must coincide (see for instance Cuenca and Rodríguez, 1995). From here, it is also clear that any isomorphism between two finite dimensional absolute valued algebras $f : A \to A'$ is isometric. Indeed, we can define a new norm on A by $|x|'_A := |f(x)|_{A'}$ making A an absolute valued algebra, finally the uniqueness of the absolute value gives us $|x|_A = |x|'_A = |f(x)|_{A'}$. The precise determination of isomorphism classes for absolute valued real algebras of dimensions 1 and 2 is contained in Rodríguez (1994), where the number of classes reduces to 1 and 4, respectively, while a detailed determination for the four-dimensional ones appears in Ramírez (1999) (see also Calderón and Martín, 2005, Theorem 5.4). However, this determination in the eight-dimensional case is still an open problem.

As usual, given an algebra A and an element $a \in A$, we denote by R_a (resp. L_a), the right, (left), product operator $R_a(x) := xa$, $(L_a(x) = ax)$.

1.2.

Let T be a vector space over $\mathbb{K}$. We say that T is a *triple system* if it is endowed with a trilinear map $\langle\ \ \rangle : T \times T \times T \to T$, called the *triple product* of T. A triple system T is called *associative* if the following identity holds for any $x, y, z, u, v \in T$: $\langle\langle xyz\rangle uv\rangle = \langle x\langle yzu\rangle v\rangle = \langle xy\langle zuv\rangle\rangle$. Let T, T' be triple systems, a bijective linear map $f : T \to T'$ is called an *isomorphism* of triple systems if it satisfies $f(\langle xyz\rangle) = \langle f(x)f(y)f(z)\rangle$ for any $x, y, z \in T$. Triple systems appear in the literature as the natural ternary extension of algebras and have been studied in the associative (Shaw, 1990a,b), nonassociative (Calderón and Martín, 2001; Castellón et al., 2000; Hopkins, 1985; Lister, 1952) and general context (Castellón and Cuenca, 1993). An absolute valued triple system is defined as follows:

Definition 1.1. An *absolute valued triple system*, (a.v.t.s.), is a non-zero triple system T over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, endowed with a norm $|\cdot|$ that satisfies $|\langle xyz\rangle| = |x||y||z|$ for any $x, y, z \in T$.

Any absolute valued algebra A can be seen as an a.v.t.s., with the same norm, by defining for instance the triple product as $\langle xyz\rangle := (xy)z$. Then we have that the class of absolute valued algebras is related to the class of a.v.t.s. Moreover, given $0 \neq u \in T$, we define the algebra T^u as the vector space underlying T with the product $xy := \langle xuy\rangle$. We will say that T^u is *the u-homotopic algebra* of T. If $|u| = 1$, then T^u becomes an absolute valued algebra. These algebras will be fundamental in our study. Since $\dim(T) = \dim(T^u)$, $|u| = 1$, Albert's result in 1.1 gives us that any finite dimensional a.v.t.s. has dimension 1 in the complex case and 1, 2, 4, or 8 in the real one, and that the absolute values are the usual euclidean norms. By considering T^u, $|u| = 1$, and taking into account the observations in 1.1, we also obtain that if we have two norms on T converting it into an a.v.t.s., then they must coincide; and that any isomorphism between two finite dimensional a.v.t.s. is isometric.

The first approach to the classification of finite dimensional a.v.t.s. is given in McCrimmon (1983). In Calderón and Martín (2004) we study some aspects of the theory of a.v.t.s. and exhibit the isomorphisms classes of a.v.t.s. of dimension 1, 2, and 4. We also relate the theory of a.v.t.s. to the one of two-graded absolute valued algebras in Calderón and Martín (2005).

1.3.

An algebra A is called *power-associative* if the subalgebra generated for any element $x \in A$ is associative. The algebra A is said to be *alternative* if it satisfies the identities $x^2y = x(xy)$ and $(xy)y = xy^2$ for every $x, y \in A$. The classical example of an alternative algebra is $\mathbb{O}$. Clearly, any alternative algebra is power-associative. We note that alternative algebras are "very nearly" associative. Indeed, by Artin's theorem (see Zhevlakov et al., 1982, Theorem 2.3.2), *the subalgebra generated by two arbitrary elements of an alternative algebra is associative*. In the framework of absolute valued algebras, El-Mallah and Micali (1980) proved that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are, up to isomorphisms, the unique absolute valued power-associative algebras. As a consequence, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are also the unique absolute valued alternative algebras, and we can assert that in the context of absolute valued algebras, power-associativity is equivalent to alternativity.

In the context of triple systems, there are two different ways of introducing power-associativity. First, a triple system is called *power-associative* if the subtriple generated for any element is associative, an arising question is whether any power associative a.v.t.s. is finite dimensional. Second, we recall that given a couple of modules over a commutative unitary ring $A = (A^+, A^-)$, A is said to be a *pair*, if it is endowed with two trilinear maps $\langle\ \rangle_\sigma : A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma$, where $\sigma \in \{+, -\}$. If $(T, \langle\ \rangle)$ is a triple system, then (T, T) is an example of a pair with the trilinear maps $\langle\ \rangle_\sigma := \langle\ \rangle$, $\sigma \in \{+, -\}$. A *pair* $A = (A^+, A^-)$ is called *power-associative*, if the subpair of A generated by any $x^+ \in A^+$ and any $x^- \in A^-$ is associative (see for instance Cuenca et al., 1989, as a reference on associative pairs).

Definition 1.2. A triple system T is said to be *strongly power-associative* if the pair (T, T) is power-associative.

As examples of strongly power-associative triple systems we can mention the alternative and the Jordan triple systems. It can be proved that in an alternative pair $A = (A^+, A^-)$, fixing $x \in A^\sigma$ and $y, z \in A^{-\sigma}$, the subpair of A generated by x, y and z is associative (this is a version of Artin's Theorem for alternative pairs). This implies that any alternative triple systems is strongly power-associative. Also Jordan triple systems are strongly power-associative taking into account for instance (Loos, 1975, Paragraph 1.9, p. 6). Thus, the class of strongly power-associative triple systems contains some of the more studied classes of triple systems. On the other hand, the research activity around alternative (in particular associative) and Jordan triple systems in the last decades can be represented by the works of Kaup (1981, 1983), Loos (1975), Neher (1980, 1981, 1985), Upmeier (1985) among (many) others. We should mention also the works of the active Spanish school under the impulse of A. Rodríguez.

2. CLASSIFICATION RESULTS

2.1.

In the present article, we are interested in investigating absolute valued strongly power-associative triple systems T (strongly power-associative a.v.t.s. for short). From now on, given $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $x, y \in \mathbb{A}$, the juxtaposition xy will mean the usual product in $\mathbb{A}$. We recall that $x \in T$ is called a *tripotent* if $\langle xxx \rangle = x$. The set of all nonzero tripotents of T will be denoted by tri (T).

Let T be a strongly power-associative a.v.t.s. It is easy to check that any of its u-homotopic algebras T^u, $|u| = 1$ (see 1.2), is an absolute valued power-associative algebra with the same dimension as T. Therefore, $\dim T^u \in \{1, 2, 4, 8\}$ (see 1.3), and we conclude that $\dim T \in \{1, 2, 4, 8\}$.

Let us consider now a strongly power-associative a.v.t.s. T, over $\mathbb{K}$, with dimension 1. By an easy argument on the triple product $\langle 111 \rangle$, it is straightforward to prove that T is isomorphic to one of the following:

1. If $\mathbb{K} = \mathbb{R}$, then either $T \cong (\mathbb{R}, \langle\ \ \rangle)$ or $T \cong (\mathbb{R}, -\langle\ \ \rangle)$ with $\langle x, y, z \rangle = xyz$ for any $x, y, z \in \mathbb{R}$. These two triple systems are not isomorphic,
2. If $\mathbb{K} = \mathbb{C}$, then $T \cong (\mathbb{C}, \langle\ \ \rangle)$ with $\langle x, y, z \rangle = xyz$ for any $x, y, z \in \mathbb{C}$.

As consequence, any 1-dimensional a.v.t.s. is necessarily associative and so a strongly power-associative a.v.t.s. As any complex finite dimensional a.v.t.s. has dimension 1, (see 1.2), we have completed the study of the complex case, therefore we confine ourselves to the real case.

Remark 1. The definition of triple system given in this work imposes the linearity of the triple product in each variable. However different approaches exist in the literature since the very beginning of the theory itself. Thus the hermitian Jordan triple systems studied by Kaup (1981) are complex Jordan triple systems whose triple product is conjugate-linear in the middle variable. With such a definition the unique up to isomorphism complex finite-dimensional absolute valued triple system would be $\mathbb{C}$ with the triple product xyz. Our study would not change in the real case hence the parallel approach using conjugate-linear triple products in the middle variable would be possible.

Consider now T a strongly power-associative a.v.t.s. with dimension 2. Though the classification of 2-dimensional a.v.t.s. is contained in Calderón and Martín (2004), Theorem 2.5, it can be directly shown that the ones satisfying the strongly power-associativity condition are isomorphic to $\mathbb{C}$ with one and only one of the following triple products:

1. $\langle xyz \rangle_1 := xyz$;
2. $\langle xyz \rangle_2 := x\bar{y}z$;
3. $\langle xyz \rangle_3 := -x\bar{y}z$.

We note that these triple systems are not isomorphic because $\mathrm{tri}((\mathbb{C}, \langle\ \ \rangle_1)) = \pm 1$, $\mathrm{tri}((\mathbb{C}, \langle\ \ \rangle_2)) = S^1$ (the 1-sphere), and $\mathrm{tri}((\mathbb{C}, \langle\ \ \rangle_3)) = \emptyset$.

2.2.

Let us consider now T a strongly power-associative a.v.t.s. with dimension 4 or 8. We identify T with $\mathbb{H}$ or $\mathbb{O}$ respectively as euclidean vector space (see 1.2).

We recall that a 2-graded absolute valued algebra, (2-graded a.v. algebra), is a nonzero 2-graded algebra $A = A_0 \oplus A_1$ over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, endowed with two norms $|\cdot| : A_i \to \mathbb{K}$, $i = 0, 1$, such that $|x_i x_j| = |x_i||x_j|$, for any $x_i, x_j \in A_0 \cup A_1$. It is proved in Calderón and Martín (2005) that any 2-graded a.v. algebra is isomorphic to $\mathbb{B} \oplus \mathbb{B}$, with $\mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and where the product is

$$(x, y)(z, t) = (\alpha_1(x, z) + \alpha_2(y, t), \alpha_3(x, t) + \alpha_4(y, z)),$$

and where the maps $\alpha_i : A \times A \to A$ $(i = 1, 2, 3, 4)$ take one of the possible forms:

1. $(x, y) \mapsto \beta_i(x)\gamma_i(y)$, or
2. $(x, y) \mapsto \beta_i(y)\gamma_i(x)$,

for some linear isometries β_i and γ_i of $\mathbb{B}$.

Given any finite dimensional a.v.t.s. V, we also have by Calderón and Martín (2005), Theorem 4.3, that there exists a 2-graded a.v. algebra $A = A_0 \oplus A_1$ such that $V = A_1$ and its triple product can be written as $\langle xyz \rangle = (xy)z$, $\langle xyz \rangle = x(yz)$, or $\langle xyz \rangle = (xz)y$.

Let us return to the strongly power-associative a.v.t.s. T. The above considerations and the triality property for the rotations of $\mathbb{A} \in \{\mathbb{H}, \mathbb{O}\}$ (see for instance Postnikov, 1985, Proposition 4, p. 227 and Proposition 2, p. 275), allow us to prove (see Calderón and Martín, 2004, Theorem 2.1), that the triple product in T adopts one of the following 12 forms $\langle x_1 x_2 x_3 \rangle = (\alpha(x_{\sigma(1)})\beta(x_{\sigma(2)}))\gamma(x_{\sigma(3)})$ or

$$\langle x_1 x_2 x_3 \rangle = \alpha(x_{\sigma(1)})(\beta(x_{\sigma(2)})\gamma(x_{\sigma(3)})),$$

with $\sigma \in S_3$ a permutation of the set $\{1, 2, 3\}$ and α, β, γ isometries of $\mathbb{A}$. If we denote by $\mathcal{T}$ and $\mathcal{T}'$ the families of a.v.t.s.

$$\mathcal{T} := \{(\mathbb{A}, \langle\ \ \rangle, \alpha_1, \alpha_2, \alpha_3) \text{ with } \langle xyz \rangle = (\alpha_1(x)\alpha_2(y))\alpha_3(z), \alpha_i \text{ isometries}\}$$

and

$$\mathcal{T}' := \{(\mathbb{A}, \langle\ \ \rangle', \alpha'_1, \alpha'_2, \alpha'_3) \text{ with } \langle xyz \rangle' = \alpha'_1(z)(\alpha'_2(y)\alpha'_3(x))\}, \alpha'_i \text{ isometries}\},$$

then $\mathcal{T} = \mathcal{T}'$ up to isomorphisms. Indeed, given any $T = (\mathbb{A}, \langle\ \ \rangle, \alpha_1, \alpha_2, \alpha_3) \in \mathcal{T}$, the map $\phi : \mathbb{A} \to \mathbb{A}$, $\phi(x) := -\bar{x}$ is an isomorphism between T and $T' := (\mathbb{A}, \langle\ \ \rangle', \alpha'_3, \alpha'_2, \alpha'_1) \in \mathcal{T}$, where $\alpha'_i(x) := \overline{\alpha_i(\bar{x})}$, $i \in \{1, 2, 3\}$. The same map ϕ shows that any a.v.t.s. in $\mathcal{T}'$ is isomorphic to one in $\mathcal{T}$. By applying this argument to the remaining 10 families of a.v.t.s., that we obtain analogously to the above ones by attending to the form the triple product adopts, we reduce the forms which could adopt the triple product of $T = \mathbb{A}$ to the following 6 ones:

1. $\langle xyz \rangle = (\alpha_1(x)\alpha_2(y))\alpha_3(z)$;
2. $\langle xyz \rangle = (\alpha_1(y)\alpha_2(x))\alpha_3(z)$;

3. $\langle xyz\rangle = (\alpha_1(z)\alpha_2(y))\alpha_3(x)$;
4. $\langle xyz\rangle = \alpha_1(x)(\alpha_2(z)\alpha_3(y))$;
5. $\langle xyz\rangle = (\alpha_1(x)\alpha_2(z))\alpha_3(y)$;
6. $\langle xyz\rangle = (\alpha_1(z)\alpha_2(x))\alpha_3(y)$;

with $\alpha_1, \alpha_2, \alpha_3 : \mathbb{A} \to \mathbb{A}$ isometries.

As any u-homotopic algebra T^u, $|u| = 1$, of T is power-associative and therefore isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ (see 1.3), then we conclude that T^u has a unit. Let us now suppose that we have T with the first possible triple product given above, that is, $\langle xyz\rangle = (\alpha_1(x)\alpha_2(y))\alpha_3(z)$. If we consider its ${\alpha_2}^{-1}(1)$-homotopic algebra $T^{{\alpha_2}^{-1}(1)}$, since this is an algebra with a unit, there exists $w \in \mathbb{A}$ such that $x = \alpha_1(x)\alpha_3(w) = \alpha_1(w)\alpha_3(x)$ for any $x \in \mathbb{A}$, therefore $\alpha_1 = R_{\alpha_3(w)^{-1}}$ and $\alpha_3 = L_{\alpha_1(w)^{-1}}$. So, the triple product can be written as $\langle xyz\rangle = ((xu)\alpha_2(y))(vz)$ for certain $|u| = |v| = 1$. A similar consideration for the remaining above triple products gives us that these are:

1. $\langle xyz\rangle = ((xu)\beta(y))(vz)$;
2. $\langle xyz\rangle = (\beta(y)(xu))(vz)$;
3. $\langle xyz\rangle = ((zv)\beta(y))(ux)$;
4. $\langle xyz\rangle = (xu)((vz)\beta(y))$;
5. $\langle xyz\rangle = ((xu)(vz))\beta(y)$;
6. $\langle xyz\rangle = ((zv)(ux))\beta(y)$;

with $|u| = |v| = 1$ and $\beta : \mathbb{A} \to \mathbb{A}$ an isometry.

Consider now $\mathbb{A}$ with the second above triple product, that is, $\langle xyz\rangle = (\beta(y)(xu))(vz)$. For any $0 \neq y \in \mathbb{A}$, we have as in the above paragraph by considering $\frac{y}{|y|}$, that the y-homotopic algebra $\mathbb{A}^y$ has a unit. Therefore, for any $y \neq 0$, there exists $y' \neq 0$ such that $\langle xyy'\rangle = \langle y'yx\rangle = x$ and so $x = (\beta(y)(xu))(vy')$ for all $x \in \mathbb{A}$. From here, $x(vy')^{-1} = \beta(y)(xu)$ and taking $x = 1$, we have $(vy')^{-1} = \beta(y)u$, then $x(\beta(y)u) = \beta(y)(xu)$. By taking $x = u$ in the last equality we get $u\beta(y)u = \beta(y)u^2$ and since $\mathbb{A}$ is an alternative division algebra this implies $u\beta(y) = \beta(y)u$, hence u belongs to the center of $\mathbb{A}$ and therefore $u \in \{\pm 1\}$. The triple product can be written now as $\langle xyz\rangle = \epsilon(\beta(y)x)(vz)$ with $\epsilon \in \{\pm 1\}$ and $|v| = 1$. The strongly power-associative identity

$$\langle xy\langle xyx\rangle\rangle = \langle\langle xyx\rangle yx\rangle, \tag{1}$$

gives

$$(\beta(y)x)[v((\beta(y)x)(vx))] = (\beta(y)[(\beta(y)x)(vx)])(vx),$$

and for $x = 1$ we get

$$\beta(y)[v(\beta(y)v)] = (\beta(y)[\beta(y)v])v.$$

By applying Artin's theorem to $\mathbb{A}$, we have

$$\beta(y)v\beta(y)v = \beta(y)^2v^2,$$

so that after simplifying we get $v\beta(y) = \beta(y)v$ and therefore v is a unit norm central element, that is, $v \in \pm 1$. The triple product is finally $\langle xyz \rangle = \epsilon(\beta(y)x)z$ with $\epsilon \in \pm 1$. By applying (1), we obtain again using Artin's theorem that $x\beta(y) = \beta(y)x$ for any $x, y \in \mathbb{A}$ and β an isometry. This is a contradiction and we conclude that the triple product $\langle xyz \rangle = (\beta(y)(xu))(vz)$ is not the one of a strongly power-associative a.v.t.s.

The same arguments can be applied to the remaining triple products in the above list, up to the cases 1 and 3, to conclude that they do not belong to a strongly power-associative a.v.t.s. Summarizing, the possible triple products for $\mathbb{A}$ are either

$$\langle xyz \rangle = ((xu)\beta(y))(vz) \quad \text{or} \quad \langle xyz \rangle = ((zv)\beta(y))(ux) \tag{2}$$

with $|u| = |v| = 1$ and $\beta : \mathbb{A} \to \mathbb{A}$ an isometry. We note that if $\mathbb{A} = \mathbb{H}$, the associativity of $\mathbb{H}$ and the fact $L_u R_v \beta$ and $L_v R_u \beta$ are isometries, allow us to write (2) as

$$\langle xyz \rangle = x\beta(y)z \quad \text{or} \quad \langle xyz \rangle = z\beta(y)x \tag{3}$$

with $\beta : \mathbb{H} \to \mathbb{H}$ an isometry.

2.3.

Let consider now a 4-dimensional strongly power-associative a.v.t.s. T. Therefore, we identify T with $\mathbb{H}$ as euclidean vector spaces. In the sequel we shall use the standard basis $\{1, i, j, k\}$ of $\mathbb{H}$ such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$.

By identifying the Lie group S^3, (the 3-sphere), with the manifold $\{q \in \mathbb{H} : |q| = 1\}$, and taking into account that this is a compact connected Lie group, we know the existence of maximal tori in S^3. It is easy to prove that one of these maximal tori is given by the set $\{a + bi : a, b \in \mathbb{R}, a^2 + b^2 = 1\} \cong S^1$. As any element in a connect compact Lie group is conjugated to some element in a fixed maximal torus (see Bröcker and Dieck, 1985, (1.7) Main Lemma, p. 159), we get that any norm one quaternion x can be written as $x = \bar{p}\exp(\theta i)p$ for $\theta \in [0, 2\pi)$ and some $p \in \mathbb{H}$, $|p| = 1$. This fact will be useful later.

We now state the classification theorem for the 4-dimensional case.

Theorem 2.1. *Let T be a 4-dimensional strongly power-associative a.v.t.s. Then T is isomorphic to $\mathbb{H}$ with one and only one of the following triple products*:

1. $\langle xyz \rangle_1 = xyz$;
2. $\langle xyz \rangle_2 = -xyz$;
3. $\langle xyz \rangle_3 = x\bar{y}z$;
4. $\langle xyz \rangle_4 = -x\bar{y}z$;
5. $\langle xyz \rangle_5 = xi\bar{y}iz$.

Proof. First, it is easy to check that any of the above triple products endows $\mathbb{H}$ with a strongly power-associative a.v.t.s. structure.

Second, by 2.2, the triple product of T is one of the two ones given in equation (3). If we denote by $\mathcal{T}$ and $\mathcal{T}'$ the families of a.v.t.s. $\mathcal{T} := \{(\mathbb{H}, \langle\ \ \rangle, \beta)$

with $\langle xyz\rangle = x\beta(y)z\}$ and $\mathcal{T}' := \{(\mathbb{H}, \langle\ \ \rangle', \beta')$ with $\langle xyz\rangle' = z\beta'(y)x\}$; $\beta, \beta' : \mathbb{H} \to \mathbb{H}$ isometries, by arguing as in 2.2 we have that each triple system in $\mathcal{T}$ is isomorphic to one in $\mathcal{T}'$ and reciprocally. Therefore, our study is reduced to the triple product $\langle xyz\rangle = x\beta(y)z$. It is well known that β is either of the form $\beta(x) = axb$ or $\beta(x) = a\bar{x}b$, for some elements $|a| = |b| = 1$. Therefore, we have two possibilities for $\langle\ \ \rangle$:

1. $\langle xyz\rangle = xaybz$; or
2. $\langle xyz\rangle = xa\bar{y}bz$.

In the first case, the strongly power-associative identity

$$\langle\langle xyx\rangle yx\rangle = \langle x\langle yxy\rangle x\rangle \tag{4}$$

gives $bxa = axb$ for any $x \in \mathbb{H}$. From here, $b^{-1}ax = xab^{-1}$ and, in particular, $b^{-1}a = ab^{-1}$. Then $b^{-1}ax = xb^{-1}a$, so $b^{-1}a$ belongs to the center of $\mathbb{H}$ and therefore $b = \pm a$. We conclude $\langle xyz\rangle = \epsilon xayaz$, $\epsilon \in \{\pm 1\}$. We can verify that the map L_a gives us an isomorphism between the a.v.t.s. $(\mathbb{H}, \langle\ \ \rangle)$ and the a.v.t.s. $(\mathbb{H}, \langle\ \ \rangle')$, where $\langle\ \ \rangle'$ is given by $\langle xyz\rangle' := \epsilon xyz$. From here, we obtain types 1 and 2 of the theorem.

In the second case, (4) gives $bxa = \bar{b}x\bar{a}$ for any $x \in \mathbb{H}$. From here, $b^2x = xa^{-2}$ and, by taking $x = 1$, $b^2 = a^{-2}$. Then $b^2x = xb^2$ for any $x \in \mathbb{H}$, and so $b^2 = a^2 \in \{\pm 1\}$. If $b^2 = a^2 = 1$, then $a, b \in \{\pm 1\}$ and T is isomorphic either to type 3 or 4 of the theorem. Finally, if $b^2 = a^2 = -1$, the comments at the beginning of 2.3 allow us to assert that there exist $p, q \in \mathbb{H}$, $|p| = |q| = 1$ such that $a = \bar{p}\exp(\theta i)p$ and $b = \bar{q}\exp(\zeta i)q$ for certain $\theta, \zeta \in [0, 2\pi)$. Now, we can check that the map $L_q R_{\bar{p}}$ gives us an isomorphism between our a.v.t.s. $(\mathbb{H}, \langle\ \ \rangle)$, $\langle xyz\rangle = xa\bar{y}bz$, and the a.v.t.s. $(\mathbb{H}, \langle\ \ \rangle')$, where $\langle\ \ \rangle'$ is given by $\langle xyz\rangle' := x\exp(\theta i)\bar{y}\exp(\zeta i)z$. As $a^2 = b^2 = -1$ implies $\exp(\theta i)^2 = \exp(\zeta i)^2 = -1$, we obtain $\exp(\theta i), \exp(\zeta i) \in \{\pm i\}$ and then the triple product can be written as $\langle xyz\rangle = \epsilon xi\bar{y}iz$, $\epsilon \in \{\pm 1\}$. If we now observe that the map L_{-j} is an isomorphism between $\mathbb{H}$ with the triple product $\langle xyz\rangle = xi\bar{y}iz$ and $\mathbb{H}$ with the triple product $\langle xyz\rangle = -xi\bar{y}iz$, we obtain type 5 of the theorem.

Finally, the five triple systems of the theorem are not isomorphic since $\mathrm{tri}(\mathbb{H}, \langle\ \ \rangle_1) = \{\pm 1\}$, $\mathrm{tri}(\mathbb{H}, \langle\ \ \rangle_2) \cong S^2$, $\mathrm{tri}(\mathbb{H}, \langle\ \ \rangle_3) = S^3$, $\mathrm{tri}(\mathbb{H}, \langle\ \ \rangle_4) = \emptyset$ and $\mathrm{tri}(\mathbb{H}, \langle\ \ \rangle_5) \cong S^1$. □

2.4.

Let us finally study the 8-dimensional case. So, in this paragraph we identify T with $\mathbb{O}$ as euclidean vector spaces. By 2.2, the triple product in $\mathbb{O}$ is one of the two ones given in equation (2). Let us suppose $\langle xyz\rangle = ((xu)\beta(y))(vz)$, with $u, v \in \mathbb{O}$, $|u| = |v| = 1$ and $\beta : \mathbb{O} \to \mathbb{O}$ an isometry. Taking into account that for any $0 \neq y \in \mathbb{O}$, the y-homotopic algebra has a unit e so that $x = [(xu)\beta(y)](ve)$. Then $x(ve)^{-1} = (xu)\beta(y)$ and for $x = 1$ we get $(ve)^{-1} = u\beta(y)$. Consequently, $x(u\beta(y)) = (xu)\beta(y)$, implying that u associates with any two elements in $\mathbb{O}$. Thus $u = \pm 1$ and the triple product can be written now as $\langle xyz\rangle = \epsilon(x\beta(y))(vz)$, $\epsilon = \pm 1$.

We now note that, by applying the Moufang identity $u(vwv) = ((uv)w)v$, which holds in any alternative algebra (see Zhevlakov et al., 1982, p. 35), we can prove the equality

$$R_a(xy) = (R_a(x)a^{-2})(aR_a(y)), \tag{5}$$

for any $x, y, a \in \mathbb{O} - \{0\}$. We assert that our a.v.t.s. $(\mathbb{O}, \langle\ \ \rangle)$ with $\langle xyz\rangle = \epsilon(x\beta(y))(vz)$, is isomorphic to an a.v.t.s. $(\mathbb{O}, \langle\ \ \rangle')$ where $\langle xyz\rangle' := \epsilon(x\beta'(y))z$ for an adequate isometry β'. Indeed, if we define $\beta' := L_v R_v \beta R_v$, then equality (5) gives us

$$\begin{aligned} R_{v^{-1}}(\langle xyz\rangle) &= \epsilon R_{v^{-1}}((x\beta(y))(vz)) = \epsilon(R_{v^{-1}}(x\beta(y))v^2)(v^{-1}R_{v^{-1}}(vz)) \\ &= \epsilon R_v(x\beta(y))R_{v^{-1}}(z) = \epsilon((R_v(x)v^{-2})(vR_v(\beta(y))))R_{v^{-1}}(z) \\ &= \epsilon(R_{v^{-1}}(x)(L_vR_v\beta R_vR_{v^{-1}}(y)))R_{v^{-1}}(z) = \langle R_{v^{-1}(x)}R_{v^{-1}}(y)R_{v^{-1}}(z)\rangle' \end{aligned}$$

and so $R_{v^{-1}}$ is an isomorphism between the above a.v.t.s. Summarizing, we can assert that T is isomorphic to $\mathbb{O}$ with the triple product $\langle xyz\rangle = \epsilon(x\beta(y))z$, $\epsilon \in \{\pm 1\}$ and β an isometry of $\mathbb{O}$.

In order to study β, we recall that given a Jordan algebra J with quadratic operator $x \to U_x$ (see Loos, 1975), the *structure group* of J, $\mathrm{Str}(J)$, is defined as the subgroup of the group $\mathrm{GL}(J)$ of linear automorphisms of J for which there is a $f^\sharp \in \mathrm{GL}(J)$ satisfying $U_{f(x)} = fU_xf^\sharp$ for any $x \in J$ (see Springer, 1973). Let V be a $\mathbb{K}$-vector space endowed with a nondegenerate quadratic form $Q : V \to \mathbb{K}$, and fix $c \in V$ such that $Q(c) = 1$. Define $\bar{x} := Q(x, c)c - x$ and

$$U_x y := Q(x, \bar{y})x - Q(x)\bar{y}, \tag{6}$$

where $Q(x, y) := Q(x + y) - Q(x) - Q(y)$ for all $x, y \in V$. Then V with the quadratic operator $x \mapsto U_x$ becomes a Jordan algebra with unit c, which is called *the Jordan algebra of Q based at c* and is denoted by $\mathrm{Jor}(V, Q, c)$. Its structure group, see Springer (1973, 2.28 Corollary, p. 82), is the generalized orthogonal group which is the group of all $f \in \mathrm{GL}(V)$ such that there exists $\lambda_f \in \mathbb{K}^\times$ satisfying $Q(f(x)) = \lambda_f Q(x)$ for any $x \in V$, it is denoted by $\mathrm{GO}(V, Q)$. If V is finite dimensional, any $f \in \mathrm{GL}(V)$ has an adjoint map $f^\square$ which satisfies $Q(f(x), y) = Q(x, f^\square(y))$ for any $x, y \in V$. If moreover $f \in \mathrm{GO}(V, Q)$, as $Q(f(x), f(y)) = \lambda_f Q(x, y)$ we have

$$f^\square = \lambda_f f^{-1}. \tag{7}$$

By (6) and (7) we can write for any $x, y \in V$:

$$\begin{aligned} U_{f(x)}y &= Q(f(x), \bar{y})f(x) - Q(f(x))\bar{y} \\ &= Q(x, f^\square(\bar{y}))f(x) - \lambda_f Q(x)\bar{y} = f(Q(x, f^\square(\bar{y}))x - \lambda_f Q(x)f^{-1}(\bar{y})) \\ &= f(Q(x, f^\square(\bar{y}))x - Q(x)f^\square(\bar{y})) = fU_x(\overline{f^\square(\bar{y})}). \end{aligned}$$

From here, taking into account $U_{f(x)} = fU_xf^\sharp$, we have $f^\sharp(x) = \overline{f^\square(\bar{x})}$ and then (7) gives us

$$f^\sharp(x) = \lambda_f(\overline{f^{-1}(\bar{x})}). \tag{8}$$

Let us consider finally our a.v.t.s. $(\mathbb{O}, \langle\ \ \rangle)$ with $\langle xyz\rangle = \epsilon(x\beta(y))z$. Identity (4) gives $\beta(y\beta(x)y) = \beta(y)x\beta(y)$ which, by denoting $U_yx = yxy$, reads

$$\beta U_y \beta = U_{\beta(y)} \tag{9}$$

for any $y \in \mathbb{O}$. If we define in $\mathbb{O}$, $Q(x) := x\bar{x}$, then $\mathrm{Jor}(\mathbb{O}, Q, 1)$ is a Jordan algebra of the above type, being $U_y x = yxy$. Equation (9) gives us that $\beta \in \mathrm{Str}(\mathrm{Jor}(\mathbb{O}, Q, 1))$ with $\beta^\sharp = \beta$, and the isometric character of β together with the fact $Q(x) = x\bar{x}$, imply that $\lambda_\beta = 1$. Finally, by applying (8), we conclude $\beta(x) = \overline{\beta^{-1}(\bar{x})}$. Summarizing, if $\mathbb{O}$ is a strongly power-associative a.v.t.s. with the triple product in the first possibility given by (2), that is $\langle xyz\rangle = ((xu)\beta(y))(vz)$, then it is isomorphic to $\mathbb{O}$ with the triple product $\langle xyz\rangle = \epsilon(x\gamma(y))z$, where $\epsilon \in \{\pm 1\}$ and γ is an isometry satisfying $\gamma(x) = \overline{\gamma^{-1}(\bar{x})}$. Moreover, taking into account the above general considerations on GO(V,Q), it is easy to check that any a.v.t.s. of this form is strongly power-associative. Finally, we may suppose that ϵ is absorbed by the isometry γ so that the triple product is given by $\langle xyz\rangle = (x\gamma(y))z$. If we consider $\mathbb{O}$ with the second possibility for the triple product given by (2), the same argument developed in the first possibility, (but taking in this case the isomorphism $\phi R_{v^{-1}}\phi$, $\phi(x) := -\bar{x}$ instead of $R_{v^{-1}}$), allows us to assert the following theorem.

Theorem 2.2. *Any 8-dimensional strongly power-associative a.v.t.s. is isomorphic to $\mathbb{O}$ with one of the triple products*

Type 1. $\langle xyz\rangle = (x\beta(y))z$,
Type 2. $\langle xyz\rangle = (z\beta(y))x$,

where $\beta : \mathbb{O} \to \mathbb{O}$ is an isometry satisfying $\beta(x) = \overline{\beta^{-1}(\bar{x})}$ for any $x \in \mathbb{O}$.

Let us now study the isomorphism problem for the above a.v.t.s.

3. ISOMORPHISM CONDITION

3.1.

In this section we shall need Elduque (2000, Lemma 1.3, p. 53) which states that for a Cayley Dickson algebra C, and any elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{GL}(C)$ satisfying $\alpha_1(x)\alpha_2(y) = \beta_1(x)\beta_2(y)$ for all $x, y \in C$, one has $\beta_i = \mu\alpha_i$ $(i = 1, 2)$ for a nonzero scalar in the ground field. It is easy to prove that the equality $\alpha_1(x)\alpha_2(y) = \beta_1(y)\beta_2(x)$ for all $x, y \in C$ is not possible.

Consider two 8-dimensional strongly power-associative a.v.t.s. of type 1, with triple products $\langle xyz\rangle_\beta = (x\beta(y))z$, and $\langle xyz\rangle_\gamma = (x\gamma(y))z$. Let f be an isomorphism from one to another so that

$$f(\langle xyz\rangle_\beta) = \langle f(x)f(y)f(z)\rangle_\gamma \tag{10}$$

for all x, y, z. We know that f is an isometry of the euclidean space $\mathbb{O}$. So we have two possibilities for f: either $f \in \mathrm{SO}(8)$ or the composition of f with the Cayley involution of $\mathbb{O}$ is an element in SO(8). Let us consider now the first possibility $f \in \mathrm{SO}(8)$. The Triality Principle claims that $f(xy) = f_1(x)f_2(y)$ for some $f_1, f_2 \in \mathrm{SO}(8)$ and for all $x, y \in \mathbb{O}$. Taking $y = 1$ we get $f(x) = f_1(x)f_2(1)$ for all x. Therefore $f_1(x) = f(x)u$ for any $x \in \mathbb{O}$, where $u = f_2(1)^{-1}$. In a similar way $f_2(y) = vf(y)$ for all y, being $v = f_1(1)^{-1}$. Thus

$$f(xy) = [f(x)u][vf(y)], \qquad \forall x, y \in \mathbb{O}. \tag{11}$$

Thus f is the first component of an Albert autotopy of $\mathbb{O}$ (see Albert, 1942 or Petersson, 2002, 2.5, (2.5.3), p. 169 for a more accessible reference). Replacing in (10) the values of the triple systems we get $f((x\beta(y))z) = [f(x)\gamma(f(y))]f(z)$ and applying (11) to the left hand side of this, we get $[f(x\beta(y))u][vf(z)] = [f(x)\gamma(f(y))]f(z)$. By applying the above-mentioned (Elduque, 2000, Lemma 1.3, p. 53), it follows that $vf(z) = \delta f(z)$, $\delta = \pm 1$ for all z. Hence $v = \delta$. Equation (11) can be rewritten as

$$f(xy) = \delta[f(x)u]f(y), \qquad \forall x, y \in \mathbb{O}. \tag{12}$$

Thus $f(x\beta(y))u = \delta f(x)\gamma(f(y))$ for all $x, y \in \mathbb{O}$. Applying again (12) we can write

$$\delta[(f(x)u)f(\beta(y))]u = \delta f(x)\gamma(f(y))$$

and one of Moufang's identities gives $\delta f(x)(uf(\beta(y))u) = \delta f(x)\gamma(f(y))$. Consequently $uf(\beta(y))u = \gamma(f(y))$. This last equation may be rewritten as

$$\gamma = T_u f\beta f^{-1} \tag{13}$$

where $T_u : \mathbb{O} \to \mathbb{O}$ is $T_u(x) := U_u x = uxu$ for all x. Now we would like to analyze the other possibility for f, namely that $x \mapsto f(\bar{x})$ is an element of SO(8). In this case the Triality Principle gives $f(xy) = [f(y)u][vf(x)]$ and equation (10) gives an absurd according to the results in the first paragraph of this section.

Therefore we have proved that if f is an isomorphism, then it is an element in SO(8) satisfying (12) and (13). Conversely, it is easy to prove that any such f is an isomorphism between the given triple products. Finally, we note that we can apply the above arguments to two 8-dimensional strongly power-associative a.v.t.s. of type 2 so as to obtain the same result; and that these arguments give us an absurd if we part from an isomorphism between two 8-dimensional strongly power-associative a.v.t.s. of different types. Summarizing these facts, we have the following proposition.

Proposition 3.1.

(a) *Two 8-dimensional strongly power-associative a.v.t.s. of different type are not isomorphic.*
(b) *Two eight-dimensional strongly power-associative a.v.t.s. of a same type,* $(\mathbb{O}, \langle\ \rangle_\beta)$ *and* $(\mathbb{O}, \langle\ \rangle_\gamma)$*, are isomorphic by an isomorphism*

$$f : (\mathbb{O}, \langle\ \rangle_\beta) \to (\mathbb{O}, \langle\ \rangle_\gamma)$$

if and only if f is an element in SO(8) *satisfying equations* (12) *and* (13) *above.*

Let now C be an alternative algebra. As in Tits and Weiss (2002, (36.5)) and Petersson (2002, 3.0, p. 172), we can denote by X_C the set of all bijective linear maps $\varphi : C \to C$ such that $\varphi(1) \in C^\times$ and $\varphi(xy) = [\varphi(x)\varphi(1)^{-1}]\varphi(y)$ for all $x, y \in C$. O. Loos uses these transformations in Loos (1975, §6) for studying the relationship between alternative pairs and algebras. The set X_C is a subgroup of GL(C) as it is proved in Tits and Weiss (2002, (36.10)). It is easy to prove that X_C acts on

$\mathrm{GL}(C)$ defining for each $f \in X_C$ and $\rho \in \mathrm{GL}(C)$ the action $f \cdot \rho := T_u f \rho f^{-1}$ where $u = f(1)^{-1}$ and, as above, $T_u := L_u R_u$. Indeed, taking $f, g \in X_C$ with $u = f(1)^{-1}$, $u' = g(1)^{-1}$, and $\rho \in \mathrm{GL}(C)$ one has

$$g \cdot (f \cdot \rho) = g \cdot (T_u f \rho f^{-1}) = T_{u'} g T_u f \rho f^{-1} g^{-1}.$$

But $gT_x = T_{g(x)} T_{u'} g$ for all x hence

$$g \cdot (f \cdot \rho) = T_{u'} T_{g(u)} T_{u'} g f \rho f^{-1} g^{-1} = T_{u' g(u) u'} g f \rho f^{-1} g^{-1} = (gf) \cdot \rho$$

since $(gf(1))^{-1} = u' g(u) u'$ as one can see easily computing $g(f(xy))$ and applying one of Moufang identities.

Returning now to our isomorphism f which satisfies (12), we see that $f \in X_{\mathbb{O}} \cap \mathrm{SO}(8)$ (since any isomorphism between a.v.t.s. is isometric). Thus $\|u\| = \|f(1)^{-1}\| = \|f(1)\| = 1$ hence the subgroup $X_{\mathbb{O}} \cap \mathrm{SO}(8)$ acts on $\mathrm{O}(8)$ by means of the previous action. We have proved furthermore that the triple systems $(\mathbb{O}, \langle\ \ \rangle_\beta)$ and $(\mathbb{O}, \langle\ \ \rangle_\gamma)$ are isomorphic by an isomorphism f if and only if $f \in \mathrm{SO}(8)$ satisfies (12) and (13). Thus the triples are isomorphic if and only if β and γ are in the same orbit of $\mathrm{O}(8)$ under the action just described.

In order to present this fact under a different light which also provides a more symmetric approach we must follow Petersson (2002). In that reference, the *structure group* of an alternative algebra C over a commutative ring of scalars k (denoted $\mathrm{Str}(C)$) is introduced as the set of triples $(v, u, \upsilon) \in \mathrm{GL}(C) \times C^\times \times C^\times$ such that $v(xy) = (v(x)u)(\upsilon v(y))$ for all $x, y \in C$. In Petersson (2002, Theorem 2.3) it is described the structure group of $\mathrm{Str}(C)$ and also its identification with the group of Albert autotopies of C defined as

$$\mathrm{Atp}(C) := \{(f, g, h) \in \mathrm{GL}(C)^3 : f(xy) = g(x)h(y),\ x, y \in C\}.$$

In Petersson (2002, Theorem 2.7, p. 170) it is proved that the map $\mathrm{Atp}(C) \to \mathrm{Str}(C)$ such that $(f, g, h) \mapsto (f, h(1)^{-1}, g(1)^{-1})$ is a group isomorphism. One important subgroup of $\mathrm{Str}(C)$ is

$$\mathrm{Str}_l(C) := \{(v, u, \upsilon) \in \mathrm{Str}(C) : \upsilon = 1\}.$$

It turns out that the map $X_C \to \mathrm{Str}_l(C)$ such that $\varphi \mapsto (\varphi, \varphi(1)^{-1}, 1)$ is a group isomorphism (Petersson, 2002, p. 173). The promised symmetry gaining comes from the subgroup

$$\mathrm{Str}_1(C) := \{(v, u, \upsilon) \in \mathrm{Str}(C) : \upsilon = u^{-1}\}$$

and the isomorphism $\mathrm{Str}_1(C) \to \mathrm{Str}_l(C)$ given by $(v, u, u^{-1}) \mapsto (R_u v, u^{-1}, 1)$ (see Petersson, 2002, Proposition 3.6, p. 174). Thus we get an identification of X_C with $\mathrm{Str}_1(C)$ by composing the previous isomorphisms. Suppose now that C is a composition algebra over a field k, relative to the norm $n : C \to k$. We can now define the subgroup

$$\mathrm{Str}_0(C) := \{(v, u, u^{-1}) \in \mathrm{Str}_1(C) : n(u) = 1\},$$

which is a normal subgroup of $\mathrm{Str}_1(C)$ (see Petersson, 2002, Proposition 4.1, p. 179). Following always the reference Petersson (2002), consider the vector representation $\chi : \mathrm{Spin}(C) \to \mathrm{O}(C)$ of the spin group. This induces an isomorphism

$$\mathrm{Spin}(C) \to \mathrm{Atp}(C) \cap \mathrm{O}(C)^3$$

and $\mathrm{Str}_0(C)$ is the subgroup of $\mathrm{Atp}(C) \cap \mathrm{O}(C)^3$ which under this isomorphism corresponds to the subgroup

$$\mathrm{Spin}_0(C) = \{x \in \mathrm{Spin}(C) : \chi(x)(1) = 1\}.$$

Particularizing now to $k = \mathbb{R}$, $C = \mathbb{O}$, the previous identification of $X_{\mathbb{O}}$ with $\mathrm{Str}_1(\mathbb{O})$ induces an identification of $X_{\mathbb{O}} \cap \mathrm{SO}(8)$ with $\mathrm{Str}_0(\mathbb{O})$ hence finally the group $X_{\mathbb{O}} \cap \mathrm{SO}(8)$ is identified with $\mathrm{Spin}_0(\mathbb{O})$. Moreover the action of $X_{\mathbb{O}} \cap \mathrm{SO}(8)$ on $\mathrm{O}(8)$ corresponds with the canonical action of $\mathrm{Spin}_0(\mathbb{O})$ on $\mathrm{O}(8)$. Consequently, we have the following theorem.

Theorem 3.2 (Isomorphism Condition). *Let* $(\mathbb{O}, \langle\ \rangle_{\beta_1})$ *and* $(\mathbb{O}, \langle\ \rangle_{\beta_2})$ *be triple systems, of a same type, as in Theorem* 2.2. *Then they are isomorphic if and only if the isometries* $\beta_1, \beta_1 \in \mathrm{O}(8)$ *are in the same orbit under the action of* $\mathrm{Spin}_0(\mathbb{O})$.

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