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Nonclassical symmetry reductions for an inhomogeneous nonlinear diffusion equation

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Abstract

In this paper we consider a class of generalised diffusion equations which are of great interest in mathematical physics. For some of these equations model, fast diffusion nonclassical symmetries are derived. We find the connection between classes of nonclassical symmetries of the equation and of an associated system. These symmetries allow us to increase the number of solutions. Some of these solutions are unobtainable by classical symmetries and exhibit an interesting behaviour.

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Keywords: Partial differential equations; Nonclassical symmetries; Potential symmetries

1. Introduction

In this paper we consider a mathematical model for diffusion processes which is the generalised inhomogeneous nonlinear diffusion equation

$$f(x)u_t = [g(x)u^n u_x]_x. \tag{1}$$

The diffusion processes appear in many physics processes such as plasma physics, kinetic theory of gases, solid state, metallurgy and transportation in porous medium [2,16,19].

In [19] Rosenau presented a number of remarkable features of the fast diffusion processes: for $f(x) = 1$, $g(x) = 1$ and $-2 \leqslant n \leqslant -1$, the family of fast diffusion (1) coexists with a subclass of superfast diffusions where the whole process terminates within a finite time. The special case with $n = -1$ emerges in plasma physics and reveals a surprising richness of new physics-mathematical phenomena.

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In (1), $u(x,t)$ is a function of position x and time t and may represent the temperature, $f(x)$ and $g(x)$ are arbitrary smooth functions of position and may denote the density and the density-dependent part of thermal diffusion, respectively.

There is no existing general theory for solving nonlinear partial differential equations and the methods of point transformations are a powerful tool. One of the most useful point transformations are those which form a continuous group. Lie classical symmetries admitted by nonlinear partial differential equations (PDE's) are useful for finding invariant solutions.

Motivated by the fact that symmetry reductions for many PDE's are known and are not obtained by using the classical Lie method, there have been several generalizations of the classical Lie group method for symmetry reductions.

Bluman and Cole [5] developed the nonclassical method to study the symmetry reductions of the heat equation. The basic idea of the method requires that the N order PDE

$$\Delta = \Delta(x,t,u,u^{(1)}(x,t),\ldots,u^{(N)}(x,t)) = 0, \tag{2}$$

where $(x,t) \in \mathbb{R}^2$ are the independent variables, $u \in \mathbb{R}$ is the dependent variable and $u^{(l)}(x,t)$ denote the set of all partial derivatives of l order of u and the invariance surface condition

$$\xi u_x + \tau u_t - \phi = 0, \tag{3}$$

which is associated with the vector field

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{4}$$

which are both invariant under the transformation with infinitesimal generator (4). Since then, a great number of papers have been devoted to the study of nonclassical symmetries of nonlinear PDE's in both one and several dimensions.

An obvious limitation of group-theoretic methods based local symmetries, in their utility for particular PDE's, is that many of these equations does not have local symmetries. It turns out that PDE's can admit nonlocal symmetries whose infinitesimal generators depend on integrals of the dependent variables in some specific manner.

Krasil'shchik and Vinogradov [14,15,20] gave criteria which must be satisfied by nonlocal symmetries of a PDE when realized as local symmetries of a system of PDE's which 'covers' the given PDE. Akhatov et al. [1] gave nontrivial examples of nonlocal symmetries generated by heuristic procedures.

In [6,7] Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by $R\{x,t,u\}$ in a conserved form, a related system denoted by $S\{x,t,u,v\}$ as additional dependent variables is obtained. Any Lie group of point transformations admitted by $S\{x,t,u,v\}$ induces a symmetry for $R\{x,t,u\}$; when atleast one of the generators of the group depends explicitly on the potential, then the corresponding symmetry is neither a point nor a Lie–Bäcklund symmetry. These symmetries of $R\{x,t,u\}$ are called *potential* symmetries.

Knowing that an associated system to the Boussinesq equation has the same classical symmetries as the Boussinesq equation, Clarkson [9] proposed as an open problem, an auxiliary system of the Boussinesq equation possesess more or less nonclassical symmetries than the equation itself. Bluman claims [4] that the ansatz to generate nonclassical solutions of the associated system could yield solutions of the original equation which are neither nonclassical solutions nor solutions arising from potential symmetries.

However as far as we know these new class of potential symmetries, which we have called *nonclassical potential* symmetries, were first derived in [11] for the Burgers equation and in [10] for the porous medium equation. After that we have derived *nonclassical potential symmetries*, in different ways for some interesting equations.

In [18], Sophocleous has classified the nonlocal potential symmetries of (1). He obtained that potential symmetries exist only if the parameter n takes the values -2 or $-\frac{2}{3}$ and also certain relations must be satisfied by the functions $f(x)$ and $g(x)$.

In [12], we have derived *nonclassical potential* symmetries for the special case of (1), with $f(x) = 1$ and $g(x) = 1$

$$u_t = [u^{-1} \quad u_x]_x. \tag{5}$$

In [17] connection between classes of nonclassical symmetries of (5), and of nonclassical symmetries of an associated system as well as some new generators have been found.

The aim of this paper is to obtain nonclassical symmetries for (1), with $n = -1$, and for the associated system given by

$$\begin{aligned} v_x &= f(x)u, \\ v_t &= g(x)u^{-1}u_x, \end{aligned} \tag{6}$$

as well as the connection between these symmetries. These symmetries lead to new solutions, some of these solution exhibit an interesting behaviour.

2. Nonclassical symmetries

2.1. Nonclassical symmetries of the PDE (1)

The basic idea of the method is that the PDE (1) is augmented with the invariance surface condition (3) which is associated with the vector field (4). By requiring that both (1) and (3) are invariant under the transformation with infinitesimal generator (4) one obtains an overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is in general, larger than for the classical method as in this method one requires only the subset of solutions of (1) and (3) to be invariant under the infinitesimal generator (4). However, the associated vector fields do not form a vector space.

To obtain nonclassical symmetries of (1), with $n = -1$, we apply the algorithm described in [8,9] for calculating the determining equations. We can distinguish two different cases:

In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,u) = 1$. The corresponding determining equations give rise to

$$\begin{aligned} \xi &= \xi(x,t), \\ \phi &= \alpha(x,t)u, \end{aligned}$$

where $\xi(x,t)$, $\alpha(x,t)$, $f(x)$ and $g(x)$ are related by the following conditions:

$$-\xi\alpha + \xi^2\left(\frac{g'}{g} - \frac{f'}{f}\right) - 2\xi\xi_x - \xi_t = 0, \tag{7}$$

$$-\xi\left(\frac{g''}{g} - \frac{g'^2}{g^2}\right) - \xi_x\frac{g'}{g} + \xi_{xx} = 0, \tag{8}$$

$$\alpha_{xx} + \frac{g'}{g}\alpha_x = 0, \tag{9}$$

$$2\alpha\xi_x - \alpha\xi\left(\frac{g'}{g} - \frac{f'}{f}\right) + \alpha_t + \alpha^2 = 0. \tag{10}$$

By solving these equations we obtain

$$\alpha = \delta(t)(\varphi(x) + k_1), \tag{11}$$

$$\xi = -g(x)\delta(t)(k_3\varphi(x) - k_1k_2), \tag{12}$$

where

$$\varphi(x) = \int \frac{1}{g(x)}\,\mathrm{d}x, \tag{13}$$

and f can be derived from the following condition

$$\frac{f'}{f}=\frac{\varphi'\delta'+\delta^2(k_3(\varphi\varphi''-2\varphi'^2)+k_1(\varphi'^2-k2\varphi'')+\varphi\varphi'^2)}{\delta^2\varphi'(k_3\varphi-k_1k_2)}. \tag{14}$$

As f depends only on x, $f=f(x)$, δ must adopt one of the following forms:

$$\delta=\frac{k}{t}\quad\text{or}\quad\delta=k.$$

These generators can be obtained by Lie classical method and consequently the nonclassical method with $\tau\neq 0$ applied to (1) gives only rise to the classical symmetries.

In the case $\tau=0$, without loss of generality, we may set $\xi=1$ and by applying the extended version of Bîlă and Niesen procedure [3,8] the surface condition (3) is written as

$$u_x=\phi.$$

By substituting this expression into the PDE (1) we get

$$f(x)u_t=\mathscr{A}(x,t,u), \tag{15}$$

where

$$\mathscr{A}(x,t,u)=g(x)((\phi_x+\phi_u\phi)u^n+n\phi^2u^{n-1})+g'\phi u^n. \tag{16}$$

Since in (15) the coefficients $\mathscr{A}(x,t,u)$ can be viewed as arbitrary functions, we can apply the Lie classical method to this equation, and setting $\tau=0$ and $\xi=1$ we obtain the following PDE linear in ϕ

$$f\mathscr{A}\phi_u+f^2\phi_t-\mathscr{A}_uf\phi+f'\mathscr{A}-f\mathscr{A}_x=0. \tag{17}$$

If we substitute the expression of $\mathscr{A}$ (16) given above we get that the nonclassical determining equation for the infinitesimal ϕ is

$$\begin{aligned}&u^{(n+2)}(fg\phi_{xx}+2fg'\phi_x-f'g\phi_x+fg\phi^2\phi_{uu}+2fg\phi\phi_{ux}+fg'\phi\phi_u-f'g\phi_u+fg''\phi-f'g'\phi)\\&\quad+n\phi u^{(n+1)}(3fg\phi_x+fg\phi\phi_u+2fg'\phi-f'g\phi)+fg(n-1)n\phi^3u^n-f^2\phi_tu^2=0.\end{aligned} \tag{18}$$

The complexity of this equation is the reason why we cannot solve (18) in general. Thus we proceed, by making an ansatz on the form of $\phi(x,t,u)$, to solve (18) for $n=-1$. In this way we found, choosing $\phi=\alpha(x,t)u^2+\beta(x,t)u$, after substituting into the determining equation and splitting with respect to u we obtain an overdetermined system for the functions α and β. So, for Eq. (1) with $n=-1$, we obtain the infinitesimal generator

$$\mathbf{v}=\partial_x+(\alpha(x,t)u^2+\beta(x,t)u)\partial_u,$$

where f, g, α and β satisfy the system

$$\begin{aligned}&fg'\alpha^2-fg\alpha\alpha_x+f^2\alpha_t=0,\\&f'g'\alpha-fg''\alpha^2+(fg'+f'g)\alpha\beta+(f'g-2fg')\alpha_x+fg(\alpha\beta_x-\alpha_x\beta-\alpha_{xx})+f^2\beta_t=0,\\&f\beta\beta_x-fg\beta_{xx}+f'g\beta_x=0,\\&\beta(f'g'-fg'')+\beta_x(f'g-2fg')+\beta^2fg'+fg(\beta\beta_x-\beta_{xx})=0.\end{aligned} \tag{19}$$

2.2. Nonclassical symmetries of the system (6)

We now consider the associated auxiliary system (6) augmented with the invariance surface condition

$$\xi v_x+\tau v_t-\psi=0, \tag{20}$$

which is associated with the vector field

$$\mathbf{w}=\xi(x,t,u,v)\partial_x+\tau(x,t,u,v)\partial_t+\phi(x,t,u,v)\partial_u+\psi(x,t,u,v)\partial_v. \tag{21}$$

By requiring both (6) and (20) to be invariant under the transformation with infinitesimal generator (21) one obtains an over determined, nonlinear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\phi(x,t,u,v)$, $\psi(x,t,u,v)$. When atleast one of the generators of the group depend explicitly on the potential, that is if

$$\xi_v^2+\tau_v^2+\phi_v^2\neq 0, \tag{22}$$

then (21) yields a nonlocal symmetry of (1).

A nonclassical potential symmetry of (1) is a nonclassical symmetry of the associated potential system (6) that satisfies (22).

We are considering $\tau\neq 0$, and without loss of generality, we set $\tau=1$. The nonclassical method, with $\tau\neq 0$, applied to (6), gives rise to nonlinear determining equations for the infinitesimals.

If we require that $\xi_u=\psi_u=0$, we obtain that

$$\phi=-f\xi_v u^2+\left(\psi_v-\frac{f_x\xi}{f}-\xi_x\right)u+\frac{\psi_x}{f}, \tag{23}$$

and $f(x)$, $g(x)$, $\xi(x,t,v)$ and $\psi(x,t,v)$, must satisfy the following equations:

$$g\xi_{vv}-\xi\xi_v=0, \tag{24}$$

$$-fg\xi\xi_x+2fg^2\xi_{vx}+fg\psi\xi_v+2f'g^2\xi_v-fg\xi_t+fg'\xi^2-fg\psi_v\xi-fg^2\psi_{vv}=0, \tag{25}$$

$$f^2g^2\xi_{xx}+f^2g\psi\xi_x+ff'g^2\xi_x-f^2g\psi_x\xi-f^2g'\psi\xi+ff''g^2\xi-f'^2g^2\xi-2f^2g^2\psi_{vx}+f^2g\psi\psi_v+f^2g\psi_t=0, \tag{26}$$

$$-fg\psi_{xx}+f\psi\psi_x+f'g\psi_x=0. \tag{27}$$

We can distinguish the following cases:

If $\xi_v\neq 0$ by solving (24) and substituting into (25)–(27) leads to generators for which (22) is satisfied, consequently they are nonclassical potential generators and have been considered in [13].

If ξ does not depend on v, by substituting $\xi=\xi(x,t)$ in (25) and (26) we obtain that

$$\psi=v\left(-\xi_x-\frac{\xi_t}{\xi}+\frac{g'\xi}{g}\right)+g\left(\frac{\xi_x}{\xi}+\frac{\xi_t}{\xi^2}\right)-g'+\theta(x,t), \tag{28}$$

By substituting (28) into (27) we obtain that

$$\psi=\delta(t)\left(v-\frac{g}{\xi}\right)+\theta(x,t). \tag{29}$$

By substituting into (23) we get that $\phi_v=0$. We observe that in this case condition (22) is not satisfied, consequently

$$\mathbf{w}=\xi(x,t)\partial_x+\partial_t+\left(\left(\delta(t)-\frac{f_x\xi}{f}-\xi_x\right)u+\frac{\psi_x}{f}\right)\partial_u+\psi\partial_v, \tag{30}$$

is not a nonclassical potential generator.

2.3. Connection between symmetries of the PDE (1) and of the system (6)

If we assume that ξ and ψ do not depend on v, the system (24)–(27) becomes

$$\begin{aligned}&-g\xi\xi_x-g\xi_t+g'\xi^2=0,\\&f^2g^2\xi_{xx}+f^2g\psi\xi_x+ff'g^2\xi_x-f^2g\psi_x\xi-f^2g'\psi\xi+ff''g^2\xi-f'^2g^2\xi+f^2g\psi_t=0,\\&-fg\psi_{xx}+f\psi\psi_x+f'g\psi_x=0.\end{aligned} \tag{31}$$

It is easy to check that denoting $\alpha=-\frac{f}{g}\xi$, $\beta=\frac{\psi}{g}$, systems (19) and (31) coincides. Consequently we can state:

$$\mathbf{w}=\xi(x,t)\partial_x+\partial_t+\psi(x,t)\partial_v-\left(\frac{f'}{f}\xi u+\frac{\psi_x}{f}\right)\partial_u,$$

is a generator for system (6) *if and only if*

$$\mathbf{v} = \partial_x + \left(-\frac{f}{g}\xi u^2 + \frac{\psi}{g}u\right)\partial_u,$$

is a generator for Eq. (1).

3. Some exact solutions

In this section we derive some exact solutions by using some generators:

(1) From generator

$$\xi = k_4\sqrt{x}, \quad \psi = 0, \tag{32}$$

for $f = \frac{k_3k_4}{2\sqrt{x}}e^{\frac{k_1k_4\sqrt{x}}{k_2}}$ and $g = \frac{2k_2\sqrt{x}}{k_4}$, and the surface condition we obtain the similarity variable and similarity solution

$$z = k_4t - 2\sqrt{x}, \quad v = h(z),$$

and the ODE

$$2k_2h'' + k_1k_4h' + k_4^2h'^2 = 0.$$

The solution is

$$h' = \frac{k_1e^{\frac{k_1k_4z}{2k_2}+\frac{k_1k_4k_5}{2k_2}} + k_1k_4}{e^{\frac{k_1k_4z}{k_2}+\frac{k_1k_4k_5}{k_2}} - k_4^2}.$$

From (6) we obtain for (1) the exact solution

$$u = \frac{2k_1}{k_3k_4\left(k_4e^{\frac{k_1k_4\sqrt{x}}{k_2}} - e^{\frac{k_1k_4^2t}{2k_2}+\frac{k_1k_4k_5}{2k_2}}\right)}. \tag{33}$$

We observe that the solution (33), for $x = \frac{(k_4t+k_5)^2}{4}$, blows up at a parabola. In Fig. 1 we plot (33) with $k_i = 1$, $i = 1, \ldots, 5$.

(2) From generator

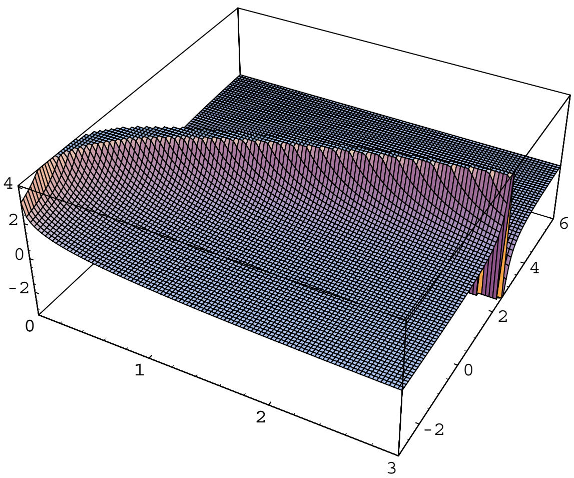

Fig. 1. Solution (33) for $f = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ and $g = 2\sqrt{x}$.

$$\xi = k, \quad \psi = \frac{-2}{x+kt},$$

for $g = 1, f = e^x$ and the surface condition we obtain the independent variable

$$z = x - kt,$$

and the similarity solution

$$v = h(z) - \frac{1}{k}\log(x+kt). \tag{34}$$

By introducing (34) into (6) we have h satisfying $h'' + kh'^2 = 0$, consequently a solution of (1) is given by

$$u = e^{-x}\left(\frac{1}{x-t+k_1} - \frac{1}{x+t}\right). \tag{35}$$

We observe that solution (35) blows up into two straight lines $x - t + k_1 = 0$ and $x + t = 0$. In Fig. 2 we plot (35) with $k_1 = 1$.

(3) From generator

$$\xi = \frac{1}{f}, \quad \psi = -2k_1 \tanh\left[k_1\left(t + \int f(x)\,dx\right)\right],$$

for $g = \frac{1}{f}$, and the surface condition we obtain the independent variable

$$z = \int f(x)\,dx - t,$$

and the similarity solution

$$v = h(z) - 2\log\left(\cosh\left(k_1\left(t + \int f(x)\,dx\right)\right)\right). \tag{36}$$

By introducing (36) into (6) we have that a solution of (1) is given by

$$u = h'(z) - k_1 \tanh\left(k_1\left(t + \int f(x)\,dx\right)\right). \tag{37}$$

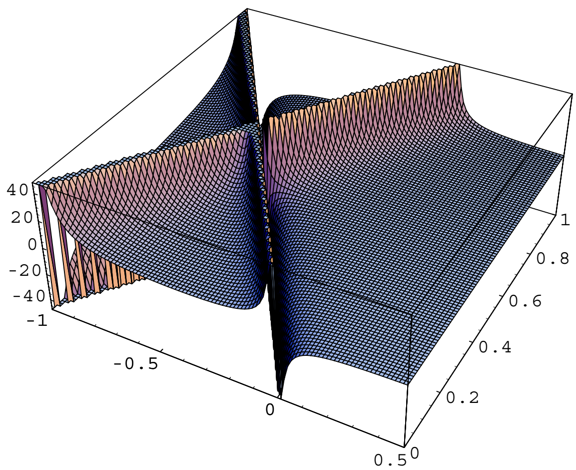

Fig. 2. Solution (35) for $f = e^x$, $g = 1$.

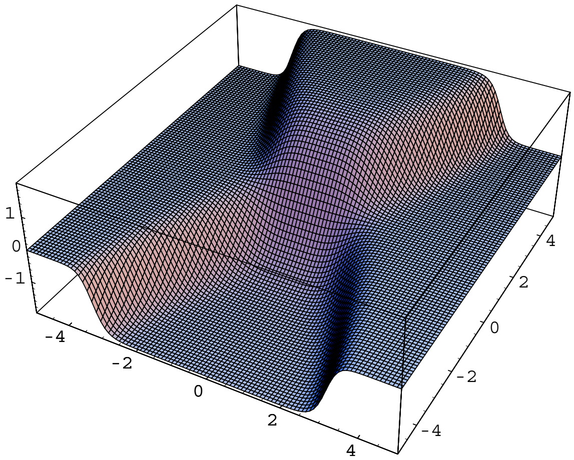

Fig. 3. Solution (38) for $g = \frac{1}{f}$, $f(x) = x$.

By introducing (37) into (1) we arrive at a reduced equation for the dependent similarity variable, and, for $k_1 = 1$, an explicit solution of (1) is the bounded solution

$$u = -k_1 \tanh\left(k_1\left(t + \int f(x)\,\mathrm{d}x\right)\right) - k_1 \tanh\left(\left(k_1 t - \int f(x)\,\mathrm{d}x\right)\right). \tag{38}$$

In Fig. 3 we plot (38) which represents two front waves that evolve changing their shape with opposite velocities with $f(x) = x$, $k_1 = 1$.

It was pointed out in [18] that the transformation

$$x' = \int \frac{\mathrm{d}x}{g(x)} = G(x), \quad t' = t, \quad u' = u, \tag{39}$$

connects Eq. (1), and the PDE

$$g(G^{-1}(x'))f(G^{-1}(x'))u'_{t'} = [u'^n u'_{x'}]_{x'}, \tag{40}$$

where G^{-1} is the inverse function of G.

It is clear that we can equivalently use an equation of the form

$$f(x')u'_{t'} = [u'^n u'_{x'}]_{x'}, \tag{41}$$

and then transform the results for Eq. (1) by using the corresponding point transformation.

We also observe that the transformation

$$x' = x, \quad t' = t, \quad u' = \frac{u}{f}, \tag{42}$$

connects Eq. (41), and the PDE (5) when $f = k_2 e^{k_1 x}$ and $n = -1$.

Consequently, it is clear that we can equivalently use for the subsequent analysis an equation of the form (5), which appear in [12] and then transform the results for equation (1) by using the transformations (39) and (42).

4. Concluding remarks

We have considered some inhomogeneous diffusion equations (1). If these equations are written in a conserved form, then a related system (6) may be obtained. Classical potential symmetries for these equations have been derived in [18]. In this paper, for $n = -1$ we have derived *nonclassical* symmetries for (1) and for (6). We have proved that the nonclassical method with $\tau \neq 0$ applied to (1) gives only rise to classical symmetries. For $\tau = 0$ we have applied an extension [8] of the Bîlă and Nielsen procedure [3]. We found the connection between classes of nonclassical symmetries of (1) and classes of nonclassical symmetries of (6). We have proved that (1), when the parameter n takes the value -1, that is when the equation model has fast diffusion, admits nonclassical symmetries that yield new solutions. Some of these solutions are unobtainable by classical symmetries an exhibit and interesting behaviour.

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