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Pseudoinvexity, optimality conditions and efficiency in multiobjective problems; duality

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Abstract

In this paper, we establish characterizations for efficient solutions to multiobjective programming problems, which generalize the characterization of established results for optimal solutions to scalar programming problems. So, we prove that in order for Kuhn–Tucker points to be efficient solutions it is necessary and sufficient that the multiobjective problem functions belong to a new class of functions, which we introduce. Similarly, we obtain characterizations for efficient solutions by using Fritz–John optimality conditions. Some examples are proposed to illustrate these classes of functions and optimality results. We study the dual problem and establish weak, strong and converse duality results.

Keywords: Multiobjective programming; Invexity; Pseudoinvexity; Kuhn–Tucker and Fritz–John optimality conditions; Efficient solutions

1. Introduction and preliminaries

The search for solutions to mathematical programming problems has been carried out through the study of optimality conditions and of the properties of the functions that are involved, as well as through the study of dual problems. In the case of optimality conditions, it is customary to use critical points of the Kuhn–Tucker or Fritz–John [12] types. In the case of the kinds of functions employed in mathematical programming problems, the tendency has been to replace convex functions with more general ones, with the objective of obtaining a solution through an optimality condition. Meanwhile the inverse result has also sometimes been sought.

With the introduction of the invex function, by Hanson [8] and Craven [5], Craven and Glober [6] established the equivalence between a global minimum of a scalar function and a stationary point; and besides, this characterizes the invex functions (see [6] and [1]). Several authors have generalized convexity and invexity, and they have continued

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the study of optimal solutions for scalar problems ([3,9,17], ...), formulated as follows:

(CP) Minimize $\theta(x)$
subject to:
$g(x) \leqq 0$
$x \in S \subseteq R^n$

with $\theta : S \subseteq R^n \to R$, $g = (g_1, \ldots, g_m) : S \subseteq R^n \to R^m$, $\theta, g_1, \ldots, g_m$ are differentiable functions on the open set $S \subseteq R^n$.

For constrained scalar problems (CP), invexity is a sufficient condition for which a critical point leads to a solution of (CP), but it is not necessary. Martin [11] defined a weaker concept: KT-invexity; and proved that it is a necessary and sufficient condition in order for a Kuhn–Tucker critical point to be an optimal solution of (CP).

Definition 1. Problem (CP) is said to be KT-invex if there exists a function $\eta : S \times S \to R^n$ such that $\forall x_1, x_2 \in S$, with $g_i(x_1) \leq 0$, $g_i(x_2) \leq 0$, $i = 1, \ldots, m$,

$$\theta(x_1) - \theta(x_2) \geq \nabla\theta(x_2)\eta(x_1, x_2),$$
$$-\nabla g_j(x_2)\eta(x_1, x_2) \geq 0, \quad \forall j \in I(x_2),$$

where

$$I(x_2) = \{j : j = 1, \ldots, m \text{ such that } g_j(x_2) = 0\}.$$

Martin [11] obtained the following result:

Theorem 1. *Every Kuhn–Tucker critical point is an optimal solution of* (CP) *if and only if* (CP) *is KT-invex.*

The following convention for equalities and inequalities will be used.
If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in R^n$, then

$$x = y \Leftrightarrow x_i = y_i, \quad \forall i = 1, \ldots, n.$$
$$x < y \Leftrightarrow x_i < y_i, \quad \forall i = 1, \ldots, n.$$
$$x \leqq y \Leftrightarrow x_i \leq y_i, \quad \forall i = 1, \ldots, n.$$
$$x \leq y \Leftrightarrow x \leqq y, \quad \text{there exists } i \text{ such that } x_i < y_i.$$

In this paper we will centre on the problem of multiobjective programming with constraints:

(CMP) Minimize $f(x)$
subject to:
$g(x) \leqq 0$
$x \in S \subseteq R^n$

where $f = (f_1, \ldots f_p) : S \subseteq R^n \to R^p$, $g = (g_1, \ldots, g_m) : S \subseteq R^n \to R^m$ are differentiable functions on the open set $S \subseteq R^n$.

We will move forward in the study and location of efficient solutions of (CMP), whose concept was introduced by Pareto [16].

Definition 2. A feasible point, $\bar{x}$, is said to be an efficient solution of (CMP) if there does not exist another feasible point, x, such that $f(x) \leq f(\bar{x})$.

There appears a more general concept such as that of the weakly efficient solution of (CMP).

Definition 3. A feasible point, $\bar{x}$, is said to be a weakly efficient solution of (CMP) if there does not exist another feasible point, x, such that $f(x) < f(\bar{x})$.

Following the same lines as scalar problems, the outline is to obtain classes of functions that make up constrained multiobjective problems (CMP), such that any class of functions which is characterized by having every critical point as an efficient solution of (CMP) must be equivalent to these classes of functions. So it is a question of extending, amongst others, the kind of KT-invex functions introduced by Martin [11], as well as the results obtained by him. And in order to do that, we are going to use Fritz–John and Kuhn–Tucker vector critical points as optimality conditions for multiobjective problems. In Section 2, we will introduce new kinds of functions based on generalized invexity for multiobjective programming problems with constraints, in which joint properties of the objective function and the functions which are involved in the constraints of the problem are formed. Along similar lines, Osuna et al. [14,15] studied weakly efficient solutions for (CMP), and provided a new class of functions which extended the KT-invex functions [11] to the multiobjective case and proved that they are characterized by all Kuhn–Tucker vector critical points being weakly efficient solutions. In Section 3 of this paper, we will extend this study to efficient points. In Section 4, we propose some examples to illustrate new classes of functions and results obtained. Finally, in Section 5, we will conclude by studying the duality of the problem (CMP) and the Mond–Weir type dual problems.

2. Optimality conditions. KT/FJ-pseudoinvexity

Just as happens in the scalar case, we are going to use Kuhn–Tucker and Fritz–John vector critical points as optimality conditions, as we shall define below.

Definition 4. A feasible point $\bar{x}$ for (CMP) is said to be a Fritz–John vector critical point, FJVCP, if there exist $\lambda \in R^p$, $\mu \in R^m$ such that

$$\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) = 0 \tag{1}$$

$$\mu^T g(\bar{x}) = 0 \tag{2}$$

$$(\lambda, \mu) \geqq 0, \qquad (\lambda, \mu) \neq 0. \tag{3}$$

This is equivalent to saying that there exists $(\lambda, \mu_I) \geq 0$ such that

$$\lambda^T \nabla f(\bar{x}) + \mu_I^T \nabla g_I(\bar{x}) = 0, \quad I = I(\bar{x}) = \{j = 1, \ldots, m : g_j(\bar{x}) = 0\}. \tag{4}$$

The same will occur in the following definition, but in this case with $(\lambda, \mu_I) \geq 0, \lambda \neq 0$.

Definition 5. A feasible point $\bar{x}$ for (CMP) is said to be a Kuhn–Tucker vector critical point, KTVCP, if there exist $\lambda \in R^p$, $\mu \in R^m$ such that

$$\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) = 0 \tag{5}$$

$$\mu^T g(\bar{x}) = 0 \tag{6}$$

$$\mu \geqq 0 \tag{7}$$

$$\lambda \geq 0. \tag{8}$$

On the basis of results by Chankong and Haimes [2] and Kanniappan [10] we obtain the following Fritz–John optimality result for efficient solutions of (CMP).

Theorem 2. *If $\bar{x}$ is an efficient solution of* (CMP)*, then $\bar{x}$ is a FJVCP.*

In a similar manner to that which appears in Kanniappan [10] and Gulati and Talaat [7], the following Kuhn–Tucker optimality result is obtained for efficient solutions of (CMP), for which we need to take on a constraint qualification.

Theorem 3. *If $\bar{x}$ is an efficient solution of* (CMP) *and a constraint qualification is satisfied at $\bar{x}$, then $\bar{x}$ is a KTVCP.*

Like for the scalar case, we have got the following classes of functions, and we can find them, amongst others, defined in [14,15].

Definition 6. Let $f = (f_1, \ldots, f_p) : S \subseteq R^n \to R^p$ be a differentiable function on the open set S. Then the vector function f is said to be invex if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that $\forall x, \bar{x} \in S$,

$$f(x) - f(\bar{x}) \geqq \nabla f(\bar{x})\eta(x, \bar{x}).$$

Next we define the class of pseudoinvex functions, which we will designate as pseudoinvex-I, to distinguish it from a new class which we are going to introduce and will designate as pseudoinvex-II, and which also generalizes the pseudoinvex class defined for scalar functions.

Definition 7. Let $f = (f_1, \ldots, f_p) : S \subseteq R^n \to R^p$ be a differentiable function on the open set S. Then the vector function f is said to be pseudoinvex-I if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that $\forall x, \bar{x} \in S$,

$$f(x) - f(\bar{x}) < 0 \Rightarrow \nabla f(\bar{x})\eta(x, \bar{x}) < 0.$$

Definition 8. Let $f = (f_1, \ldots, f_p) : S \subseteq R^n \to R^p$ be a differentiable function on the open set S. Then the vector function f is said to be pseudoinvex-II if there exists a vector function $\eta : R^n \times R^n \to R^n$ such that $\forall x, \bar{x} \in S$,

$$f(x) - f(\bar{x}) \leq 0 \Rightarrow \nabla f(\bar{x})\eta(x, \bar{x}) < 0.$$

In the scalar case, i.e., $p = 1$, these three definitions are equivalent [1]. In the multiobjective case, the relationship between pseudoinvexity-I and pseudoinvexity-II is as follows.

Proposition 1. *Let $f = (f_1, \ldots, f_p) : S \subseteq R^n \to R^p$ be a differentiable function on the open set S. If the vector function f is pseudoinvex-II, then the vector function f is pseudoinvex-I.*

Egudo and Hanson [4] establish results for multiobjective problems under invexity. However, finding a class of functions for which it is possible to verify not only that a vector critical point is an efficient solution, but also that the functions are characterized by this, was still left pending. Osuna et al. [15] proved that for weakly efficient solutions, this search was resolved by the class KT-pseudoinvex-I, which they defined in the following way.

Definition 9. Problem (CMP) is said to be KT-pseudoinvex-I if there exists a vector function $\eta : S \times S \to R^n$ such that for all feasible points x_1, x_2,

$$f(x_1) - f(x_2) < 0 \Rightarrow \begin{cases} \nabla f(x_2)\eta(x_1, x_2) < 0 \\ \nabla g_j(x_2)\eta(x_1, x_2) \leqq 0, & \forall j \in I(x_2) \end{cases}$$

where $I(x_2) = \{j = 1, \ldots, m : g_j(x_2) = 0\}$.

And Osuna et al. [15] established the following theorem for (CMP).

Theorem 4. *Every KTVCP is a weakly efficient solution of* (CMP) *if and only if problem* (CMP) *is KT-pseudoinvex-I.*

For the study of efficient points of (CMP) from the condition of optimality of Kuhn–Tucker vector critical points, we are going to need a new kind of function, one that is contained in the KT-pseudoinvex-I class, and which we present below.

Definition 10. Problem (CMP) is said to be KT-pseudoinvex-II if there exists a vector function $\eta : S \times S \to R^n$ such that for all feasible points x_1, x_2,

$$f(x_1) - f(x_2) \leq 0 \Rightarrow \begin{cases} \nabla f(x_2)\eta(x_1, x_2) < 0 \\ \nabla g_j(x_2)\eta(x_1, x_2) \leqq 0, & \forall j \in I(x_2) \end{cases}$$

where $I(x_2) = \{j = 1, \ldots, m : g_j(x_2) = 0\}$.

In the same way, for the study of efficient points from the Fritz–John optimality condition we need a new kind of function which we will designate as FJ-pseudoinvex-II.

Definition 11. Problem (CMP) is said to be FJ-pseudoinvex-II if there exists a vector function $\eta : S \times S \to R^n$ such that for all feasible points x_1, x_2,

$$f(x_1) - f(x_2) \leq 0 \Rightarrow \begin{cases} \nabla f(x_2)\eta(x_1, x_2) < 0 \\ \nabla g_j(x_2)\eta(x_1, x_2) < 0, & \forall j \in I(x_2) \end{cases}$$

where $I(x_2) = \{j = 1, \ldots, m : g_j(x_2) = 0\}$.

And the relationship between these classes of functions is as follows.

Proposition 2. *If* (CMP) *is FJ-pseudoinvex-II, then* (CMP) *is KT-pseudoinvex of type II.*

With all these premises we have the conditions with which to characterize efficient solutions of the constrained multiobjective problem (CMP), in the next section.

3. Characterization of efficient solutions

Let us begin with Kuhn–Tucker optimality conditions. The Kuhn–Tucker optimality condition is necessary for a point to be an efficient solution for (CMP), as we have already seen. Let us also observe that under KT-pseudoinvexidad-II, the Kuhn–Tucker optimality condition is sufficient for a point to be an efficient solution. But moreover, KT-pseudoinvexity-II is a necessary condition, as we demonstrate below.

Theorem 5. *Every KTVCP is an efficient solution of* (CMP) *if and only if* (CMP) *is KT-pseudoinvex-II.*

Proof. (i) Let $\bar{x}$ be a KTVCP and (CMP) KT-pseudoinvex-II. We have to prove that $\bar{x}$ is an efficient solution of (CMP), and to do so let us suppose that it is not. Then there exists a feasible point x such that

$$f(x) - f(\bar{x}) \leq 0.$$

Since (CMP) is KT-pseudoinvex-II, there exists a vector function η such that

$$\left.\begin{array}{l} \nabla f(\bar{x})\eta(x,\bar{x}) < 0 \\ \nabla g_I(\bar{x})\eta(x,\bar{x}) \leqq 0, \quad I = I(\bar{x}) \end{array}\right\}. \tag{9}$$

On the other hand, $\bar{x}$ is a KTVCP; then $\exists(\bar{\lambda},\bar{\mu}) \geq 0$, $\bar{\lambda} \neq 0$ such that

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}_I^T \nabla g_I(\bar{x}) = 0, \quad I = I(\bar{x})$$

and by multiplying this equality by $\eta(x,\bar{x})$, we get

$$\bar{\lambda}^T \nabla f(\bar{x})\eta(x,\bar{x}) + \bar{\mu}_I^T \nabla g_I(\bar{x})\eta(x,\bar{x}) = 0. \tag{10}$$

Since $\bar{\lambda} \geq 0$, $\bar{\mu}_I \geqq 0$ and from (9), it follows that

$$\bar{\lambda}^T \nabla f(\bar{x})\eta(x,\bar{x}) < 0$$
$$\bar{\mu}_I^T \nabla g_I(\bar{x})\eta(x,\bar{x}) \leqq 0$$

and then

$$\bar{\lambda}^T \nabla f(\bar{x})\eta(x,\bar{x}) + \bar{\mu}_I^T \nabla g_I(\bar{x})\eta(x,\bar{x}) < 0$$

which stands in contradiction to (10), and therefore, $\bar{x}$ is an efficient solution of (CMP).

(ii) Let us suppose that there exist two feasible points x and $\bar{x}$ such that

$$f(x) - f(\bar{x}) \leq 0,$$

since otherwise (CMP) would be KT-pseudoinvex-II, and the result would be proved. This means that $\bar{x}$ is not an efficient solution, and by using the initial hypothesis, $\bar{x}$ is not a KTVCP, i.e.,

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}_I^T \nabla g_I(\bar{x}) = 0$$

has no solution $\bar{\lambda} \geq 0$, $\bar{\mu}_I \geqq 0$. Therefore, by Motzkin's theorem, the system

$$\left.\begin{array}{l} \nabla f(\bar{x})\eta(x,\bar{x}) < 0 \\ \nabla g_I(\bar{x})\eta(x,\bar{x}) \leqq 0 \quad I = I(\bar{x}) \end{array}\right\}$$

has the solution $\eta(x,\bar{x}) \in R^n$. In consequence, (CMP) is KT-pseudoinvex-II. □

From Theorem 5 it is easy to see that KT-pseudoinvexity-II, in the case of restriction and application to the scalar problem (CP), is equivalent to the KT-invexity given by Martin [11], and moreover, Theorem 5 is a generalization of Theorem 1. Therefore, the results for efficient solutions of (CMP) can be considered a generalization of results for the scalar problem (CP) given by Martin [11].

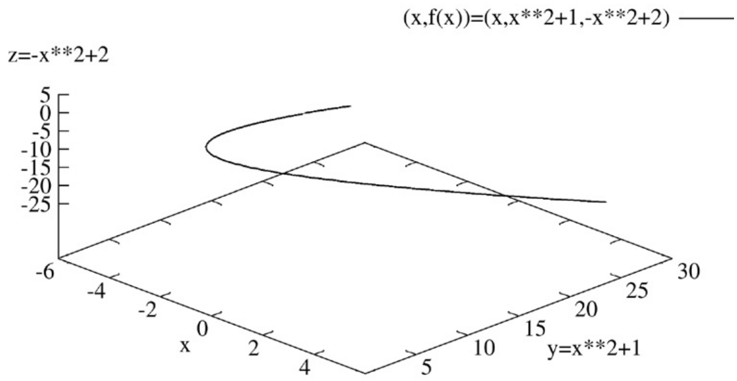

Fig. 1. Objective function graph for (CMP1).

Proceeding in the same way as in the demonstrations of the above result, we are going to prove the following theorem by using Fritz–John vector critical points.

Theorem 6. *Every FJVCP is an efficient solution of* (CMP) *if and only if* (CMP) *is FJ-pseudoinvex-II.*

Therefore, FJ-pseudoinvexity-II is characterized by the FJVCP being identified with the efficient solutions to our multiobjective problem.

4. Examples

The relationship between efficient solutions and vector critical points has been studied from KT-pseudoinvexity-II and FJ-pseudoinvexity-II of (CMP). This relationship becomes an equivalence under this kind of pseudoinvexity and even characterizes KT-pseudoinvex-II and FJ-pseudoinvex-II multiobjective problems. KT-pseudoinvexity-II generalizes KT-invexity introduced by Martin [11], who extended the invexity characterization result established by Craven and Glover [6] to constrained scalar problems. KT-invexity is a stronger condition than invexity of f and g with respect to the same η, although invexity provided a sufficient condition for a Kuhn–Tucker critical point to be an optimal solution, such as Hanson [8] proved. In this sense, we consider the following examples to illustrate these optimality results related to KT-pseudoinvexity, FJ-pseudoinvexity and invexity.

Example 1. We present an example of a KT-pseudoinvex-II and FJ-pseudoinvex-II multiobjective problem, in which we show that all KTVCP and FJVCP are efficient solutions. Besides, we prove that invexity of the functions involved in this problem, f and g, is not verified.

(CMP1) Minimize $(x^2+1, -x^2+2)$
subject to:
$3 - x \leq 0$
$x \in R$

where $f = (f_1, f_2) : R \to R^2$, with $f(x) = (f_1(x), f_2(x)) = (x^2+1, -x^2+2)$, and $g : R \to R$, with $g(x) = 3-x$, are differentiable functions on R. An objective function graph is shown in Fig. 1, and its projections on the XY and XZ planes to provide a better view in Fig. 2.

(i) (CMP1) is KT-pseudoinvex-II and FJ-pseudoinvex-II. Let us prove it.
The inequality $f(x) - f(\bar{x}) = (x^2 - \bar{x}^2, \bar{x}^2 - x^2) \leq 0$ is equivalent to

$$\begin{cases} x^2 - \bar{x}^2 < 0 \quad \text{and} \quad \bar{x}^2 - x^2 \leq 0 \\ \text{or} \\ x^2 - \bar{x}^2 \leq 0 \quad \text{and} \quad \bar{x}^2 - x^2 < 0. \end{cases}$$

But this last condition is not verified since if $x^2 - \bar{x}^2 < 0$, then $\bar{x}^2 - x^2 > 0$ and if $\bar{x}^2 - x^2 < 0$, then $x^2 - \bar{x}^2 > 0$. Therefore, the inequality $f(x) - f(\bar{x}) \leq 0$ is not verified, and we conclude that (CMP) is KT-pseudoinvex-II and FJ-pseudoinvex-II.

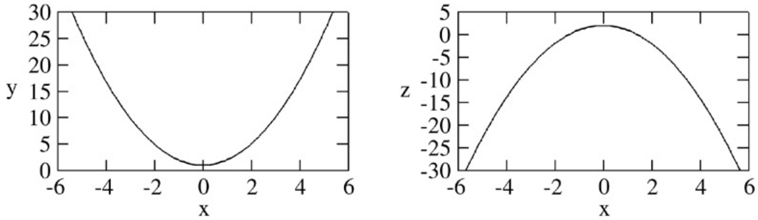

Fig. 2. Projections of the objective function graph for (CMP1).

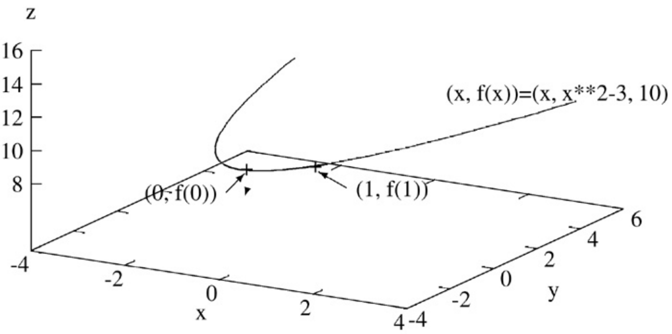

Fig. 3. Objective function graph for (CMP2).

(ii) Let us see that f is not invex. In contrast, from the invexity definition, f_1 and f_2 should be invex, but $f_2(x) = -x^2 + 2$, which implies that f_2 is not invex, and neither is f.

(iii) To study KTVCP and FJVCP, let us consider $x \geq 3$, and $\nabla f(x) = (2x, -2x)$, $\nabla g(x) = -1$. If x is a KTVCP, then there exist $\lambda = (\lambda_1, \lambda_2)$ and μ such that (x, λ, μ) has to satisfy the equations

$$2\lambda_1 x - 2\lambda_2 x - \mu = 0$$
$$\mu(3 - x) = 0$$
$$\mu \geq 0, \qquad (\lambda_1, \lambda_2) \geq 0.$$

From the second equation, it follows that $\mu = 0$ or $x = 3$. If $\mu = 0$, from the first equation we have that $2x(\lambda_1 - \lambda_2) = 0$, which is verified if $\lambda_1 = \lambda_2 > 0$. Therefore given x, $\mu = 0$ and $\lambda_1 = \lambda_2 > 0$ we obtain (x, λ, μ), which verifies the equations before. So, all feasible points x are KTVCP. Moreover, all feasible points are FJVCP.

(iv) Seeing that given $x, \bar{x}$ feasible points, $f(x) - f(\bar{x}) \leq 0$ is not verified (such as has been proved at (i)), it follows that all feasible points are efficient solutions for (CMP1).

Consequently, (CMP1) is a KT-pseudoinvex-II and FJ-pseudoinvex-II problem, where all KTVCP and FJVCP are efficient solutions; however the objective function f is not invex.

Example 2. Conversely, we present a new problem (CMP2) with f and g invex, which is not KT-pseudoinvex-II, and we provide a KTVCP that is not an efficient solution.

(CMP2) Minimize $(x^2 - 3, 10)$
subject to:
$-1 - x \leq 0$
$x \in R$

where $f = (f_1, f_2) : R \to R^2$, with $f(x) = (f_1(x), f_2(x)) = (x^2 - 3, 10)$, and $g : R \to R$, with $g(x) = -1 - x$, are differentiable functions on R. There is shown an objective function graph in Fig. 3.

(i) Let us see that f and g are invex. Firstly, let us study f. We have that $\nabla f(x) = (2x, 0)$. And we define

$$\eta(x,\bar{x}) = \begin{cases} \dfrac{x^2-\bar{x}^2}{2\bar{x}}, & \bar{x} \neq 0 \\ 1, & \bar{x} = 0. \end{cases}$$

It follows that

$$f(x) - f(\bar{x}) = (x^2 - \bar{x}^2, 0) = \begin{cases} \left(\dfrac{x^2-\bar{x}^2}{2\bar{x}} \cdot 2\bar{x}, 0\right), & \bar{x} \neq 0 \\ (x^2, 0), & \bar{x} = 0 \end{cases} \geqq \nabla f(\bar{x})\eta(x,\bar{x}).$$

And in consequence, f is invex with respect to η. On the other hand, it is easy to see that g is invex.

(ii) (CMP2) is not KT-pseudoinvex-II, for which we select the feasible points $x = 1$ and $\bar{x} = 2$. $f(x) - f(\bar{x}) = (-3, 0) \leq 0$. But $\nabla f(\bar{x})u = (4u, 0) \nless 0$, for all $u \in R$. Then there exists no η such that $\nabla f(\bar{x})\eta(x,\bar{x}) < 0$, and therefore (CMP2) is not KT-pseudoinvex-II.

(iii) Let us consider the feasible point $\bar{x} = 1$. The point $\bar{x}$ is a KTVCP because it verifies the corresponding conditions:

$$2\lambda_1 x - \mu = 0$$
$$\mu(-1 - x) = 0$$
$$\mu \geq 0, \qquad (\lambda_1, \lambda_2) \geq 0$$

for every $\mu = 0$, $\lambda_1 = 0$ and $\lambda_2 > 0$. Otherwise, $\bar{x}$ is not an efficient solution for (CMP2), since given the feasible point $x = 0$, we have that $f(x) - f(\bar{x}) = (-1, 0) \leq 0$. They are both illustrated in Fig. 3.

So, there exists a KTVCP, which is not an efficient solution for the multiobjective problem (CMP2), and however f and g are invex functions.

5. Duality

Let us move on to the study of duality. In order to do so, we are going to tackle duality between the multiobjective problem (CMP), and two associated problems of the Mond–Weir [13] type, defined for the multiobjective case by Egudo and Hanson [4], but with the differences that the complementarity condition has been restricted and that of the non-negativity of (λ, μ) has been extended now that $\lambda > 0$ is not necessary. Let us begin with the first problem (DM1) as the dual of (CMP), and formulated thus:

(DM1) Maximize $f(u)$
subject to:

$$\lambda^T \nabla f(u) + \mu^T \nabla g(u) = 0 \tag{11}$$
$$\mu_j g_j(u) = 0, \quad j = 1\ldots, m \tag{12}$$
$$\mu \geqq 0 \tag{13}$$
$$\lambda \geq 0 \tag{14}$$
$$u \in S \subseteq R^n.$$

We will need a new class of pseudoinvex functions, which differs slightly from those already defined and which we present below.

Definition 12. The pair of functions (f, g) is said to be KT-pseudoinvex-II on S if there exists a vector function $\eta : S \times S \to R^n$ such that for all $x_1, x_2 \in S$,

$$f(x_1) - f(x_2) \leq 0 \Rightarrow \begin{cases} \nabla f(x_2)\eta(x_1,x_2) < 0 \\ \nabla g_I(x_2)\eta(x_1,x_2) \leqq 0 \end{cases}$$

with $I = I(x_2) = \{j = 1, \ldots, m : g_j(x_2) = 0\}$.

The relationship between this class of functions and KT-pseudoinvexity-II of (CMP) is as follows.

Proposition 3. *If* (f, g) *is KT-pseudoinvex-II on S, then* (CMP) *is KT-pseudoinvex-II.*

Let us move on to the result of duality; in order to do so we begin with weak duality.

Theorem 7 (*Weak Duality*). *Let* x *be a feasible point for (CMP), and* (u, λ, μ) *a feasible point for (DM1). If* (f, g) *is KT-pseudoinvex-II on S then* $f(x) \leq f(u)$ *is not verified.*

Proof. Let us suppose that (f, g) is KT-pseudoinvex-II with respect to a vector function η. Let x be a feasible point for (CMP), (u, λ, μ) a feasible point for (DM1), such that $f(x) \leq f(u)$ as, if not, the result would be proved. Then, there exist $\lambda \in R^p$, $\mu \in R^m$ such that

$$\lambda^T \nabla f(u) + \mu^T \nabla g(u) = 0$$
$$\mu_j g_j(u) = 0, \quad j = 1 \ldots, m$$
$$\mu \geqq 0$$
$$\lambda \geq 0$$

i.e.,

$$\lambda^T \nabla f(u) + \mu_I^T \nabla g_I(u) = 0$$

with $(\lambda, \mu_I) \geq 0$, $\lambda \neq 0$, $I = I(u) = \{j = 1, \ldots, m : g_j(u) = 0\}$. In consequence, on multiplying this expression by $\eta(x, u)$, it follows that

$$\lambda^T \nabla f(u)\eta(x, u) + \mu_I^T \nabla g_I(u)\eta(x, u) = 0. \tag{15}$$

Since $f(x) \leq f(u)$, from the KT-pseudoinvexity-II of (f, g) it follows that

$$\left.\begin{array}{l} \nabla f(\bar{u})\eta(x, u) < 0 \\ \nabla g_I(\bar{u})\eta(x, u) \leqq 0 \quad I = I(\bar{x}) \end{array}\right\}$$

and by multiplying by (λ, μ_I) we have

$$\lambda^T \nabla f(u)\eta(x, u) + \mu_I^T \nabla g_I(u)\eta(x, u) < 0,$$

which stands in contradiction to (15), and therefore, $f(x) \leq f(u)$ is not verified. □

This weak duality result allows us to prove the strong duality result, as follows.

Theorem 8 (*Strong Duality*). *Let (f,g) be KT-pseudoinvex-II on S. If* $\bar{x}$ *is an efficient solution for* (CMP) *and a constraint qualification is satisfied at* $\bar{x}$*, then there exists* λ, μ*, such that* $(\bar{x}, \lambda, \mu)$ *is an efficient solution of* (DM1).

Proof. Let us suppose that $\bar{x}$ is an efficient solution for (CMP). From Theorem 3, $\bar{x}$ is a KTVCP, i.e., $\exists(\lambda, \mu) \geq 0$, $\lambda \neq 0$ such that

$$\lambda^T \nabla f(\bar{x})^T + \mu^T \nabla g(\bar{x})^T = 0$$
$$\mu^T g(\bar{x}) = 0.$$

Since $g(\bar{x}) \leqq 0$, and $\mu^T g(\bar{x}) = 0$, it follows that

$$\mu_j g_j(\bar{x}) = 0 \quad j = 1, \ldots, m.$$

Then $\bar{x}$ is a feasible point for (DM1), and from the weak duality theorem $f(\bar{x}) \leq f(u)$ is not verified, where u is a feasible point for (DM1). Therefore, $\bar{x}$ is an efficient solution of (DM1). □

The converse result is also verified, as we demonstrate below.

Theorem 9 (*Converse Duality*). *Let* (CMP) *be KT-pseudoinvex-II, and* $\bar{x}$ *a feasible point for* (CMP). *If* $(\bar{x}, \lambda, \mu)$ *is a feasible point for* (DM1) *then* $\bar{x}$ *is an efficient solution of* (CMP).

Proof. Let us suppose that $\bar{x}$ is a feasible point for (CMP). If $(\bar{x}, \lambda, \mu)$ is a feasible point for (DM1), then

$$\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) = 0$$
$$\mu_j g_j(\bar{x}) = 0, \quad j = 1, \ldots, m$$

and therefore $\bar{x}$ is a KTVCP. Since (CMP) is KT-pseudoinvex-II, from Theorem 5 it follows that $\bar{x}$ is an efficient solution of (CMP). □

Similarly, we can obtain duality results for (CMP) and the following dual problem.

(DM2) Maximize $f(u)$
subject to:

$$\lambda^T \nabla f(u) + \mu^T \nabla g(u) = 0 \tag{16}$$
$$\mu_j g_j(u) = 0, \quad j = 1 \ldots, m \tag{17}$$
$$(\lambda, \mu) \geq 0 \tag{18}$$
$$u \in S \subseteq R^n.$$

For this, we introduce the following definition.

Definition 13. The pair of the functions (f, g) is said to be FJ-pseudoinvex-II on S if there exists a vector function η such that for all $x_1, x_2 \in S$

$$f(x_1) - f(x_2) \leq 0 \Rightarrow \begin{cases} \nabla f(x_2)\eta(x_1, x_2) < 0 \\ \nabla g_I(x_2)\eta(x_1, x_2) < 0 \end{cases}$$

with $I = I(x_2) = \{j = 1, \ldots, m : g_j(x_2) = 0\}$.

The relationship between this class of functions and the FJ-pseudoinvexity-II of (CMP) is as follows.

Proposition 4. *If (f, g) is FJ-pseudoinvex-II on S, then* (CMP) *is FJ-pseudoinvex-II.*

Following the same lines as the above results and demonstrations, we put forward the following duality theorems.

Theorem 10 (*Weak Duality*). *Let x be a feasible point for* (CMP)*, and (u, λ, μ) a feasible point for* (DM2)*. If (f, g) is FJ-pseudoinvex-II on S then $f(x) \leq f(u)$ is not verified.*

Theorem 11 (*Strong Duality*). *Let (f,g) be FJ-pseudoinvex-II on S. If $\bar{x}$ is an efficient solution for* (CMP)*, then there exists λ, μ, such that $(\bar{x}, \lambda, \mu)$ is an efficient solution of* (DM2).

Theorem 12 (*Converse Duality*). *Let* (CMP) *be FJ-pseudoinvex-II, and $\bar{x}$ a feasible point for* (CMP)*. If $(\bar{x}, \lambda, \mu)$ is a feasible point for* (DM2) *then $\bar{x}$ is an efficient solution of* (CMP).

6. Conclusion

In multiobjective mathematical programming problems we can tackle the search for efficient solutions from optimality conditions of the Kuhn–Tucker type or Fritz–John type. For the first of these cases, we have introduced a new kind of function, the KT-pseudoinvex-II, which is characterized by every Kuhn–Tucker point being an efficient solution. Analogously, the Fritz–John optimality condition is a necessary and sufficient condition for a point to be an efficient solution beneath FJ-pseudoinvexity-II, and moreover this kind of function is characterized by this. With this, we generalize the characterizations of optimality provided by Martin [11] for scalar problems and complete work carried out by Osuna et al. [15,14] for weakly efficient solutions. We illustrate these optimality results with some examples.

On the other hand, we establish results of duality beneath KT-pseudoinvexity-II, both weak and strong as well inverse. In the case of the class of functions being FJ-pseudoinvex-II we also establish results of weak, strong and inverse duality.

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