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Applying a new algorithm to derive nonclassical symmetries

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Abstract

In this paper we extend the procedure described for Bîlă and Niesen in [Bîlă N, Niesen J. On a new procedure for finding nonclassical symmetries. J Symbol Comp 2004;38:1523–33], to obtain the determining equations of the nonclassical symmetries associated with a partial differential equation system, to a different case. We offer some examples of how our method works. By using this procedure we obtain a new nonclassical symmetry for the $2+1$-dimensional shallow water wave equation.

Keywords: Partial differential equations; Symmetries; Nonclassical method

1. Introduction

The application of Lie transformations group theory for the construction of solutions of nonlinear partial differential equations (PDEs) is one of the most active fields of research in the theory of nonlinear PDEs and applications.

Motivated by the fact that symmetry reductions for many PDEs are known that are not obtained by using the classical symmetries, there have been several generalizations of the classical Lie group method for symmetry reductions. The notion of nonclassical symmetries was firstly introduced by Bluman and Cole [1] to study the symmetry reductions of the heat equation. The description of the method can be found in [1,3,7]. In [4] Clarkson and Mansfield proposed an algorithm for calculating the determining equations associated with the nonclassical method: the PDE system is augmented with the invariant surface conditions, the nonclassical symmetries are found by seeking the classical symmetries of the augmented system while demanding that the symmetries operator be related to the invariant surface condition.

Bîlă and Niesen in [2] dropped this requirement. Their procedure consists in reducing the augmented PDE system to its involutive form and then applying the classical Lie method to the reduced PDE system, but with an arbitrary symmetry operator which is not related anymore to the invariant surface condition.

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In this paper we extend the procedure described in [2] to a different case. We apply the procedure to a Cahn–Hilliard equation, to a Boussinesq equation and to a $2+1$-dimensional shallow water wave equation.

2. Nonclassical symmetries

Bluman and Cole [1], in their study of symmetry reductions of the heat equation, proposed the called nonclassical method. The basic idea of the method applied to the general nth order PDE, with p independent variables, $x=(x_1,\ldots,x_p)$, and one dependent variables, $u=u(x)$,

$$\Delta \equiv \Delta(x,u,\mathbf{u}^{(l)}(x),\ldots,\mathbf{u}^{(n)}(x)) = 0, \tag{1}$$

where $\mathbf{u}^{(l)}(x)$ denotes the set of all the partial derivatives of order l of u, is the following:

The PDE (1) is augmented with the invariance surface condition

$$\Psi \equiv \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial u}{\partial x_i} - \eta(x,u) = 0, \tag{2}$$

which is associated with the vector field

$$V = \sum_{i=1}^{p} \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}. \tag{3}$$

Let us consider the submanifold

$$S_\Delta = \{u(x) : \Delta = 0\}, \tag{4}$$

i.e., the set of solutions of the system (1). In the nonclassical method one requires that the subset of S_Δ given by

$$S_{\Delta,\Psi} = \{u(x) : \Delta = 0, \Psi = 0\}, \tag{5}$$

are invariant under the transformation with infinitesimal generator (3).

The application of the criterion for infinitesimal invariance to the equation (1) and the invariant surface condition (2) require that

$$\mathrm{pr}^{(n)}V(\Delta)_{\Delta=0,\Psi=0} = 0, \qquad \mathrm{pr}^{(n)}V(\Psi)_{\Delta=0,\Psi=0} = 0, \tag{6}$$

and we obtain an overdetermined nonlinear system of equations for the infinitesimals. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is, in general, larger than for the classical method [4].

• If $\xi_p \neq 0$ Bîlă and Niesen proposed an algorithm for finding nonclassical symmetries which is based on the following procedure: Since if V is a vector field then so is λV, for any function $\lambda=\lambda(x,u)$, if $\xi_p \neq 0$ we can multiply V for $\frac{1}{\xi_p}$ and the invariant surface conditions is,

$$\frac{\partial u}{\partial x_p} = \eta(x,u) - \sum_{i=1}^{p-1} \xi_i(x,u)\frac{\partial u}{\partial x_i}. \tag{7}$$

Substituting (7) and its derivatives with respect to x in (1) we obtain a new PDE

$$\Delta' \equiv \Delta'(\mathscr{A}_v(x,u),\mathbf{u}^{[l]},\ldots,\mathbf{u}^{[n]}) = 0, \tag{8}$$

for the unknown function $u=u(x_1,\ldots,x_{p-1};\,x_p)$ of $x_1,\ldots,x_{p-1}$ (here x_p is considered as a parameter); where $\mathscr{A}_v(x,u)$ are the coefficients of $u^{[l]}$, and $u^{[l]}$ denotes the set of all the partial derivatives of u with respect to $x=(x_1,x_2,\ldots,x_{p-1})$ up to order N. Applying the classical Lie method to (8), if (8) is of maximal rank. Invariance of (8) under a Lie group of point transformation, with infinitesimal generator

$$W = \sum_{i=1}^{p} s_i(x,u)\frac{\partial}{\partial x_i} + r(x,u)\frac{\partial}{\partial u}, \tag{9}$$

leads to the determining equations. Substituting $s_p = 1$, $s_i = \xi_i$, $i = 1,\ldots,p-2$, $r = \eta$ and the functions $\mathscr{A}_v$ into the determining equations we obtain the determining equations of the nonclassical symmetries of the original PDE (1).

• The case for $\xi_p = 0$ has not been considered in [2] because the authors claim needs to be handled separately. If $\xi_p = 0$ and at least two $\xi_i \neq 0$, $i = 1,\ldots,p-1$ we can procedure in a similar way to the one described in [2] and we take $\xi_{p-1} = 1$, without loss of generality. The vector field (3) can be expressed in the form

$$V = \sum_{i=1}^{p-2} \xi_i(x,u)\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{p-1}} + \eta(x,u)\frac{\partial}{\partial u},$$

and the associated invariant surface condition is

$$u_{p-1} = \eta(x,u) - \sum_{i=1}^{p-2} \xi_i(x,u)\frac{\partial u}{\partial x_i}.$$

By substituting u_{p-1} and its differential consequences in (1) we obtain a new PDE where the coefficients of $u^{(n)}$ are functions which depends of (x,u),

$$\Omega = \Omega\left(x,u^{(n)}\right). \tag{10}$$

To apply the classical Lie method to equation (10) with infinitesimal generator (9) we require that

$$\mathrm{pr}^{(n)}(W)(\Omega)|_{\Omega=0} = 0,$$

where

$$\mathrm{pr}^{(n)}(W)(\Omega) = \sum\nolimits_{i=1}^{p} s_i(x,u)\frac{\partial\Omega}{\partial x_i} + r(x,u)\frac{\partial\Omega}{\partial u} + \sum\nolimits_I \left[D_I\left(r - \sum_{i=1}^{p} s_i u_i\right) + \sum_{i=1}^{p} s_i u_{I,i}\right]\frac{\partial\Omega}{\partial u_I} \tag{11}$$

with $s_i = \frac{\partial s}{\partial x_i}$, $u_{I,i} = \frac{\partial u_I}{\partial x_i}$, $I = (i_1,\ldots,i_k)$, $1 \leqslant i_k \leqslant p$ and $1 \leqslant k \leqslant n$. Substituting $s_i = \xi_i$, for $i = 1,\ldots,p$, and $r = \eta$ in (11) we obtain

$$\begin{aligned}\mathrm{pr}^{(n)}(W)(\Omega) = {} & \sum\nolimits_{i=1}^{p-2} \xi_i(x,u)\frac{\partial\Omega}{\partial x_i} + \xi_{p-1}(x,u)\frac{\partial\Omega}{\partial x_{p-1}} + \xi_p(x,u)\frac{\partial\Omega}{\partial x_p} + \eta(x,u)\frac{\partial\Omega}{\partial u} \\ & + \sum\nolimits_I \left[D_I\left(\eta - \sum_{i=1}^{p-2} \xi_i u_i - \xi_{p-1}u_{p-1} - \xi_p u_p\right) + \sum_{i=1}^{p-2} \xi_i u_{I,i} + \xi_{p-1}u_{I,p-1} + \xi_p u_{I,p}\right]\frac{\partial\Omega}{\partial u_I}.\end{aligned} \tag{12}$$

For $\xi_{p-1} = 1$ and $\xi_p = 0$, since $-D_I(\xi_{p-1}u_{p-1}) + \xi_{p-1}u_{I,p-1} = 0$, (12) gives

$$\begin{aligned}\mathrm{pr}^{(n)}(W)(\Omega) = {} & \sum\nolimits_{i=1}^{p-2} \xi_i(x,u)\frac{\partial\Omega}{\partial x_i} + \frac{\partial\Omega}{\partial x_{p-1}} + \eta(x,u)\frac{\partial\Omega}{\partial u} \\ & + \sum_I \left[D_I\left(\eta - \sum_{i=1}^{p-2} \xi_i u_i - u_{p-1}\right) + \sum_{i=1}^{p-2} \xi_i u_{I,i} + u_{I,p-1}\right]\frac{\partial\Omega}{\partial u_I}.\end{aligned}$$

As Ω is obtained from Δ, changing functions, so that the prolongation formula leads to a system which is equivalent to the system obtained by the nonclassical method.

• If one generator is different from zero and $p-1$ generators are equal to zero, without loss of generality, we can set $\xi_1 = 1$, and the invariant surface conditions for $x_1 = x$ becomes

$$u_x = \eta. \tag{13}$$

By substituting u_x and its derivatives in (1) we obtain a new PDE where the coefficients of $u^{(n)}$ are functions which depends of (x,u),

$$\Xi = \Xi(x,u^{(n)}). \tag{14}$$

To apply the classical Lie method to equation (14) with infinitesimal generator (9) we require that

$$\mathrm{pr}^{(n)}(W)(\Xi)|_{\Xi=0}=0,$$

where $\mathrm{pr}^{(n)}(W)$ is given in (11). Substituting $s_1=\xi_1$, $s_i=0$ for $i=1,\ldots,p$, and $r=\eta$ in (11) we obtain

$$\mathrm{pr}^{(n)}(W)(\Xi)=\xi_1(x,u)\frac{\partial\Xi}{\partial x}+\eta(x,u)\frac{\partial\Xi}{\partial u}+\sum_I[D_I(\eta-\xi_1u_x)+\xi_1u_{I,x}]\frac{\partial\Xi}{\partial u_I}. \tag{15}$$

This expression lead to the determining equations of the nonclassical symmetries.

In the following we consider different PDEs for which we have applied the procedure with $\xi_1\neq 0$ and $\xi_i=0$, $i=2,\ldots,p$.

- Case in which (1) can be written, by using (13), in the equivalent form

$$u_t=\mathscr{A}(x,t,u), \tag{16}$$

where $\mathscr{A}$ is a arbitrary function which depend of x, t and u.

Invariance of Eq. (16) under a Lie group of point transformations with infinitesimal generator

$$V=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\phi(x,t,u)\frac{\partial}{\partial u} \tag{17}$$

leads, for $\xi=1$, $\tau=0$ and $\phi=\eta$, to the following determining equation:

$$\mathscr{A}\eta_u+\eta_t-\mathscr{A}_u\eta-\mathscr{A}_x=0. \tag{18}$$

- In the case in which (1) can be written, by using (13), in the equivalent form

$$u_{tt}=\mathscr{A}(x,t,u), \tag{19}$$

where $\mathscr{A}$ is a arbitrary function which depend of x, t and u.

Invariance of this equation under a Lie group of point transformations with infinitesimal generator (17) leads, for $\xi=1$, $\tau=0$ and $\phi=\eta$, to the following determining equations:

$$\begin{aligned}&\eta_{uu}=0,\\&\eta_{tu}=0,\\&\mathscr{A}\eta_u+\eta_{tt}-\mathscr{A}_u\eta-\mathscr{A}_x=0.\end{aligned} \tag{20}$$

- Case in which (1) can be written, by using (13), in the equivalent form

$$\mathscr{A}(x,y,t,u)u_y+u_{ty}+\mathscr{B}(x,y,t,u)=0, \tag{21}$$

where $\mathscr{A}$ and $\mathscr{B}$ are arbitrary functions which depend of x, y, t and u. We apply the classical Lie method to (21). Invariance of this equation under a Lie group of point transformations with infinitesimal generator (9) with $p=3$ leads, for $s_1=1$, $s_2=s_3=0$ and $r=\eta$, to the following determining equations:

$$\begin{aligned}&\eta_{uy}=0,\\&\eta_{uu}=0,\\&\mathscr{A}_u\eta+\eta_{tu}+\mathscr{A}_x=0,\\&\mathscr{A}\eta_y-\mathscr{B}\eta_u+\eta_{ty}+\mathscr{B}_u\eta+\mathscr{B}_x=0.\end{aligned} \tag{22}$$

We observe that for any equation which can be expressed in the form (16), (19) or (21) the nonclassical determining equations can be derived by substituting the corresponding functions $\mathscr{A}$ and $\mathscr{B}$ into (18), (20) or (22).

3. Some examples

3.1. The Cahn–Hilliard equation

The Cahn–Hilliard equation

$$u_t+ku_{xxxx}-f(u)u_{xx}-f'(u)u_x^2=0 \tag{23}$$

describes diffusion for decomposition of a one-dimensional binary solution. In [5,6] the authors have analyzed the classical and nonclassical symmetries of (23).

In order to apply the nonclassical method to the equation (23) we consider a one-parameter group of transformation generated by the vector field (3), which in our case is,

$$V = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{24}$$

We can distinguish two cases:

Case (i): $\xi_2 \neq 0$, we set $\xi_2(x,t,u) = 1$. The equivalent form of (23), by reducing the initial system, using the invariant surface condition $u_t = \eta - \xi_1 u_x$, is

$$k u_{xxxx} - f(u)u_{xx} - f'(u)u_x^2 + \mathscr{A}_1(x,t,u)u_x + \mathscr{A}_2(x,t,u) = 0, \tag{25}$$

where $\mathscr{A}_1 = -\xi_1$ and $\mathscr{A}_2 = \eta$. The classical method applied to (25) gives only rise to the classical symmetries of (23).

Case (ii): $\xi_2 = 0$, we may set $\xi_1 = 1$, without loss of generality. Subsequently, we find the equivalent form of (23) using the invariant surface condition (13) and its derivatives. This yields a new equation in the form (16) where

$$\begin{aligned}\mathscr{A}(x,t,u) = {} & -k(\eta_u\eta_{xx} + 3\eta\eta_{uu}\eta_x + 3\eta_{ux}\eta_x + (\eta_u)^2\eta_x + \eta^3\eta_{uuu} + 3\eta^2\eta_{uux} + 4\eta^2\eta_u\eta_{uu} \\ & + 3\eta\eta_{uxx} + 5\eta\eta_u\eta_{ux} + \eta(\eta_u)^3 + \eta_{xxx}) + f_u\eta^2 + f(\eta_x + \eta_u\eta).\end{aligned} \tag{26}$$

Substituting (26) into (18) we obtain the determining equation for the infinitesimal η derived in [5,6].

3.2. The Boussinesq equation

We apply the new procedure to the Boussinesq equation

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0. \tag{27}$$

The Boussinesq equation arises in several physical applications: propagation of long waves in shallow water, one-dimensional nonlinear lattice-waves, vibrations in a nonlinear string and ion sound waves in a plasma. In [3], symmetry reductions and exact solutions of this equation using the classical Lie method of infinitesimals, the direct method due Clarkson and Kruskal and the nonclassical method due to Bluman and Cole were derived.

In order to apply the new procedure to Eq. (27) we consider a one-parameter group of transformation generated by the vector field (24).

We can distinguish two cases:

Case (i): $\xi_2 \neq 0$, we set $\xi_2(x,t,u) = 1$. Substituting

$$u_{tt} = u_x\xi_1\xi_{1,x} + 2(u_x)^2\xi_1\xi_{1,u} - \eta u_x\xi_{1,u} - u_x\xi_{1,t} + u_{xx}\xi_1^2 - 2\eta_u u_x\xi_1 - \eta_x\xi_1 + \eta\eta_u + \eta_t,$$

which is obtained from the invariant surface condition, in (27) we obtain that the equivalent form is

$$u_{xxxx} + \mathscr{A}(x,t,u)u_{xx} + \mathscr{B}(x,t,u)u_x^2 + \mathscr{C}(x,t,u)u_x + \mathscr{D}(x,t,u) = 0, \tag{28}$$

where $\mathscr{A} = \xi_1^2 + u$ and $\mathscr{B} = 2\xi_1\xi_{1,u} + 1$, $\mathscr{C} = -2\xi_1\eta_u - \eta\xi_{1,u} + \xi_1\xi_{1,x} - \xi_{1,t}$ and $\mathscr{D} = -\xi_1\eta_x + \eta\eta_u + \eta_t$.We apply the classical Lie method to (28). Invariance of Eq. (28) under a Lie group of point transformations with infinitesimal generator (17). Substituting $\xi = \xi_1$, $\tau = 1$, $\phi = \eta$ and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ we obtain the determining equations derived in ([3], Case 2.3.1).

Case (ii): $\xi_2 = 0$, we may set $\xi_1 = 1$, without loss of generality. The equivalent form of (27), using the invariant surface conditions (13) and its derivatives, is given in (19) where

$$\mathcal{A}(x,t,u) = -(\eta_x + \eta\eta_u)u - \eta_{xxx} - \eta_u\eta_{xx} - 3\eta\eta_{uu}\eta_x - 3\eta_{ux}\eta_x - (\eta_u)^2\eta_x - \eta^3\eta_{uuu} \\ - 3\eta^2\eta_{uux} - 4\eta^2\eta_u\eta_{uu} - 3\eta\eta_{uxx} - 5\eta\eta_u\eta_{ux} - \eta(\eta_u)^3 - \eta^2. \quad (29)$$

Substituting (29) in (20) we obtain the determining equations derived in ([3] Case 2.3.2).

3.3. The 2 + 1-dimensional shallow water wave equation

We apply the procedure described to the 2 + 1-dimensional shallow water wave equation

$$u_{yt} + \alpha u_x u_{xy} + \beta u_y u_{xx} + u_{xxxy} = 0, \quad (30)$$

where α and β are arbitrary, nonzero, constants. Classical and nonclassical reductions of this equation are classified in [8].

In order to apply the new procedure to equation (30) to obtain the nonclassical symmetries we consider a one-parameter group of transformations generated by the vector field

$$V = \xi_1(x,y,t,u)\frac{\partial}{\partial x} + \xi_2(x,y,t,u)\frac{\partial}{\partial y} + \xi_3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}. \quad (31)$$

There are three cases to consider: (i) $\xi_3 \equiv 1$, (ii) $\xi_3 = 0$ and $\xi_2 \equiv 1$ and (iii) $\xi_2 = \xi_3 = 0$ $\xi_1 \equiv 1$.

Case (i): the infinitesimals are equivalent to the classical infinitesimals [8].

Case (ii): $\xi_3 = 0$ and $\xi_2 \equiv 1$. Substituting $u_{yt} = -\xi_{1,u}u_t u_x - \xi_{1,t}u_x - \xi_1 u_{tx} + \eta_u u_t + \eta_t$ and $u_{xy} = -\xi_{1,x}u_x - \xi_{1,u}(u_x)^2 - \xi_1 u_{xx} + \eta_u u_x + \eta_x$, which are obtained from the invariant surface condition $u_y = \eta - \xi_1 u_x$, in (30) we obtain

$$\mathcal{A}_1 u_{xxxx} + \mathcal{A}_2 u_{xxx} + \mathcal{A}_3 u_x u_{xxx} + \mathcal{A}_4 u_x u_{xx} + \mathcal{A}_5 u_x + \mathcal{A}_6 (u_{xx})^2 + \mathcal{A}_7 (u_x)^2 u_{xx} + \mathcal{A}_8 (u_x)^2 + \mathcal{A}_9 u_{xx} \\ + \mathcal{A}_{10}(u_x)^4 + \mathcal{A}_{11}(u_x)^3 + \mathcal{A}_{12}u_{xt} + \mathcal{A}_{13}u_t + \mathcal{A}_{14}u_x u_t + \mathcal{A}_{15} = 0, \quad (32)$$

where

$$\begin{aligned} &\mathcal{A}_1 = \xi_1, \quad \mathcal{A}_2 = 3\xi_{1,x} - \eta_u, && \mathcal{A}_3 = 4\xi_{1,u}, \\ &\mathcal{A}_4 = 9\xi_{1,ux} + (\beta+\alpha)\xi - 3\eta_{uu}, && \mathcal{A}_5 = \alpha\eta_x + 3\eta_{uxx} - \xi_{xxx} - \xi_t, \\ &\mathcal{A}_6 = 3\xi_u, \quad \mathcal{A}_7 = 6\xi_{1,uu}, && \mathcal{A}_8 = 3\xi_{1,uxx} + \alpha(\xi_{1,x} - \eta_u) - 3\eta_{uux}, \\ &\mathcal{A}_9 = \beta\eta - 3\xi_{1,xx} - 3\eta_{ux}, \quad \mathcal{A}_{10} = \xi_{uuu}, && \mathcal{A}_{11} = \eta_{uuu} - \alpha\xi_{1,u} - 3\xi_{1,uux}, \\ &\mathcal{A}_{12} = -\xi_1, \quad \mathcal{A}_{13} = \eta_u, && \mathcal{A}_{14} = -\xi_{1,u}, \quad \mathcal{A}_{15} = -\eta_{xxx} - \eta_t. \end{aligned}$$

We apply the classical Lie method to (32). Invariance of Eq. (32) under a Lie group of point transformations with infinitesimal generator (9), $p = 3$, leads to the determining equations of the classical method. Substituting $s_1 = \xi_1$, $s_2 = 1$, $s_3 = 0$ and $\mathcal{A}_i$, $i = 1,\ldots,15$, we deduce the determining equations of the nonclassical method obtained in [8].

Case (iii): $\xi_2 = \xi_3 = 0$ $\xi_1 \equiv 1$, we consider the equivalent form of (30) given in (21), where

$$\mathcal{A} = \Big((\eta_{uu} + \beta)\eta_x + \eta^2\eta_{uuu} + 2\eta\eta_{uux} + 4\eta\eta_u\eta_{uu} + \eta_{uxx} + 3\eta_u\eta_{ux} + (\eta_u)^3 + (\beta+\alpha)\eta\eta_u\Big),$$

$$\mathcal{B} = (2\eta\eta_{uu} + 2\eta_{ux} + (\eta_u)^2 + \alpha\eta)\eta_y + \eta_{xxy} + \eta_u\eta_{xy} + \eta_{uy}\eta_x + \eta^2\eta_{uuy} + 2\eta\eta_u\eta_{uy} + 2\eta\eta_{uxy}.$$

Substituting $\mathcal{A}$ and $\mathcal{B}$ in (22) we obtain the determining equations for the infinitesimal η,

$$
\begin{aligned}
&\eta_{uy} = 0,\\
&\eta_{uu} = 0,\\
&\beta\eta_{xx} + (\beta+\alpha)\eta_u\eta_x + \eta_{uxxx} + 3\eta_u\eta_{uxx} + 3(\eta_{ux})^2 + \eta(\beta\eta_{ux} + (\beta+\alpha)(\eta_u)^2)\\
&\quad + 3(\eta_u)^2\eta_{ux} + (\beta+\alpha)\eta\eta_{ux} + \eta_{tu} = 0,\\
&-\eta_u((2\eta_{ux} + (\eta_u)^2 + \alpha\eta)\eta_y + \eta_{xxy} + \eta_u\eta_{xy}) + (\beta\eta_x + \eta_{uxx} + 3\eta_u\eta_{ux} + (\eta_u)^3 + (\beta+\alpha)\eta\eta_u)\eta_y\\
&\quad + (\alpha\eta_x + 2\eta_{uxx} + 2\eta_u\eta_{ux})\eta_y + \alpha\eta\eta_u\eta_y + \eta_{xxxy} + \eta_u\eta_{xxy} + (2\eta_{ux} + (\eta_u)^2 + \alpha\eta)\eta_{xy} + \eta_{ux}\eta_{xy} + \eta_{ty} = 0.
\end{aligned}
\tag{33}
$$

The complexity of this system is the reason why we cannot solve (33) in general. Thus we proceed, by making ansatz on the form of $\eta(x,t,u)$, to solve (33). If $\alpha+\beta=0$ one solution of the determining equations is

$$\xi_1 = 1, \qquad \xi_2 = \xi_3 = 0, \qquad \eta = k_1u + k_2x + f(y) + g(t).$$

It is easy to check that these generators do not satisfy Lie classical determining equations and that it is a new nonclassical symmetry. Therefore we obtain the nonclassical symmetry reduction

$$z = y, \qquad r = t, \qquad u = e^{k_1x}h(y,t) - \frac{k_2}{k_1}x - \frac{1}{k_1}(g(t) + f(y)) - \frac{k_2}{k_1^2},$$

where $h(y,t)$ satisfies the PDE

$$h_{yt} + (k_1^3 + \beta k_2)h_y - \beta k_1 f_y h = 0.$$

4. Conclusions

Finding the nonclassical symmetries of PDEs involves a large amount of tedious calculations which can become virtually unmanageable if attempted manually. Some authors have designed computer programs for their obtaining. Bîlă and Niesen [2] developed a procedure to obtain the nonclassical symmetries of a PDEs system. The authors claim that the case $\xi_p = 0$ needs to be handled separately.

In this work we have extended the algorithm described for Bîlă and Niesen to determine the nonclassical symmetries of a PDE for this last case.

We observe that for any equation which can be expressed in the form (16), (19) or (21) the nonclassical determining equation can be derived by substituting the corresponding functions $\mathscr{A}$ and $\mathscr{B}$ into (18), (20) or (22).

We also apply the described algorithm to a Cahn–Hilliard equation, to a Boussinesq equation and to a $2+1$-dimensional shallow water wave equation. For the $2+1$-dimensional shallow water wave equation the method yields a new symmetry reduction which is unobtainable by using Lie classical method.

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